847.

ON THE THEORY OF SEMINVARIANTS.

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IN my "Mémoire sur les Hyperdéterminants," *Crelle*, t. XXX. (1846), pp. 1—37, [13, 14, 16*], I gave an investigation for the number of and relations between the quartic invariants of a given binary quantic: the very same formulæ apply to the cubic covariants of a given binary quantic, or what is the same thing to the cubic seminvariants of a given weight, and I propose at present (considering the formulæ in this point of view) to further develope the theory in regard to the solution of the systems of linear equations obtained in the memoir in question. But I first reproduce the investigation as it stands: I recall that the notation is the ordinary hyperdeterminant notation: for instance,

$$U = \tfrac{1}{2}(a,\ b,\ c \between x_1,\ y_1)^2, \quad V = \tfrac{1}{2}(a,\ b,\ c \between x_2,\ y_2)^2.$$

$$\begin{aligned} T\overline{12}^2 UV &= (\partial_{x_1}\partial_{y_2} - \partial_{y_1}\partial_{x_2})^2\, UV \\ &= (\xi_1\eta_2 - \eta_1\xi_2)^2\, UV \\ &= \xi_1^2 U \,.\, \eta_2^2 V - 2\xi_1\eta_1 U \,.\, \xi_2\eta_2 V + \eta_1^2 U \,.\, \xi_2^2 V \\ &= \quad a\,.\,c \quad - \quad 2b\,.\,b \quad + \quad c\,.\,a \\ &= 2(ac - b^2)\,; \end{aligned}$$

and that, when the variables do not disappear, each set $(x_1,\ y_1)$, $(x_2,\ y_2)$, ... is ultimately replaced by $(x,\ y)$, so that these are the variables of any covariant.

The investigation* is as follows:

"...We pass to the derivatives of the fourth degree, considering the forms in which all the differential coefficients are of the same order. It is easy to see that we may write

$$\begin{aligned} \square\, UVWX &= (\overline{12}\,.\,\overline{34})^\alpha\, (\overline{13}\,.\,\overline{42})^\beta\, (\overline{14}\,.\,\overline{23})^\gamma\, UVWX \\ &= D_{\alpha,\,\beta,\,\gamma} UVWX \ \text{ or } \ D_{\alpha,\,\beta,\,\gamma}. \end{aligned}$$

[* This Collection, vol. I., p. 104: see also, vol. I., p. 117.]

Writing for shortness

$$\overline{12}\,.\,\overline{34}=\mathfrak{A},\quad \overline{13}\,.\,\overline{42}=\mathfrak{B},\quad \overline{14}\,.\,\overline{23}=\mathfrak{C},$$

we have

$$D_{\alpha,\,\beta,\,\gamma}=\mathfrak{A}^{\alpha}\mathfrak{B}^{\beta}\mathfrak{C}^{\gamma}\,UVWX.$$

Suppose $U=V=W=X$, and consider the derivatives which correspond to the same value f of α, β, γ: we have to find how many of these derivatives are independent, and to express the others in terms of these. Since the functions are equal after the differentiations, we may before the differentiations interchange in any manner the symbolic numbers 1, 2, 3, 4: this gives

$$D_{\alpha,\,\beta,\,\gamma}=D_{\beta,\,\gamma,\,\alpha}=D_{\gamma,\,\alpha,\,\beta}=(-)^f D_{\alpha,\,\gamma,\,\beta}=(-)^f D_{\gamma,\,\beta,\,\alpha}=(-)^f D_{\beta,\,\alpha,\,\gamma}.$$

But we have identically

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}=0.$$

Multiplying by $\mathfrak{A}^a\mathfrak{B}^b\mathfrak{C}^c$, and applying it to the product $UVWX$, we have

$$D_{a+1,\,b,\,c}+D_{a,\,b+1,\,c}+D_{a,\,b,\,c+1}=0,$$

which, putting $a+b+c=f-1$, gives a system of equations between the derivatives $D_{\alpha,\,\beta,\,\gamma}$ for which $\alpha+\beta+\gamma=f$. Reducing these by the conditions just obtained, suppose that Θf denotes the number of partitions of f into three parts, zero included, and the permutations of the parts being disregarded, we have a number Θf of derivatives $D_{\alpha,\,\beta,\,\gamma}$, and a number $\Theta(f-1)$ of linear relations between these derivatives. There remains therefore a number $\Theta f-\Theta(f-1)$ of independent derivatives: but when f is an even number, we have included among these the functions $D_{f,\,0,\,0}$, that is, $\overline{12}^f.\overline{34}^f UVWX$, which evidently reduces itself to

$$\overline{12}^f UV\,.\,\overline{34}^f WX,\quad\text{or}\quad B_f(U,\,V)\,.\,B_f(W,\,X),$$

that is to say, to the square of $B_f(U,\,V)$. We must therefore diminish by unity this number $\Theta f-\Theta(f-1)$, when f is even. Let $E\left(\dfrac{a}{b}\right)$ denote the integer part of the fraction $\dfrac{a}{b}$: it can be shown that the required number is equal to

$$E\left(\frac{f}{6}\right)\quad\text{or}\quad E\left(\frac{f+3}{6}\right)$$

according as f is even or odd. Giving to f the six forms

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

we obtain for the independent derivatives of the fourth degree the corresponding numbers

$$g,\quad g,\quad g,\quad g+1,\quad g,\quad g+1;$$

there is for example a single derivative for each of the orders 3, 5, 6, 7, 8, 10, two for each of the orders 9, 11, 12, 13, 14, 16, &c.

