867.

NOTE ON THE JACOBIAN SEXTIC EQUATION.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 78—84.]

IN the Jacobian sextic equation

$$(z-a)^6 - 4a(z-a)^5 + 16b(z-a)^3 - 4c(z-a) + 5b^2 - 4ac = 0,$$

if A, B, C, D, E, F are the square roots (each with a determinate sign) of the roots z_0, z_1, z_2, z_3, z_4, z_∞ of the equation, and if ϵ be an imaginary fifth root of unity, such that $\epsilon + \epsilon^4 - \epsilon^2 - \epsilon^3 = \sqrt{5}$, then we have between the square roots a system of three linear equations

$$\begin{aligned}
0 &= -A\sqrt{5} + B + C + D + E + F, \\
0 &= B + \epsilon^3 C + \epsilon D + \epsilon^4 E + \epsilon^2 F, \\
0 &= B + \epsilon^2 C + \epsilon^4 D + \epsilon E + \epsilon^3 F,
\end{aligned}$$

see Brioschi's *Funzioni ellittiche* (Milan, 1880), third appendix, p. 402.

The sextic equation is irreducible, and there is thus nothing to distinguish any one square root from any other square root. But in the foregoing system of equations, A is distinguished from the other square roots; hence the system must be one of 6 systems, wherein the place occupied by A is occupied by A, B, C, D, E, F respectively. Observe however that the letters being square roots, it may very well happen, and (as will appear) it does in fact happen, that the passage from the first to another system implies a change of sign in certain of the letters A, B, C, D, E, F.

The selection of one of the square roots as $=A$, does not determine which of the other square roots shall be $=B$, C, D, E, F respectively: and in fact each system of equations may be represented in 10 different forms: viz. by multiplying the second and third equations by ϵ, ϵ^2, ϵ^3, ϵ^4 respectively, we obtain four new forms: we have thus five forms: and then in each of them transposing the second and third equations, we have five other forms: we have thus in all 10 forms.

Write $ABCDEF$ to denote the foregoing system of three equations

$$\begin{aligned}
0 &= -A\sqrt{5} + B + C + D + E + F, \\
0 &= B + \epsilon^3 C + \epsilon D + \epsilon^4 E + \epsilon^2 F, \\
0 &= B + \epsilon^2 C + \epsilon^4 D + \epsilon E + \epsilon^3 F;
\end{aligned}$$

the 10 forms of the system are

$$\begin{array}{c|c}
A\,B\,C\,D\,E\,F & A\,B\,F\,E\,D\,C \\
A\,C\,D\,E\,F\,B & A\,C\,B\,F\,E\,D \\
A\,D\,E\,F\,B\,C & A\,D\,C\,B\,F\,E \\
A\,E\,F\,B\,C\,D & A\,E\,D\,C\,B\,F \\
A\,F\,B\,C\,D\,E & A\,F\,E\,D\,C\,B,
\end{array}$$

where observe that the five forms in the same column are connected by cyclical substitutions on the last five letters; and that the forms in each line are connected by two inversions of the last four letters.

It is hardly necessary to remark that $ABFEDC$ means the system of three equations

$$\begin{aligned}
0 &= -A\sqrt{5} + B + F + E + D + C, \\
0 &= B + \epsilon^3 F + \epsilon E + \epsilon^4 D + \epsilon^2 C, \\
0 &= B + \epsilon^2 F + \epsilon^4 E + \epsilon D + \epsilon^3 C,
\end{aligned}$$

which are the same with the equations of the original form; and similarly that $ACDEFB$ means the system of three equations

$$\begin{aligned}
0 &= -A\sqrt{5} + C + D + E + F + B, \\
0 &= C + \epsilon^3 D + \epsilon E + \epsilon^4 F + \epsilon^2 B, \\
0 &= C + \epsilon^2 D + \epsilon^4 E + \epsilon F + \epsilon^3 B,
\end{aligned}$$

where the first equation is the same with the first equation, and the second and third equations only differ by a factor from the second and third equations respectively of the original form. And similarly as regards the others of the 10 forms.

