# 849.

## ON THE INVARIANTS OF A LINEAR DIFFERENTIAL EQUATION.

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CONSIDER the equation of the second order

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = 0;$$

if we effect a transformation of the dependent variable $y$, say $y = f.Y$, where $f$ is an arbitrary function of $x$, we obtain a new equation

$$\frac{d^2Y}{dx^2} + 2P\frac{dY}{dx} + QY = 0,$$

where

$$P = p + \frac{f'}{f},$$

$$Q = q + 2p\frac{f'}{f} + \frac{f''}{f}$$

(the accents denoting differentiation in regard to $x$); and we thence establish the identity

$$Q - P^2 - P' = q - p^2 - p',$$

viz. we have $q - p^2 - p'$ a function possessing this invariantive property, but, as remarked by Mr Harley, it is the analogue rather of a seminvariant than of an invariant; I will call it an $\alpha$-seminvariant of the differential equation. The class of function was considered long ago by Sir James Cockle, and more recently by Mr Malet; see Mr Harley's paper "Professor Malet's Classes of Invariants identified with Sir James Cockle's Criticoids," *Proc. R. Soc.*, vol. XXXVIII. (1884), pp. 45—57.

Effecting for the same equation a transformation of the independent variable, say $x=\phi$, an arbitrary function of $X$, we obtain a new equation

$$\frac{d^2y}{dX^2}+2P\frac{dy}{dX}+Qy=0,$$

where

$$P=p\phi_1-\tfrac{1}{2}\frac{\phi_2}{\phi_1},$$
$$Q=q\phi_1^2,$$

(the subscript numbers denoting differentiation in regard to $X$). There is not in the present case any function possessing a like invariantive property, say there is not any $\beta$-seminvariant; but such functions exist for differential equations of the third and higher orders, and have been considered by Sir James Cockle and Mr Malet. It is to be noticed that, attempting to obtain a $\beta$-seminvariant, we are led to the equation

$$Q-P^2-P_1=\phi_1^2(q-p^2-p')+\tfrac{1}{2}\left\{\frac{\phi_3}{\phi_1}-\tfrac{3}{2}\left(\frac{\phi_2}{\phi_1}\right)^2\right\},$$

which but for the last term would be invariantive. It is curious to find this last term presenting itself in the form of a Schwarzian derivative.

Passing to the differential equation of the third order

$$\frac{d^3y}{dx^3}+3p\frac{d^2y}{dx^2}+3q\frac{dy}{dx}+r=0,$$

here the transformation $y=f.Y$ of the dependent variable gives the new equation

$$\frac{d^3Y}{dx^3}+3P\frac{d^2Y}{dx^2}+3Q\frac{dY}{dx}+R=0,$$

where

$$P=p+\frac{f'}{f},$$
$$Q=q+2p\frac{f'}{f}+\frac{f''}{f},$$
$$R=r+3q\frac{f'}{f}+3p\frac{f''}{f}+\frac{f'''}{f};$$

and we thence derive as well the before-mentioned $\alpha$-seminvariant $q-p^2-p'$, as also a new $\alpha$-seminvariant $r-3pq+2p^3-p''$.

Again, effecting the transformation $x=\phi$ of the independent variable, we have the new equation

$$\frac{d^3y}{dX^3}+3P\frac{d^2y}{dX^2}+3Q\frac{dy}{dX}+R=0,$$

where

$$P=p\phi_1-\frac{\phi_2}{\phi_1},$$
$$Q=q\phi_1^2-p\phi_2+\left(\frac{\phi_2}{\phi_1}\right)^2-\tfrac{1}{3}\frac{\phi_3}{\phi_1},$$
$$R=r\phi_1^3.$$

We thence obtain the identities

$$\frac{P_1+2P^2-3Q}{R^{\frac{2}{3}}}=\frac{p'+2p^2-3q}{r^{\frac{2}{3}}},$$

$$\frac{R_1+3PR}{R^{\frac{4}{3}}}=\frac{r'+3pr}{r^{\frac{4}{3}}};$$

so that we have $\dfrac{p'+2p^2-3q}{r^{\frac{2}{3}}}$ and $\dfrac{r'+3pr}{r^{\frac{4}{3}}}$ as $\beta$-seminvariants of the differential equation of the third order.

