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NOTE ON A FORMULA RELATING TO THE ZERO-VALUE OF A THETA-FUNCTION.

[From *Crelle's Journal der Mathem.*, t. c. (1887), pp. 87, 88.]

I HAD some difficulty in verifying for the case of a single theta-function, a formula given in Herr Thomae's paper "Beitrag zur Theorie der ϑ-Functionen," *Crelle's Journal,* vol. LXXI. (1870), pp. 201—222. The formula in question (see p. 216) is given as follows:

$$(11) \qquad \vartheta(0,\ 0, \ldots 0) = \sqrt{\frac{|A_\lambda^{(\lambda')}|}{(2\pi i)^p}}\sqrt[4]{\text{Discr.}\,(0,\ 0, \ldots 0)\ \text{Discr}'.\,(0,\ 0, \ldots 0)},$$

but the denominator factor should I think be $(\pi i)^p$ instead of $(2\pi i)^p$. Making this alteration, then in the case of a single theta-function, $p=1$, and the function belongs to the radical

$$\sqrt{x-k_1 \,.\, x-k_2 \,.\, x-k_3 \,.\, x-k_4},$$

where

$$(k_1,\ k_2,\ k_3,\ k_4) = \left(-\frac{1}{k},\ -1,\ +1,\ +\frac{1}{k}\right).$$

The determinant $|A_\lambda^{(\lambda')}|$ is a single term $=A$, and the formula becomes

$$\vartheta(0) = \sqrt{\frac{A}{\pi i}}\sqrt[4]{(k_3-k_1)(k_4-k_2)},$$

where k_3-k_1, k_4-k_2 are each $=1+\frac{1}{k}$, and we have therefore

$$\vartheta(0) = \sqrt{\frac{A}{\pi i}\left(1+\frac{1}{k}\right)};$$

also A denotes the integral

$$\int_{k_1}^{k_2} \frac{dx}{\sqrt{x-k_1 \,.\, x-k_2 \,.\, x-k_3 \,.\, x-k_4}},\ = \int_{-\frac{1}{k}}^{-1} \frac{k\,dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}$$

$$= \int_1^{\frac{1}{k}} \frac{k\,dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}} = ikK',$$

and the formula thus is

$$\vartheta(0)=\sqrt{\frac{K'(1+k)}{\pi}}.$$

But observe that, in the theta-function as defined by the equation

$$\vartheta(x)=\Sigma e^{am^2+2mx},$$

a is used to denote the value

$$a=a_1'B,\quad =\frac{2\pi}{A}B,$$

where A is the above-mentioned integral, and B is the integral

$$B=\int_{k_2}^{k_3}\frac{dx}{\sqrt{x-k_1\,.\,x-k_2\,.\,x-k_3\,.\,x-k_4}},\quad =\int_{-1}^{1}\frac{k\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}}=2kK,$$

which value must however be taken negatively, viz. we must write $B=-2kK$, and we then have

$$a=-\frac{2\pi K}{K'},$$

viz. writing as usual

$$q=e^{-\frac{\pi K'}{K}},\quad r=e^{-\frac{\pi K}{K'}},$$

the e^a of the theta-function is not $=q$, but it is $=r^2$; and the zero-value $\vartheta(0)$ is $=1+2r^2+2r^8+2r^{18}+\ldots$. The equation thus is

$$1+2r^2+2r^8+2r^{18}+\ldots=\sqrt{\frac{K'(1+k)}{\pi}},$$

which is right; in fact, writing k' in place of k, and consequently K, q in place of K', r respectively, the equation becomes

$$1+2q^2+2q^8+2q^{18}+\ldots=\sqrt{\frac{K(1+k')}{\pi}};$$

we have

$$1+2q\ +2q^4+2q^9\ +\ldots=\sqrt{\frac{2K}{\pi}},$$

and changing q into q^2, then (*Fund. Nova*, p. 92) K is changed into $\frac{K(1+k')}{2}$, and we have the formula in question. As a verification for small values of q, observe that we have

$$\frac{2K}{\pi}=1+4q+4q^2,\quad \frac{1+k'}{2}=1-4q+16q^2,$$

and thence

$$\frac{K(1+k')}{\pi}=1+4q^2\ \text{ or }\ \sqrt{\frac{K(1+k')}{\pi}}=1+2q^2.$$

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