I say that we have a second system $BCAF\bar{D}\bar{E}$ (where $\bar{D}$, $\bar{E}$ mean $-D$, $-E$ respectively): in fact, this denotes the system of three equations

$$\begin{aligned}
0 &= -B\sqrt{5} + C + A + F - D - E, \\
0 &= C + \epsilon^3 A + \epsilon F - \epsilon^4 D - \epsilon^2 E, \\
0 &= C + \epsilon^2 A + \epsilon^4 F - \epsilon D - \epsilon^3 E,
\end{aligned}$$

which are deducible from the original three equations.

We have, in fact, the identities

$$\begin{aligned}
&-\sqrt{5}\,\{-B\sqrt{5} + C + A + F - D - E\} \\
&= 1\,(B + C - A\sqrt{5} + F + D + E) \\
&\quad + 2\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
&\quad + 2\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}$$

$$\begin{aligned}
&(1 - \epsilon - \epsilon^3 + \epsilon^4)\,\{C + \epsilon^3 A + \epsilon F - \epsilon^4 D - \epsilon^2 E\} \\
&= 1\,(B + C - A\sqrt{5} + F + D + E) \\
&\quad + (\epsilon + \epsilon^4)\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
&\quad + (\epsilon^2 + \epsilon^3)\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}$$

and

$$\begin{aligned}
&(1 + \epsilon - \epsilon^2 - \epsilon^4)\,\{C + \epsilon^2 A + \epsilon^4 F - \epsilon D - \epsilon^3 E\} \\
&= 1\,(B + C - A\sqrt{5} + F + D + E) \\
&\quad + (\epsilon^2 + \epsilon^3)\,(B + \epsilon^3 C + \epsilon^2 F + \epsilon D + \epsilon^4 E) \\
&\quad + (\epsilon + \epsilon^4)\,(B + \epsilon^2 C + \epsilon^3 F + \epsilon^4 D + \epsilon E),
\end{aligned}$$

all which identities are at once verified on recollecting that

$$1 = -\epsilon - \epsilon^2 - \epsilon^3 - \epsilon^4 \text{ and } \sqrt{5} = \epsilon - \epsilon^2 - \epsilon^3 + \epsilon^4.$$

We can now write down the 6 systems, each of them under its 10 forms: these, in fact, are

$A\,B\,C\,D\,E\,F$	$A\,B\,F\,E\,D\,C$
$A\,C\,D\,E\,F\,B$	$A\,C\,B\,F\,E\,D$
$A\,D\,E\,F\,B\,C$	$A\,D\,C\,B\,F\,E$
$A\,E\,F\,B\,C\,D$	$A\,E\,D\,C\,B\,F$
$A\,F\,B\,C\,D\,E$	$A\,F\,E\,D\,C\,B$
$B\,C\,A\,F\,\bar{D}\,\bar{E}$	$B\,C\,\bar{E}\,\bar{D}\,F\,A$
$B\,A\,F\,\bar{D}\,\bar{E}\,C$	$B\,A\,C\,\bar{E}\,\bar{D}\,F$
$B\,F\,\bar{D}\,\bar{E}\,C\,A$	$B\,F\,A\,C\,\bar{E}\,\bar{D}$
$B\,\bar{D}\,\bar{E}\,C\,A\,F$	$B\,\bar{D}\,F\,A\,C\,\bar{E}$
$B\,\bar{E}\,C\,A\,F\,\bar{D}$	$B\,\bar{E}\,\bar{D}\,F\,A\,C$
$C\,D\,A\,B\,\bar{E}\,\bar{F}$	$C\,D\,\bar{F}\,\bar{E}\,B\,A$
$C\,A\,B\,\bar{E}\,\bar{F}\,D$	$C\,A\,D\,\bar{F}\,\bar{E}\,B$
$C\,B\,\bar{E}\,\bar{F}\,D\,A$	$C\,B\,A\,D\,\bar{F}\,\bar{E}$
$C\,\bar{E}\,\bar{F}\,D\,A\,B$	$C\,\bar{E}\,B\,A\,D\,\bar{F}$
$C\,\bar{F}\,D\,A\,B\,\bar{E}$	$C\,\bar{F}\,\bar{E}\,B\,A\,D$