We may take for independent derivatives, when f is even, the terms of the series $D_{f-3,3,0}$, $D_{f-6,6,0}$, ..., and when f is odd, those of the series $D_{f-1,1,0}$, $D_{f-4,4,0}$, $D_{f-7,7,0}\ldots$: continuing each series until the last term in which the first suffix exceeds or is equal to the second suffix. We have in each case the right number of terms; for example, in the case $f=9$, we take for independent derivatives the two functions D_{810} and D_{540}, and we form the equations

$$\begin{aligned}
D_{900}+D_{810}+D_{801}&=0, & D_{621}+D_{531}+D_{522}&=0,\\
D_{210}+D_{720}+D_{711}&=0, & D_{540}+D_{450}+D_{441}&=0,\\
D_{720}+D_{630}+D_{621}&=0, & D_{531}+D_{441}+D_{432}&=0,\\
D_{711}+D_{621}+D_{612}&=0, & D_{522}+D_{432}+D_{423}&=0,\\
D_{630}+D_{540}+D_{531}&=0, & D_{432}+D_{342}+D_{333}&=0,
\end{aligned}$$

which are to be reduced by means of the formulæ

$$D_{900}=-D_{900}=0, \qquad D_{810}=-D_{801}, \text{ \&c.}$$

We will presently give the solution of these equations; but beginning with the second order and going successively to the ninth order, we form easily the following table:

$$\begin{aligned}
D_{200}&=B_2^2,\\
D_{110}&=-\tfrac{1}{2}B_2^2.\\
D_{300}&=0,\\
&D_{210},\\
D_{111}&=0.\\
D_{400}&=B_4^2,\\
D_{310}&=-\tfrac{1}{2}B_4^2,\\
D_{220}&=\tfrac{1}{2}B_4^2,\\
D_{211}&=0.\\
D_{500}&=0,\\
&D_{410},\\
D_{320}&=-D_{410},\\
D_{311}&=0,\\
D_{221}&=0,\\
D_{600}&=B_6^2,\\
D_{510}&=-\tfrac{1}{2}B_6^2,\\
D_{420}&=\tfrac{1}{6}B_6^2-\tfrac{2}{3}D_{330},\\
D_{411}&=\tfrac{1}{3}B_6^2+\tfrac{2}{3}D_{330},\\
&D_{330},\\
D_{321}&=-\tfrac{1}{6}B_6^2-\tfrac{1}{3}D_{330},\\
D_{222}&=\tfrac{1}{3}B_6^2+\tfrac{2}{3}D_{330}.
\end{aligned}$$

$$\begin{aligned}
D_{700}&=0,\\
&D_{610},\\
D_{520}&=-D_{610},\\
D_{511}&=0,\\
D_{430}&=D_{610},\\
D_{421}&=0,\\
D_{331}&=0,\\
D_{322}&=0.\\
D_{800}&=B_8^2,\\
D_{710}&=-\tfrac{1}{2}B_8^2,\\
D_{620}&=\tfrac{1}{6}B_8^2-\tfrac{2}{3}D_{530},\\
D_{611}&=-\tfrac{1}{3}B_8^2-\tfrac{2}{3}D_{530},\\
&D_{530},\\
D_{521}&=-\tfrac{1}{12}B_8^2-\tfrac{1}{3}D_{530},\\
D_{440}&=-\tfrac{1}{30}B_8^2-\tfrac{16}{15}D_{530},\\
D_{431}&=\tfrac{1}{30}B_8^2+\tfrac{1}{15}D_{530},\\
D_{422}&=\tfrac{2}{15}B_8^2+\tfrac{4}{15}D_{530},\\
D_{332}&=-\tfrac{1}{15}B_8^2-\tfrac{2}{15}D_{530}.
\end{aligned}$$

$$\begin{aligned}
D_{900}&=0,\\
&D_{810},\\
D_{720}&=-D_{810},\\
D_{711}&=0,\\
D_{630}&=\tfrac{1}{2}D_{810}-\tfrac{1}{2}D_{540},\\
D_{621}&=\tfrac{1}{2}D_{810}-\tfrac{1}{2}D_{540},\\
&D_{540},\\
D_{531}&=-\tfrac{1}{2}D_{810}-\tfrac{1}{2}D_{540},\\
D_{522}&=0,\\
D_{441}&=0,\\
D_{432}&=\tfrac{1}{2}D_{810}+\tfrac{1}{2}D_{540},\\
D_{333}&=0.
\end{aligned}$$

For any value of f except $f=2$, 3, or 4, the table commences, for f even and f odd respectively, in the following manner:

$$\begin{array}{ll} D_{f,0,6} = B_f^2, & D_{f,0,0} = 0, \\ D_{f-1,1,0} = -\tfrac{1}{2} B_f^2, & D_{f-1,1,0}, \\ D_{f-2,2,0} = \tfrac{1}{6} B_f^2 - \tfrac{2}{3} D_{f-3,3,0}, & D_{f-2,2,0} = -D_{f-1,1,0}, \\ D_{f-2,1,1} = \tfrac{2}{3} B_f^2 + \tfrac{3}{2} D_{f-3,3,0}, & D_{f-2,1,1} = 0, \end{array}$$

but beyond this I do not know the law of the series."

