853.

NOTE ON A FORMULA FOR $\Delta^n 0^i/n^i$ WHEN n, i ARE VERY LARGE NUMBERS.

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THE following formula

$$\frac{\Delta^n 0^i}{n^i} = e^{-nq}\left[1 + \left(\frac{i+1-2n}{2n}\right)q - \left(\frac{n+i+2}{2}\right)q^2\right],\ q = e^{-(i+1)/n},$$

is given by Laplace (*Théorie Analytique des Probabilités*, 2nd ed., Paris, 1814, p. 195) as an approximate value of $\Delta^n 0^i/n^i$, when n and i are very large numbers, and is applied immediately afterwards to the case where i is of the order $n \log n$. As remarked by Professor Tait, it is certainly not applicable to the case where i is of the order n; for taking $i = An$, where A is a given number however large, then q is indefinitely near to the very small value e^{-A}, but nevertheless the last term $-\frac{1}{2}(n+i+2)q^2$, by taking n sufficiently large, may be made as large as we please, and the value would thus come out negative. It is thus necessary that i should be at least of the order $n \log n$; but it may be of any higher order.

Writing for greater convenience $r = ne^{-i/n}$ (where r is not very large), then $nq = re^{-1/n} = r(1-X)$, if $X = 1 - e^{-1/n}$; and the formula becomes

$$\frac{\Delta^n 0^i}{n^i} = e^{-r(1-X)}\left[1 + \frac{i+1-2n}{2n}\, e^{-1/n}\frac{r}{n} - \frac{n+i+2}{2}\, e^{-2/n}\frac{r^2}{n^2}\right].$$

Here $X = \frac{1}{n} - \frac{1}{1\,.\,2}\frac{1}{n^2} + \frac{1}{1\,.\,2\,.\,3}\frac{1}{n^3} + \&c.$, and the exponential $e^{rX} = 1 + rX + \frac{r^2X^2}{1\,.\,2} + \ldots$ is thus also expansible in negative powers of n; the formula becomes

$$\frac{\Delta^n 0^i}{n^i} = e^{-r}\left(1 + rX + \frac{r^2X^2}{1\,.\,2} + \ldots\right)\left[1 + \frac{i+1-2n}{2n}\,.\,e^{-1/n}\frac{r}{n} - \frac{n+i+2}{2}e^{-2/n}\frac{r^2}{n^2}\right];$$

viz. putting for X its value,

$$= \quad e^{-r}\Big\{1$$

$$+ r\left(\frac{i+1-2n}{2n^2}e^{-1/n} + 1 - e^{-1/n}\right)$$

$$+ r^2\left(\frac{-n-i-2}{2n^2}e^{-2/n} + \frac{i+1-2n}{2n^2}(1-e^{-1/n})\,e^{-1/n} + \tfrac{1}{2}(1-e^{-1/n})^2\right)$$

$$+ \text{\&c.}\Big\};$$

or finally, expanding $e^{-1/n}$ and taking the whole result as far as $\frac{1}{n^2}$, the coefficient of r is

$$\left(-\frac{1}{n} + \frac{i+1}{2n^2}\right)\left(1-\frac{1}{n}\right) + \frac{1}{n} - \frac{1}{2n^2}, \quad = \frac{1+\frac{1}{2}i}{n^2};$$

the coefficient of r^2 is

$$\left(-\frac{1}{n} + \frac{-2-i}{n^2}\right)\left(1-\frac{2}{n}\right) + \left(-\frac{1}{n} + \frac{4i}{2n^2}\right)\frac{1}{n} + \frac{1}{2n^2}, \quad = -\frac{\frac{1}{2}}{n} + \frac{-\frac{1}{2}-\frac{1}{2}i}{n^2};$$

whence the formula becomes

$$\frac{\Delta^n 0^i}{n^2} = e^{-r}\left\{1 + r\,\frac{1+\frac{1}{2}i}{n^2} + \frac{r^2}{1\,.\,2}\left(-\frac{1}{n} + \frac{-1-i}{n^2}\right) + \ldots\right\}.$$

It seems to me that the correct result up to this order of approximation is

$$\frac{\Delta^n 0^i}{n^2} = e^{-r}\left\{1 + r\,\frac{\frac{1}{2}i}{n^2} + \frac{r^2}{1\,.\,2}\left(-\frac{1}{n} + \frac{-i}{n^2}\right)\right\}.$$