But these are by no means the best conclusions; it is shown by M. Halphen in his great Memoir, "Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables," *Sav. Etrang.* t. XXVIII. (1884), pp. 1—297, see p. 127, that there is a function invariantive in regard to each of the two transformations, and which is thus an invariant, viz. this is the function

$$p''-3(q'-2pp')+2(r-3pq+2p^3);$$

this is for the first transformation unaltered, viz. it is

$$=P''-3(Q'-2PP')+2(R-3PQ+2P^3),$$

and for the second transformation it is only altered by the factor $\phi_1^{-3}$, viz. it is

$$=\phi_1^{-3}\{P_2-3(Q_1-2PP_1)+2(R-3PQ+2P^3)\}.$$

It is interesting to directly verify this last result. Performing the differentiations in regard to $X$, we find without difficulty

$$P_2=p_2\phi_1+2p_1\phi_2+p\phi_3-\frac{\phi_4}{\phi_1}+\frac{3\phi_2\phi_3}{\phi_1^2}-\frac{2\phi_2^3}{\phi_1^3},$$

$$Q_1-2PP_1=(q_1-2pp_1)\,\phi_1^2+p_1\phi_2-2p^2\phi_1\phi_2+2q\phi_1\phi_2+p\phi_3-\frac{\phi_4}{\phi_1}+\frac{\phi_2\phi_3}{\phi_1^2},$$

$$R-3PQ+2Q^3=(r-3pq+2p^3)\,\phi_1^3+p\phi_3-3p^2\phi_1\phi_2+3q\phi_1\phi_2-\frac{\phi_2\phi_3}{\phi_1^2}+\frac{\phi_2^2}{\phi_1^3};$$

and thence

$$P_2-3(Q_1-2PP_1)+2(R-3PQ+2Q^3)=p_2\phi_1-3(q_1-2pp_1)\,\phi_1^2+2(r-3pq+2p^3)\,\phi_1^3-p_1\phi_2.$$

But, introducing herein the derived functions in regard to $x$, we have

$$p_1=p'\phi_1,\quad q_1=q'\phi_1,\quad p_2=p'\phi_2+p''\phi_1^2,$$

whence

$$p_2\phi_1-p_1\phi_2=p'''\phi_1^3;$$

and the right-hand side becomes

$$=\phi_1^3\{p''-3(q'-2pp')+2(r-3pq+2p^3)\},$$

which is the required result.

We have thus

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3)$$

as an "invariant" of the differential equation

$$\frac{d^3y}{dx^3} + 3p\frac{d^2y}{dx^2} + 3q\frac{dy}{dx} + r = 0.$$

It is to be remarked that this is *not* what M. Halphen calls a "differential invariant;" he uses this expression not in regard to a differential equation, but in regard to a curve defined by an equation between the coordinates $(x, y)$, and the differential invariant is a function of the derivatives $y''$, $y'''$,... which is either $=0$ in virtue of the equation of the curve, or else, being put $=0$, it determines certain singularities of the curve. Thus $y''$ is a differential invariant; the equation $y''=0$ determines the points of inflexion. Again

$$\tfrac{1}{3}\left(\frac{y'''}{y''}\right)'' - \tfrac{2}{3}\frac{y'''}{y''}\left(\frac{y'''}{y''}\right)' + \tfrac{4}{27}\left(\frac{y'''}{y''}\right)^3$$

is a differential invariant, vanishing identically if the variables $(x, y)$ are connected by any quadric equation whatever; it is thus the differential invariant of a conic.

This last differential invariant is intimately connected with the above-mentioned invariant

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3),$$

viz. writing with M. Halphen

$$p = -\tfrac{1}{3}\frac{y'''}{y''},\ q = 0,\ r = 0,$$

we have

$$p'' - 3(q' - 2pp') + 2(r - 3pq + 2p^3) = p'' + 6pp' + 4p^3$$

$$= -\left\{\tfrac{1}{3}\left(\frac{y'''}{y''}\right)'' - \tfrac{2}{3}\frac{y'''}{y''}\left(\frac{y'''}{y''}\right)' + \tfrac{4}{27}\left(\frac{y'''}{y''}\right)^3\right\}.$$

It is moreover noticed by him that, writing

$$y'',\ y''',\ y'''',\ y''''' = 2a,\ 6b,\ 24c,\ 20d,$$

respectively, the function in { } becomes

$$= \frac{20}{a^2}(a^2d - 3abc + 2b^3);$$

the form under which he had previously obtained the differential invariant of the conic. As remarked by Sylvester, it is mentioned pp. 19 and 20 in Boole's *Differential Equations* (Cambridge, 1859), that the general differential equation of a conic was obtained by Monge in the form

$$9y''^2y''''' - 45y''y'''y'''' + 40(y''')^3 = 0,$$

which (representing the differential coefficients as just mentioned) becomes $a^2d - 3abc + 2b^3 = 0$, but putting them $=a, b, c, d$ respectively, it becomes $9a^2d - 45abc + 40b^3 = 0$; the last-mentioned form presented itself to Sylvester in his theory of Reciprocants.