Before going further, I remark that if $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, instead of the foregoing values, denote respectively

$$\begin{aligned} \mathfrak{A} &= (x\partial_{x_1} + y\partial_{y_1})\,\overline{23}, &&= (x\xi_1 + y\eta_1)(\xi_2\eta_3 - \xi_3\eta_2), \\ \mathfrak{B} &= (x\partial_{x_2} + y\partial_{y_2})\,\overline{31}, &&= (x\xi_2 + y\eta_2)(\xi_3\eta_1 - \xi_1\eta_3), \\ \mathfrak{C} &= (x\partial_{x_3} + y\partial_{y_3})\,\overline{12}, &&= (x\xi_3 + y\eta_3)(\xi_1\eta_2 - \xi_2\eta_1), \end{aligned}$$

then identically

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

and the same theory applies to the cubic derivatives $\mathfrak{A}^\alpha\mathfrak{B}^\beta\mathfrak{C}^\gamma . UVW$, that is, to the cubic covariants, or, attending only to the coefficients of the highest powers of x, to the cubic seminvariants.

Or, we may use the Clebschian notation, for instance

$$(a,\ b,\ c\between x,\ y)^2 = (a_0x + a_1y)^2 = (b_0x + b_1y)^2 = \&c.,$$

that is,

$$\begin{aligned} a_0^2 &= b_0^2 = \ldots = a, \\ a_0a_1 &= b_0b_1 = \ldots = b, \\ a_1^2 &= b_1^2 = \ldots = c, \end{aligned}$$

viz. in the language of Prof. Sylvester, $a_0, a_1, b_0, b_1, \ldots$ are here *umbræ*. The invariant is

$$\begin{aligned} (a_0b_1 - a_1b_0)^2 &= a_0^2b_1^2 - 2a_0b_0a_1b_1 + b_0^2a_1^2, \\ &= ac \quad - 2b\,.\,b \quad + ac, \\ &= 2\,(ac - b^2), \end{aligned}$$

and so in other cases. To apply this directly to the theory of seminvariants, it is more convenient to write

$$(1,\ b,\ c\between x,\ y)^2 = (x + \alpha y)^2 = (x + \beta y)^2 = \&c.,$$

that is,

$$\begin{aligned} \alpha &= \beta = \ldots = b, \\ \alpha^2 &= \beta^2 = \ldots = c, \end{aligned}$$

where $\alpha, \beta, \ldots$ are *umbræ*.

The invariant is

$$\begin{aligned} (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= c - 2b\,.\,b + c \\ &= 2\,(c - b^2); \end{aligned}$$

or, to take the case of a cubic function

$$(1,\ b,\ c,\ d) = (x + \alpha y)^3 = (x + \beta y)^3 = (x + \gamma y)^3,$$

that is,

$$\alpha = \beta = \gamma = b,$$
$$\alpha^2 = \beta^2 = \gamma^2 = c,$$
$$\alpha^3 = \beta^3 = \gamma^3 = d,$$

a seminvariant is

$$\begin{aligned}(\alpha-\beta)^2(\alpha-\gamma) &= \alpha^3 - 2\alpha^2\beta + \alpha\beta^2 - \alpha^2\gamma + 2\alpha\beta\gamma - \beta^2\gamma,\\ &= d - 2bc \ + bc \ - bc \ + 2b^3 \ - bc,\\ &= d - 3bc \ + 2b^3,\end{aligned}$$

and so in other cases. Writing here

$$\mathfrak{A} = \beta - \gamma,\quad \mathfrak{B} = \gamma - \alpha,\quad \mathfrak{C} = \alpha - \beta,$$

the general form of a cubic seminvariant of the weight ω is now $\mathfrak{A}^p\mathfrak{B}^q\mathfrak{C}^r$, or say $D_{p,q,r}$, where $p+q+r=\omega$ (the new letters p, q, r being used for the exponents, since α, β, γ are now employed in a different sense; and since f will be required as a coefficient, I have also written ω instead of f for the weight). We have, as before,

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0,$$

and we again see that the theory as to the number of and relations between the cubic seminvariants is identical with that of the quartic invariants. Observe also that for ω odd, $D_{\omega,0,0}$ is $=0$, while for ω even, $D_{\omega,0,0}$, $=\mathfrak{A}^\omega$, is a quadric seminvariant, which is of course not to be reckoned among the proper cubic seminvariants; this exactly corresponds to the B_f^2 which is not a proper quartic invariant.

The choice of the functions $D_{f,0,0}$, $D_{f-3,3,0}, \ldots$, or $D_{f-1,1,0}$, $D_{f-4,4,0}, \ldots$, for expressing in terms of them the other functions has its advantages, but also its disadvantages; in what follows, I in effect replace them by $D_{f,0,0}$, $D_{f-2,2,0}, \ldots$ and $D_{f-1,1,0}$, $D_{f-3,3,0}, \ldots$ respectively. And instead of considering the whole series of the functions $D_{p,q,r}$ for a given weight ω, I consider only those of the form $D_{p,q,0}$; these form a single series $D_{\omega,0,0}$, $D_{\omega-1,1,0}$, $D_{\omega-2,2,0}, \ldots$; and for shortness, I call them A, B, C, D, &c. respectively, viz. I write

$$A,\ B,\ C,\ D \ldots = (\mathfrak{A}^\omega,\ \mathfrak{A}^{\omega-1}\mathfrak{B},\ \mathfrak{A}^{\omega-2}\mathfrak{B}^2,\ \mathfrak{A}^{\omega-3}\mathfrak{B}^3, \ldots).$$

