858.

COMPARISON OF THE WEIERSTRASSIAN AND JACOBIAN ELLIPTIC FUNCTIONS.

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THE Weierstrassian function $\sigma(u)$ corresponds of course with Jacobi's $H(u)$, but it is worth while to establish the actual formulæ of transformation.

Writing, for a moment,

$$\omega = \omega_1 + iv_1,$$
$$\omega' = \omega_2 + iv_2,$$

it is convenient to assume $\omega_1 v_2 - \omega_2 v_1$ positive; we then have

$$2(\eta\omega' - \eta'\omega) = +\pi i;$$

in particular, this will be the case if $\omega = \omega_1$, $\omega' = iv_2$, where ω_1, v_2 are each positive.

To reduce the periods into the Jacobian form, we may assume

$$\omega = \lambda K,$$
$$\omega' = \lambda i K',$$

(where observe that, if as usual k, K, K' are each real and positive, and if as above $\omega = \omega_1$, $\omega' = iv_2$, ω_1 and v_2 positive, then also λ will be real and positive). We have

$$\frac{iK'}{K} = \frac{\omega'}{\omega}, \text{ or say } q = e^{-\frac{\pi K'}{K}} = e^{\frac{i\pi\omega'}{\omega}},$$

which determines first q, and thence k, K, K', as functions of $\frac{\omega'}{\omega}$, and we then have $\lambda = \frac{\omega}{K} = \frac{-i\omega'}{K'}$, either of which equations gives λ as a function of ω, ω'. Conversely, starting with k, λ, the original equations give the values of ω, ω'; those of η, η' will be determined presently.

The form of relation is at once seen to be

$$H(u) = Ae^{Bu^2}\sigma(\lambda u),$$

and observing that, for u small, we have $H(u) = \sqrt{\left(\frac{2kk'K}{\pi}\right)}\,u$ and $\sigma(\lambda u) = \lambda u$, we have $A = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}$; I first write down and afterwards verify the value of B, viz. this is $= -\frac{1}{2}\frac{\lambda^2\eta}{\omega}$; and the formula thus is

$$H(u) = \frac{1}{\lambda}\sqrt{\left(\frac{2kk'K}{\pi}\right)}\, e^{-\frac{1}{2}\frac{\lambda^2\eta}{\omega}u^2}\,\sigma(\lambda u).$$

In fact, for u writing successively $u + 2K$, and $u + 2iK'$, we obtain

$$\frac{H(u+2K)}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2K(u+K)}\;\frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)},$$

$$\frac{H(u+2iK')}{H(u)} = e^{-\frac{\lambda^2\eta}{\omega}2iK'(u+iK')}\;\frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)},$$

which should be satisfied in virtue of

$$\frac{H(u+2K)}{H(u)} = -1, \qquad \frac{\sigma(\lambda u + 2\omega)}{\sigma(\lambda u)} = -e^{2\eta(\lambda u+\omega)},$$

$$\frac{H(u+2iK')}{H(u)} = -e^{-\frac{i\pi}{K}(u+iK')}, \qquad \frac{\sigma(\lambda u + 2\omega')}{\sigma(\lambda u)} = -e^{2\eta'(\lambda u+\omega')};$$

viz. we ought to have

$$0 = -\frac{\lambda^2\eta}{\omega}2K\;(u+\;K\;) + 2\eta\;(\lambda u + \omega\;)$$

$$-\frac{i\pi}{K}(u + iK') = -\frac{\lambda^2\eta}{\omega}2iK'(u+iK') + 2\eta'(\lambda u + \omega').$$

The first of these is

$$0 = -\frac{\lambda K}{\omega}(u+K) + \left(u + \frac{\omega}{\lambda}\right),$$

that is,

$$0 = (-1+1)(u+K);$$

and the second is

$$0 = \left(\frac{\frac{1}{2}iK}{\lambda K} - \frac{\lambda iK'\eta}{\omega}\right)(u+iK') + \eta'\left(u + \frac{\omega'}{\lambda}\right);$$

viz. for $i\pi$ writing $2(\eta\omega' - \eta'\omega)$, this is

$$0 = \left(\frac{\eta\omega' - \eta'\omega}{\omega} - \frac{\eta\omega'}{\omega} + \eta'\right)(u+iK'),$$

and the two equations are thus each of them an identity.

The Weierstrassian function $\wp(u)$ is defined as

$$= -\frac{d^2}{du^2}\log\sigma(u);$$

or, what is the same thing, we have

$$\wp(u) = -\frac{d}{du}\frac{\sigma'(u)}{\sigma(u)}.$$

Hence

$$\wp(\lambda u) = -\frac{1}{\lambda}\frac{d}{du}\frac{\sigma'(\lambda u)}{\sigma(\lambda u)}.$$

But

$$\frac{H'(u)}{H(u)} = -\frac{\lambda^2\eta u}{\omega} + \frac{\lambda\sigma'(\lambda u)}{\sigma(\lambda u)},$$

or

$$\wp(\lambda u) = -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\frac{d}{du}\frac{H'(u)}{H(u)}.$$

But from the equation $\sqrt{k}\,\mathrm{sn}\,u = \dfrac{H(u)}{\Theta(u)}$, we obtain

$$\frac{d}{du}\frac{H'(u)}{H(u)} = Z'(u) - \frac{1}{\mathrm{sn}^2 u} + k^2\,\mathrm{sn}^2 u, \quad = 1 - \frac{E}{K} - \frac{1}{\mathrm{sn}^2 u},$$

and consequently

$$\wp(\lambda u) = -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

where, on the right-hand side, expanding in ascending powers of u, the constant term is

$$= -\frac{\eta}{\omega} - \frac{1}{\lambda^2}\left(1 - \frac{E}{K}\right) + \frac{1}{\lambda^2}\tfrac{1}{3}(1 + k^2).$$

But in the function $\wp(\lambda u)$ this constant term is $= 0$, and we thus have

$$\frac{\eta}{\omega} = \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

and then, since

$$\frac{\eta}{\omega} - \frac{\eta'}{\omega'} = \frac{\frac{1}{2}\pi i}{\omega\omega'}, = \frac{\frac{1}{2}\pi}{\lambda^2 KK'},$$

we have

$$\frac{\eta'}{\omega'} = -\frac{\frac{1}{2}\pi}{\lambda^2 KK'} + \frac{1}{\lambda^2}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\};$$

or, as these equations may also be written,

$$\eta = \frac{K}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

$$\eta' = \frac{-\frac{1}{2}\pi i}{\lambda K} + \frac{2K'}{\lambda}\left\{\tfrac{1}{3}(1 + k^2) - 1 + \frac{E}{K}\right\},$$

which are the values of η, η'. And we then have

$$\wp(\lambda u) = -\frac{1}{\lambda^2}\tfrac{1}{3}(1 + k^2) + \frac{1}{\lambda^2\,\mathrm{sn}^2 u},$$

the equation connecting $\wp(\lambda u)$ and $\mathrm{sn}\,u$.