My investigation is as follows: we have

$$\frac{\Delta^n 0^i}{n^i} = 1 - \frac{n}{1}\left(1-\frac{1}{n}\right)^i + \frac{n\,.\,n-1}{1\,.\,2}\left(1-\frac{2}{n}\right)^i + \ldots,$$

the series being a finite one; but the number of terms is very large. But observe that, however large n is, we can take i so large that the second term $n\left(1-\frac{1}{n}\right)^i$ may be as small as we please; taking this term to be of moderate amount, say $= r_1$, the subsequent terms will be not very different from $\frac{r_1^2}{1\,.\,2}, \frac{r_1^3}{1\,.\,2\,.\,3}, \ldots$, and the approximate value is $1 - r_1 + \frac{r_1^2}{1\,.\,2} - \text{\&c.}$, which is a convergent series having its sum $=e^{-r_1}$. To work this properly out, I represent the successive terms by $r_1, \frac{r_2}{1.2}, \frac{r_3}{1.2.3}, \ldots$, so that the series is

$$=1-r_1+\frac{r_2}{1.2}-\frac{r_3}{1.2.3}+\ldots$$

Taking r a value at pleasure not very different from r_1, and multiplying by

$$(1=)\, e^{-r}.\, e^r = e^{-r}.\left(1+r+\frac{r^2}{1.2}+\ldots\right),$$

the sum is

$$=e^{-r}.\left\{1+(r-r_1)+\frac{1}{1.2}(r^2-2rr_1+r_2)\right.$$

$$\left.+\frac{1}{1.2.3}(r^3-3r^2r_1+3rr_2-r_3)+\ldots\right\}.$$

Assuming now $r=ne^{-i/n}$, we have

$$r_1=n\left(1-\frac{1}{n}\right)^i = ne^{i\log\left(1-\frac{1}{n}\right)} = r(1+X_1),$$

where $X_1=e^{-\frac{1}{2}\frac{i}{n^2}-\frac{1}{3}\frac{i}{n^3}-\ldots}$; and similarly

$$r_2=n.n-1.\left(1-\frac{2}{n}\right)^i = n^2\left(1-\frac{1}{n}\right)e^{i\log\left(1-\frac{2}{n}\right)}$$

$$=\left(1-\frac{1}{n}\right)r^2(1+X_2),$$

where $X_2=e^{-\frac{14i}{2n^2}-\frac{18i}{3n^3}-\ldots}$; also

$$r_3=n.n-1.\left(1-\frac{2}{n}\right)^i = n^2\left(1-\frac{1}{n}\right).e^{i\log\left(1-\frac{2}{n}\right)}$$

$$=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)r^3(1+X_3),$$

where $X_3=e^{-\frac{19i}{2n^2}-\frac{127i}{3n^3}-\ldots}$, and so on. It is now easy to calculate the successive terms $r-r_1$, $r^2-2rr_1+r_2$, &c.; and it is to be observed that, in the parts independent of the X's, we have only terms divided by n, n^2, or higher powers of n: thus in $r^4-4r^3r_1+6r^2r_2-4r^3r_3+r_4$, we have r^4 multiplied by

$$1-4+6\left(1-\frac{1}{n}\right)-4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)$$

$$=\frac{3}{n^2}-\frac{6}{n^3}.$$

We thus obtain the formula

$$\begin{aligned}\frac{\Delta^n 0^i}{n^i} = e^{-r}\Big\{1 & \\ &+ r \quad (\qquad -1X_1) \\ &+ \frac{r^2}{1.2}\left\{-\frac{1}{n} \qquad -2X_1 + \left(1-\frac{1}{n}\right)X_2\right\} \\ &+ \frac{r^3}{1.2.3}\left\{-\frac{2}{n^2} \qquad -3X_1 + 3\left(1-\frac{1}{n}\right)X_2 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)X_3\right\} \\ &+ \frac{r^4}{1.2.3.4}\left\{-\frac{3}{n^2}-\frac{6}{n^3} - 4X_1 + 6\left(1-\frac{1}{n}\right)X_2 - 4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)X_3\right. \\ &\qquad\qquad \left. + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)X_4\right\} + \dots, \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \Big\},\end{aligned}$$

where $r = ne^{-i/n}$ as above, and X_1, X_2, ... have the above-mentioned values, the exponentials being expanded in negative powers of n.

Writing

$$X_1 = \frac{-\tfrac{1}{2}i}{n^2}, \quad X_2 = \frac{-2i}{n^2},$$

we have

$$\frac{\Delta^n 0^i}{n^2} = e^{-r}\left\{1 + r\,\frac{\tfrac{1}{2}i}{n^2} + \frac{r^2}{2}\left(-\frac{1}{n} + \frac{-i}{n^2}\right)\right\},$$

which is the foregoing approximate value.