Suppose first that ω is odd; we have

$$D_{\omega-q-r,q,r} = (-)^\omega D_{\omega-q-r,r,q};$$

and, in particular,

$$D_{\omega-2q,q,q} = - \ D_{\omega-2q,q,q};$$

that is,

$$D_{\omega-2q,q,q} = 0;$$

that is,

$$\mathfrak{A}^\omega = 0,\quad \mathfrak{A}^{\omega-2}\mathfrak{B}\mathfrak{C} = 0,\quad \mathfrak{A}^{\omega-4}\mathfrak{B}^2\mathfrak{C}^2 = 0, \ldots.$$

Hence

$$A = 0;$$

moreover

$$B+C=\mathfrak{A}^{\omega-2}\mathfrak{B}(\mathfrak{A}+\mathfrak{B})=-\mathfrak{A}^{\omega-2}\mathfrak{B}\mathfrak{C}=0;$$

and similarly $C+2D+E=0$, &c.; that is, we have

$$\begin{aligned}
&A &&=0,\\
&B+C &&=0,\\
&C+2D+\ E &&=0,\\
&D+3E+3F+G &&=0,\\
&\&c.
\end{aligned}$$

The readiest way of solving these is to express the other functions in terms of B, D, F, H, &c.; viz. we thus have

	A	B	C	D	E	F	G	H	I	J	K	L	M
B	0	1	−1	0	1	0	−3	0	17	0	−155	0	2073
D				1	−2	0	5	0	−28	0	255	0	−3410
F						1	−3	0	14	0	−126	0	1683
H								1	−4	0	30	0	−396
J										1	−5	0	55
L												1	−6

read according to the columns

$$\begin{aligned}
A&=0,\\
B&=B,\\
C&=-B,\\
D&=D,\\
E&=B-2D,\\
F&=F,\\
G&=-3B+5D-3F,\\
&\vdots
\end{aligned}$$

and, by way of verification of the numbers, observe that the sum of the numbers in a column is 1 and −1 alternately.

The formulæ are true for any odd value of ω whatever; but they require an explanation, viz. for any finite value of ω, the terms $B, D, F,\ldots$ are not all of them arbitrary. For any given value of ω the number of arbitrary terms is in fact given by what precedes, and it is also known by Captain MacMahon's theorem, viz. the number of cubic seminvariants is = that of the non-unitary symmetric functions $3^\alpha 2^\beta$ of the proper weight $3\alpha+2\beta=\omega$; thus for $\omega=9$, the forms are 3222 and 333; so that there should be only two arbitrary terms.

We have here, writing for shortness 900, 810, &c., instead of D_{900}, D_{810}, &c. respectively,

$$\begin{aligned}
900 &= 0,\\
810 &= B,\\
720 &= -B,\\
630 &= D,\\
540 &= B - 2D,\\
450 &= F,\\
360 &= -3B + 5D - 3F,\\
270 &= H,\\
180 &= 17B - 28D + 14F - 4H,\\
090 &= L.
\end{aligned}$$

But we have $900=0$, $810+180=0$, $720+270=0$, $630+360=0$, $540+450=0$; that is,

$$\begin{aligned}
L &= 0,\\
18B - 28D + 14F - 4H &= 0,\\
-B + H &= 0,\\
-3B + 6D - 3F &= 0,\\
B - 2D + F &= 0,
\end{aligned}$$

all satisfied by $F=-B+2D$, $H=B$, $L=0$. The proper course is to stop at the equation for 540, viz. the system is

$$\begin{aligned}
900 &= 0,\\
810 &= B,\\
720 &= -B,\\
630 &= D,\\
540 &= B-2D,
\end{aligned}$$

equations which may be considered as expressing the several functions 900, 810, 720, 630, 540 in terms of $B=810$ and $D=630$. They agree, as they should do, with a foregoing result,

$$\begin{aligned}
&900, = 0,\\
&810,\\
&720, = -810,\\
&630, = \tfrac{1}{2}810 - \tfrac{1}{2}540,\\
&540,
\end{aligned}$$

and it would of course be possible to express the remaining forms 711, 621, 522, 441, 432, and 333 in terms of $B=810$ and $D=630$. But observe that the speciality of the present process is that, instead of the whole series of forms 900, 810, 720, 711, 630, 621, 540, 531, 522, 441, 432, 330, we work only with the forms 900, 810, 720, 630, 540, for which the last element is $=0$.

A more simple solution of the system of linear equations in $A, B, C, \ldots$ is the following:

	A	B	C	D	E	F	G	H	I	J	K	L	M
x	0	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
y				1	−2	3	−4	5	−6	7	−8	9	−10
z						1	−3	6	−10	15	−21	28	−36
w								1	−4	10	−20	35	−56
t										1	−5	15	−35
u												1	−6

read

$$\begin{aligned} A &= 0, \\ B &= x, \\ C &= -x, \\ D &= x + y, \\ E &= -x - 2y, \\ F &= x + 3y + z, \\ G &= -x - 4y - 3z, \end{aligned}$$

where $x, y, z, \ldots$ are arbitrary; the numbers in the table are the binomial coefficients with the signs + and − alternately.

We may, it is clear, express $x, y, z, \ldots$ in terms of $B, D, F, \ldots$, viz. we thus have

$$\begin{aligned} x &= B, \\ y &= -B + D, \\ z &= 2B - 3D + F, \\ &\vdots \end{aligned}$$

and substituting these values, we reproduce the foregoing expressions of $A, B, C, \ldots$ in terms of $B, D, F, \ldots$.