$$\begin{array}{c|c}
D\,E\,A\,C\,\bar{F}\,\bar{B} & D\,E\,\bar{B}\,\bar{F}\,C\,A \\
D\,A\,C\,\bar{F}\,\bar{B}\,E & D\,A\,E\,\bar{B}\,\bar{F}\,C \\
D\,C\,\bar{F}\,\bar{B}\,E\,A & D\,C\,A\,E\,\bar{B}\,\bar{F} \\
D\,\bar{F}\,\bar{B}\,E\,A\,C & D\,\bar{F}\,C\,A\,E\,\bar{B} \\
D\,\bar{B}\,E\,A\,C\,\bar{F} & D\,\bar{B}\,\bar{F}\,C\,A\,E \\
\hline
E\,F\,A\,D\,\bar{B}\,\bar{C} & E\,F\,\bar{C}\,\bar{B}\,D\,A \\
E\,A\,D\,\bar{B}\,\bar{C}\,F & E\,A\,F\,\bar{C}\,\bar{B}\,D \\
E\,D\,\bar{B}\,\bar{C}\,F\,A & E\,D\,A\,F\,\bar{C}\,\bar{B} \\
E\,\bar{B}\,\bar{C}\,F\,A\,D & E\,\bar{B}\,D\,A\,F\,\bar{C} \\
E\,\bar{C}\,F\,A\,D\,\bar{B} & E\,\bar{C}\,\bar{B}\,D\,A\,F \\
\hline
F\,A\,E\,\bar{C}\,\bar{D}\,B & F\,A\,B\,\bar{D}\,\bar{C}\,E \\
F\,E\,\bar{C}\,\bar{D}\,B\,A & F\,E\,A\,B\,\bar{D}\,\bar{C} \\
F\,\bar{C}\,\bar{D}\,B\,A\,E & F\,\bar{C}\,E\,A\,B\,\bar{D} \\
F\,\bar{D}\,B\,A\,E\,\bar{C} & F\,\bar{D}\,\bar{C}\,E\,A\,B \\
F\,B\,A\,E\,\bar{C}\,\bar{D} & F\,B\,\bar{D}\,\bar{C}\,E\,A,
\end{array}$$

where, as before, the superscript line denotes a change of sign, $\bar{A} = -A$, &c.

As to the formation of this table, observe that we have $ABCDEF$ and $BCAF\bar{D}\bar{E}$; repeating the substitution, we have

$$\begin{array}{c}
A\,B\,C\,D\,E\,F \\
B\,C\,A\,F\,\bar{D}\,\bar{E} \\
C\,A\,B\,\bar{E}\,\bar{F}\,D \\
\hline
A\,B\,C\,D\,E\,F;
\end{array}$$

viz. we have thus the $CAB\bar{E}\bar{F}D$ which belongs to the third system; but nothing further, for the substitution is periodic of the third order, and the three forms are merely repeated. But if, upon

$$\begin{array}{c}
A\,B\,C\,D\,E\,F \\
B\,C\,A\,F\,\bar{D}\,\bar{E},
\end{array}$$

we operate successively with the substitutions which change the upper line into the three forms of the first system

$$A\,D\,E\,F\,B\,C,\quad A\,E\,F\,B\,C\,D,\quad A\,F\,B\,C\,D\,E,$$

then the lower line is changed into the three forms

$$D\,E\,A\,C\,\bar{F}\,\bar{B},\quad E\,F\,A\,D\,\bar{B}\,\bar{C},\quad F\,B\,A\,E\,\bar{C}\,\bar{D},$$

which are forms belonging to the fourth, fifth, and sixth systems respectively. By way of verification, observe that, for instance repeating upon the second line the substitution

$$A\,B\,C\,D\,E\,F,$$

in place of

$$D\,E\,A\,C\,\bar{F}\,B,$$

we obtain

$$C\,\bar{F}\,D\,A\,B\,\bar{E},$$

which is one of the forms of the third system, assumed to have been previously found; and so in other instances.