For a given finite value of ω, we have of course the same number of arbitrary coefficients $x, y, z, \ldots$ as there were of arbitrary coefficients $B, D, F, \ldots$; thus for $\omega = 9$, only x and y are arbitrary, and the remaining coefficients z, &c., are determined in terms of these. Or, what is the same thing, we stop with the equation $E = -x - 2y$, next preceding an equation with the non-arbitrary coefficient z.

For the case ω even, I write

$$A+2B=A',$$
$$B+2C=B',$$
$$C+2D=C',$$
$$\&c.;$$

and I say that the new functions $A', B', C', \ldots$ are related together precisely in the same manner as are the functions $A, B, C, \ldots$ belonging to an odd weight ω; viz. we have

$$\begin{aligned} A' \qquad\qquad &= 0, \\ B'+C' \qquad &= 0, \\ C'+2D'+E' &= 0, \\ \vdots \qquad & \end{aligned}$$

which being so, the theory is included in what precedes, and there is no occasion to consider it separately.

To prove this, observe that ω being even, we have

$$D_{\omega-q-r,\,q,\,r}=D_{\omega-q-r,\,r,\,q},$$

that is,

$$\mathfrak{A}^{\omega-q-r}\mathfrak{B}^{q}\mathfrak{C}^{r}=\mathfrak{A}^{\omega-q-r}\mathfrak{B}^{r}\mathfrak{C}^{q},$$

whence, in particular,

$$\mathfrak{A}^{\omega-1}\mathfrak{B}=\mathfrak{A}^{\omega-1}\mathfrak{C},\ \ \mathfrak{A}^{\omega-3}\mathfrak{B}^{2}\mathfrak{C}=\mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}^{2},\ \&c.;$$

we thus have

$$A'=\mathfrak{A}^{\omega}+2\mathfrak{A}^{\omega-1}\mathfrak{B},\ =\mathfrak{A}^{\omega}+\mathfrak{A}^{\omega-1}\mathfrak{B}+\mathfrak{A}^{\omega-1}\mathfrak{C},\ =\mathfrak{A}^{\omega-1}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}),$$

which is $=0$. Similarly

$$B'+C'=B+3C+2D=\mathfrak{A}^{\omega-1}\mathfrak{B}+3\mathfrak{A}^{\omega-2}\mathfrak{B}^{2}+2\mathfrak{A}^{\omega-3}\mathfrak{B}^{3},$$
$$=\mathfrak{A}^{\omega-3}\mathfrak{B}(\mathfrak{A}^{2}+3\mathfrak{A}\mathfrak{B}+2\mathfrak{B}^{2}),\ =\mathfrak{A}^{\omega-3}\mathfrak{B}(\mathfrak{A}+\mathfrak{B})(\mathfrak{A}+2\mathfrak{B}),$$

or, since

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}=0,$$

this is

$$=-\mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}(\mathfrak{A}+2\mathfrak{B}),$$

which, in virtue of

$$\mathfrak{A}^{\omega-3}\mathfrak{B}^{2}\mathfrak{C}=\mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}^{2},$$

is

$$=-\mathfrak{A}^{\omega-3}\mathfrak{B}\mathfrak{C}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}),$$

which is $=0$. And similarly

$$C'+2D'+E'=0,\ \&c.$$

I come to a new part of the theory: to fix the ideas, consider the weight $\omega=27$; and write also

$$a_0,\ a_1,\ a_2,\ a_3,\ldots=1,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m,\ldots,$$

using, if we please, the suffixed a for the higher terms. The cubic seminvariants are 32^{12}, $3^3 2^9$, $3^5 2^6$, $3^7 2^3$, 3^9, viz. the number of them is $=5$; we have thus the 5 terms B, D, F, H, J. Here $B = (\beta-\gamma)^{26}(\gamma-\alpha) = (\alpha-\beta)^{26}(\beta-\gamma)$, or, interchanging the α and β, $=(\alpha-\beta)^{26}(\alpha-\gamma)$, there is an initial term $\alpha^{27}=a_{27}$, and a final term $\alpha^{13}\beta^{13}\gamma = bn^2$, or we may write

$$B = (\alpha-\beta)^{26}(\alpha-\gamma) = a_{27} + bn^2;$$

and similarly for the other forms. We have thus

$$\begin{aligned} B &= (\alpha-\beta)^{26}(\alpha-\gamma)^1 = a_{27} + bn^2, \\ D &= (\alpha-\beta)^{24}(\alpha-\gamma)^3 = a_{27} + dm^2, \\ F &= (\alpha-\beta)^{22}(\alpha-\gamma)^5 = a_{27} + fl^2, \\ H &= (\alpha-\beta)^{20}(\alpha-\gamma)^7 = a_{27} + hk^2, \\ J &= (\alpha-\beta)^{18}(\alpha-\gamma)^9 = a_{27} + j^3, \end{aligned}$$

where the initial term $\alpha^{27}=a_{27}$ has in each case the coefficient unity, but the final terms have each of them the proper numerical coefficient.

It must be possible to form linear combinations

$$\begin{aligned} &B &&= a_{27} &&+ bn^2, \\ &(B,\ D) &&= a_{25}(c + b^2) &&+ dm^2, \\ &(B,\ D,\ F) &&= a_{23}(e + c^2) &&+ fl^2, \\ &(B,\ D,\ F,\ H) &&= a_{21}(g + d^2) &&+ hk^2, \\ &(B,\ D,\ F,\ H,\ J) &&= a_{19}(i + e^2) &&+ j^3, \end{aligned}$$

where $c+b^2$, $e+c^2$, $g+d^2$, $i+e^2$ denote the seminvariants

$$c-b^2,\ e-4bd+3c^2,\ g-6bf+15ce-10d^2,$$
$$i-8bh+28cg-56df+35e^2,$$

respectively. The expressions indicated by their initial and final terms $a_{27}+bn^2$, $a_{25}c+dm^2$, &c., are what I call "columns;" see my paper "Seminvariant Tables," *American Journal of Mathematics*, t. VII. (1885), pp. 59—73, [831], where, however, the theory is not by any means completely developed.

As to this, observe that B, D are each of them of the form $a_{27}+a_{26}b+a_{25}(c, b^2)+\ldots$, the proper combination in order to eliminate a_{27} is obviously $B-D$, the term in $a_{26}b$ will then disappear of itself, and the terms in $a_{25}(c, b^2)$ will combine into a term $a_{25}(c-b^2)$, viz. in the function *quà* seminvariant the coefficient of the highest letter a_{25} will be the seminvariant $c-b^2$. Similarly, in the combination B, D, F, the two coefficients will be determinable so that there are no terms in a_{27}, a_{26}, a_{25} or a_{24}, and this being so, the term in a_{23} will be $a_{23}(e-4bd+3c^2)$, viz. in the function, *quà* seminvariant, the coefficient of the highest letter a_{23} will be the seminvariant $e-4bd+3c^2$. And similarly for the remaining two linear combinations.

As a partial verification, I write

$$B = (\alpha-\beta)^{26}(\alpha-\gamma)$$
$$= \{\alpha^{26}+\beta^{26}-26(\alpha^{25}\beta+\alpha\beta^{25})+325(\alpha^{24}\beta^2+\alpha^2\beta^{24})\ldots\}(\alpha-\gamma),$$

or, retaining only the terms which have an index at least $=25$, this is

$$\begin{aligned} B = {} & \alpha^{27}+\alpha\beta^{26} - 26(\alpha^{26}\beta+\alpha^2\beta^{25})+325\alpha^{25}\beta^2 \\ & -(\alpha^{26}+\beta^{26})\gamma - 26(\alpha^{25}\beta+\alpha\beta^{25})\gamma, \\ = {} & a_{27} + a_{26}b \\ & -26a_{26}b - 26a_{25}c \\ & + 325a_{25}c \\ & - 2a_{26}b + 52a_{25}b^2 \\ = {} & a_{27} - 27a_{26}b + a_{25}(299c + 52b^2); \end{aligned}$$

and similarly

$$D = (\alpha-\beta)^{24}(\alpha-\gamma)^3$$
$$= \{\alpha^{24}+\beta^{24}-24(\alpha^{23}\beta+\alpha\beta^{23})+276(\alpha^{22}\beta^2+\alpha^2\beta^{22})\ldots\}(\alpha^3-3\alpha^2\gamma+3\alpha\gamma^2-\gamma^3),$$

or, retaining only the terms which have an index at least $=25$, this is

$$\begin{aligned} D = {} & \alpha^{27} - 24\alpha^{26}\beta + 276\alpha^{25}\beta^2 \\ & -3\alpha^{26}\gamma + 72\alpha^{25}\beta\gamma \\ & +3\alpha^{25}\gamma^2 \\ = {} & a_{27} - 24a_{26}b + 276a_{25}c \\ & - 3a_{26}b + 72a_{25}b^2 \\ & + 3a_{25}c \\ = {} & a_{27} - 27a_{26}b + a_{25}(279c + 72b^2), \end{aligned}$$

and thence

$$B - D = a_{25}(20c - 20b^2),$$

viz. the coefficient of a_{25} is $=20(c-b^2)$.

But instead of the foregoing forms B, D, F, H, J, we may start with five other forms which are in fact linear combinations of these, viz. these are the forms

$$\begin{aligned} X &= (\alpha-\beta)^{26}(\alpha+\beta-2\gamma) &&= a_{27} + bn^2, \\ Y &= (\alpha-\beta)^{24}(\alpha+\beta-2\gamma)(\alpha-\gamma)(\beta-\gamma) &&= a_{26}(c+b^2)+dm^2, \\ Z &= (\alpha-\beta)^{22}(\alpha+\beta-2\gamma)(\alpha-\gamma)^2(\beta-\gamma)^2 &&= a_{25}(c+b^2)+fl^2, \\ W &= (\alpha-\beta)^{20}(\alpha+\beta-2\gamma)(\alpha-\gamma)^3(\beta-\gamma)^3 &&= a_{24}(e+c^2)+hk^2, \\ T &= (\alpha-\beta)^{18}(\alpha+\beta-2\gamma)(\alpha-\gamma)^4(\beta-\gamma)^4 &&= a_{23}(e+c^2)+j^3, \end{aligned}$$

where as well the initial as the final terms should have their proper numerical coefficients.

As to this, observe that in Y we have a term $\alpha^{26}(\beta-\gamma)$ which is $=0$, and similarly a term $\beta^{25}(\alpha-\gamma)$ which is $=0$; the highest powers are thus α^{25} and β^{25}, giving a term in a_{25} which can only be $a_{25}(c-b^2)$. In Z we have the terms $\alpha^{25}(\beta-\gamma)^2$, and $\beta^{25}(\alpha-\gamma)^2$, giving the term $a_{25}(c-b^2)$; in W we have the terms $\alpha^{24}(\beta-\gamma)^3$ and $\beta^{24}(\alpha-\gamma)^3$ which are each $=0$, the highest powers thus are α^{23}, β^{23} giving a term in a_{23} which can only be $a_{23}(e-4bd+3c^2)$; and in T we have the terms $\alpha^{23}(\beta-\gamma)^4$ and $\beta^{23}(\alpha-\gamma)^4$, which give the term $a_{23}(e-4bd+3c^2)$.

The combinations which give the columns thus are

$$\begin{aligned} X &= a_{27} + bn^2, \\ (Y) &= a_{25}(c+b^2)+dm^2, \\ (Y,\ Z) &= a_{23}(e+c^2)+fl^2, \\ (Y,\ Z,\ W) &= a_{21}(g+d^2)+hk^2, \\ (Y,\ Z,\ W,\ T) &= a_{19}(i+e^2)+j^3. \end{aligned}$$

I start with the form

$$u=(Y,\ Z,\ W,\ T)=Y+\lambda Z+\mu W+\nu T.$$

We know that it is possible to determine the coefficients λ, μ, ν, in suchwise that the linear function shall be of the proper form $a_{19}(i+e^2)+j^3$; or, what is the same thing, that there shall be no term with an index exceeding 19; the conditions to be satisfied are apparently more than three, but of course the number of independent conditions must be $=3$. We have, in particular, to get rid of all the terms of the second degree which precede $a_{19}i$, viz. the coefficients of $a_{25}c$, $a_{24}d$, $a_{23}e$, $a_{22}f$, $a_{21}g$, $a_{20}h$ must all of them vanish. Now the terms of u which produce terms of the second degree are the terms containing any two but not all three of the symbolic letters α, β, γ, say the biliteral terms; we have for instance $\alpha^{25}\beta^2$, $\alpha^{25}\gamma^2$, $\beta^{25}\alpha^2$, $\beta^{25}\gamma^2$ (there are no terms $\gamma^{25}\alpha^2$ or $\gamma^{25}\beta^2$, since the highest power of γ is γ^9), each of which is $=a_{25}c$. But in the development of u, we may in any biliteral term by an interchange of the letters α, β, γ make the letter of highest index to be α, and the other letter to be β; thus the several terms $\alpha^{25}\beta^2$, $\alpha^{25}\gamma^2$, $\beta^{25}\alpha^2$, $\beta^{25}\gamma^2$ may be each of them converted into $\alpha^{25}\beta^2$, and so in other cases. Imagining this to be done, the term in $a_{25}c$ is simply the term in $\alpha^{25}\beta^2$, and in like manner the term in $a_{24}d$ is the term in $\alpha^{24}\beta^3$, and so in other cases; the condition as to the disappearance of the terms $a_{25}c, \ldots, a_{20}h$ is the condition that the terms in $\alpha^{25}\beta^2$, $\alpha^{24}\beta^3$, $\alpha^{23}\beta^4$, $\alpha^{22}\beta^5$, $\alpha^{21}\beta^6$, $\alpha^{20}\beta^7$ shall all of them disappear. And if, in the function u transformed in the foregoing manner, we write for convenience $\alpha=1$, so that u has become a function of β only (and observe that u will contain the factor β^2), then the condition is that the terms in β^2, β^3, β^4, β^5, β^6, β^7 shall all of them disappear. The conditions may in fact be written $u=0$; viz. it is assumed that u is in the first instance transformed into a function of β as just explained, and the equation is then to be understood as denoting that the coefficients of u are to be so determined that as many as possible of the terms (beginning with that which contains the lowest power of β) shall each of them vanish.

Consider any term of u, for instance the term

$$Y=(\alpha-\beta)^{24}(\alpha+\beta-2\gamma)(\alpha-\gamma)(\beta-\gamma).$$

To obtain the biliteral terms in the proper form 1° we write therein $1=0$, 2° we write $\beta=0$, changing γ into β, 3° we write $\alpha=0$, changing β, γ into α, β; we thus obtain

$$(\alpha-\beta)^{24}(\alpha+\beta)\,\alpha\beta,$$
$$\alpha^{24}(\alpha-2\beta)(\alpha-\beta)(\ -\beta),$$
$$\alpha^{24}(\alpha-2\beta)(\ -\beta)(\alpha-\beta);$$

viz. the second and third terms are identical; the first term requires, however, the factor 2 (for any term $\alpha^p\beta^q$ is accompanied by a corresponding term $\alpha^q\beta^p$), so that, omitting this factor throughout, the terms are

$$(\alpha-\beta)^{24}(\alpha+\beta)\,\alpha\beta-\alpha^{24}(\alpha-2\beta)(\alpha-\beta)\,\beta,$$

or, putting therein $\alpha=1$, the terms are

$$(1-\beta)^{24}(1+\beta)\,\beta-(1-2\beta)(1-\beta)\,\beta.$$

Similarly from

$$Z=(\alpha-\beta)^{22}(\alpha+\beta-2\gamma)(\alpha-\gamma)^2(\beta-\gamma)^2,$$

we have

$$(1-\beta)^{22}(1+\beta)\,\beta^2+(1-2\beta)(1-\beta)^2\beta^2,$$

and the like as regards W and T. The result thus is

$$(1+\beta)\,\beta\,(1-\beta)^{24}+\lambda\,(1-\beta)^{22}\beta^2+\mu\,(1-\beta)^{20}\beta^3+\nu\,(1-\beta)^{18}\beta^4$$
$$-(1-2\beta)(1-\beta)\,\beta\,\{1-\lambda\beta\,(1-\beta)+\mu\beta^2(1-\beta)^2-\nu\beta^3(1-\beta)^3\}=0,$$

or, as this may be written,

$$\frac{(1+\beta)(1-\beta)^{23}}{1-2\beta}\left\{1+\lambda\frac{\beta}{(1-\beta)^2}+\mu\frac{\beta^2}{(1-\beta)^4}+\nu\frac{\beta^3}{(1-\beta)^6}\right\}$$
$$-1+\lambda\,(\beta-\beta^2)-\mu\,(\beta-\beta^2)^2+\nu\,(\beta-\beta^2)^3=0,$$

viz. each side is to be expanded in ascending powers of β, and the coefficients λ, μ, ν are then to be determined so that as many terms as possible shall vanish.

Expanding, we have

$$(1-20\beta+190\beta^2-1138\beta^3+4808\beta^4-15178\beta^5+\ldots)$$
$$\times\{1+\beta\lambda+\beta^2(2\lambda+\mu)+\beta^3(3\lambda+4\mu+\nu)+\beta^4(4\lambda+10\mu+6\nu)+\beta^5(5\lambda+20\mu+21\nu)+\ldots\}$$
$$-1+\beta\lambda+\beta^2(-\lambda-\mu)+\beta^3(2\mu+\nu)+\beta^4(-3\nu)+\beta^5\,.\,3\nu+\ldots=0,$$

or finally, as far as β^5, the equation is

$$0=0+\beta\,(2\lambda-20)+\beta^2(-19\lambda+190)+\beta^3(153\lambda-14\mu+2\nu-1138)$$
$$+\beta^4(-814\lambda+119\mu-17\nu+4808)+\beta^5(3027\lambda-558\mu+94\nu-15178).$$

We have in the first place $\lambda=10$, which makes the coefficient of β^2 to be $=0$; and we then have

$$-\ 14\mu+\ 2\nu+\ \ 392=0,\ \text{or say}\quad 7\mu-\ \ \nu-\ 196=0,$$
$$119\mu-17\nu-\ 3332=0,\qquad 119\mu-17\nu-3332=0,$$
$$-\ 558\mu+94\nu+15092=0,\qquad 272\mu-47\nu-7546=0,$$

where the second equation is equivalent to the first: the three equations give $\mu = \frac{833}{25}$, $\nu = \frac{931}{25}$. I have since found that proceeding one step further, that is, to β^6, the coefficient is $-8310\lambda + 1791\mu - 363\nu + 36942$, viz. putting this $= 0$, and for λ substituting its value $= 10$, we have $597\mu - 121\nu = 15386$, an equation which is in fact satisfied by the values just obtained for λ and μ. For the function (Y, Z, W) we have the same equations with $\nu = 0$; and therefore $\lambda = 10$, $\mu = 28$; and for the function (Y, Z) the same equations with $\mu = 0$, $\nu = 0$; and therefore $\lambda = 10$. The linear combinations thus are

$$\begin{aligned}
&X &&= a_{27} + bn^2,\\
&Y &&= a_{25}(c + b^2) + dm^2,\\
&Y + 10Z &&= a_{23}(e + c^2) + fl^2,\\
&Y + 10Z + 28W &&= a_{21}(g + d^2) + hk^2,\\
&25Y + 250Z + 823W + 931T &&= a_{19}(i + e^2) + j^3,
\end{aligned}$$

where remark that in $Y + 10Z$, by means of the coefficient 10, we make to disappear the two terms $a_{25}c$, $a_{24}d$; in the next function, by means of the coefficients 10 and 28, the four terms $a_{25}c$, $a_{24}d$, $a_{23}e$, $a_{22}f$; and in the last function, by means of the three coefficients, the six terms $a_{25}c$, $a_{24}d$, $a_{23}e$, $a_{22}f$, $a_{21}g$ and $a_{20}h$.

It is possible that a larger number of terms will disappear; but if this is so, the general form shown by the combinations on p. 353 will require modification.

The investigation applies without alteration to any odd weight ω whatever; the condition for the determination of the coefficients $\lambda, \mu, \nu, \ldots$ is obviously

$$\frac{(1+\beta)(1-\beta)^{\omega-4}}{1-2\beta}\left\{1 + \lambda\frac{\beta}{(1-\beta)^2} + \mu\frac{\beta^2}{(1-\beta)^4} + \nu\frac{\beta^3}{(1-\beta)^6} + \ldots\right\}$$
$$-1 + \lambda(\beta-\beta^2) - \mu(\beta-\beta^2)^2 + \nu(\beta-\beta^2)^4 - \ldots = 0.$$

For the case of an even weight ω, the series of functions is

$$\begin{aligned}
X &= (\alpha-\beta)^{\omega},\\
Y &= (\alpha-\beta)^{\omega-2}(\alpha-\gamma)\,(\beta-\gamma),\\
Z &= (\alpha-\beta)^{\omega-4}(\alpha-\gamma)^2(\beta-\gamma)^2,
\end{aligned}$$

and the condition for the determination of the coefficients $\lambda, \mu, \nu, \ldots$ is in like manner found to be

$$(1-\beta)^{\omega-3}\left\{1 + \lambda\frac{\beta}{(1-\beta)^2} + \mu\frac{\beta^2}{(1-\beta)^4} + \nu\frac{\beta^3}{(1-\beta)^6} + \ldots\right\}$$
$$-1 + \lambda(\beta-\beta^2) - \mu(\beta-\beta^2)^2 + \nu(\beta-\beta^2)^3 + \ldots = 0,$$

which is to be understood as explained above.