Reverting to the three equations belonging to the form *ABCDEF*, by subtracting the third from the second equation, we obtain a linear relation between the four square roots *C*, *D*, *E*, *F*, viz. this is

$$0 = (\epsilon^3 - \epsilon^2)(C - F) + (\epsilon - \epsilon^4)(D - E);$$

and the same equation is obtained by means of the form *ABFEDC*. We thus obtain from the thirty lines of either column of the preceding table, thirty such equations, but obviously the number of such equations should be fifteen, for there can be but one relation between any four square roots *C*, *D*, *E*, *F*; consequently each equation will be obtained twice; and, in fact, it is clear that the forms *ABCDEF* and $BAF\bar{D}\bar{E}C$, and so in general any two forms which begin with the same pair of letters, give the same equation. But for greater symmetry, I write down the thirty equations in the order in which they are given by the left-hand column of the table: the equations are

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+ (\epsilon - \epsilon^4)$ multiplied by
$C - F$	$D - E$
$D - B$	$E - F$
$E - C$	$F - B$
$F - D$	$B - C$
$B - E$	$C - D$
$A + E$	$F + D$
$F - C$	$- D + E$
$- D - A$	$- E - C$
$- E - F$	$C - A$
$C + D$	$A - F$
$A + F$	$B + E$
$B - D$	$- E + F$
$- E - A$	$- F - D$
$- F - B$	$D - A$
$D + E$	$A - B$

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+ (\epsilon - \epsilon^4)$ multiplied by
$A + B$	$C + F$
$C - E$	$-F + B$
$-F - A$	$-B - E$
$-B - C$	$E - A$
$E + F$	$A - C$
$A + C$	$D + B$
$D - F$	$-B + C$
$-B - A$	$-C - F$
$-C - D$	$F - A$
$F + B$	$A - D$
$E - B$	$-C + D$
$-C - A$	$-D - B$
$-D - E$	$B - A$
$B + C$	$A - E$
$A + D$	$E + C$

where it is to be observed that, adding together the five equations given by any one of the systems, we obtain the identical result $0 = 0$.

I write down the 15 equations in a different order, in some cases changing the sign of the whole equation, as follows

$0 = (\epsilon^3 - \epsilon^2)$ multiplied by	$+ (\epsilon - \epsilon^4)$ multiplied by
$A + C$	$B + D$
$B + C$	$A - E$
$A + B$	$C + F$
$D + E$	$A - B$
$B + F$	$A - D$
$A + F$	$B + E$
$A + D$	$C + E$
$C + D$	$A - F$
$E + F$	$A - C$
$A + E$	$D + F$
$B - E$	$C - D$
$F - D$	$B - C$
$E - C$	$F - B$
$D - B$	$E - F$
$C - F$	$D - E$

The first three of these equations, or writing $\lambda = \epsilon^3 - \epsilon^2$, $\mu = \epsilon - \epsilon^4$, say

$$\lambda (A + C) + \mu (B + D) = 0,$$
$$\lambda (B + C) + \mu (A - E) = 0,$$
$$\lambda (A + B) + \mu (C + F) = 0,$$

constitute the entire system of independent linear relations between the square roots A, B, C, D, E, F. The coefficients λ, μ are such that

$$\lambda^2 - \mu^2 = \lambda\mu \, (= \epsilon - \epsilon^2 - \epsilon^3 + \epsilon^4, \ = \sqrt{5}),$$

and it is hence easy to verify that the remaining twelve equations can be deduced from these by the elimination of one or two of the square roots A, B, C. For instance, to eliminate A from the first and second equations, multiplying by $-\mu$, λ and adding, we obtain

$$(-\lambda\mu + \lambda^2) C + (-\mu^2 + \lambda^2) B - \mu^2 D - \lambda\mu E = 0,$$

that is,

$$\mu^2 C + \qquad \lambda\mu B - \mu^2 D - \lambda\mu E = 0,$$

or finally

$$\lambda (B - E) + \mu (C - D) = 0,$$

which is one of the equations. And so again eliminating A from the first and third equations, we find

$$\lambda (B - C) + \mu (C + F - B - D) = 0,$$

that is,

$$(\lambda - \mu)(B - C) + \mu (F - D) = 0,$$

or multiplying by λ,

$$\mu^2 (B - C) + \lambda\mu (F - D) = 0,$$

that is, finally

$$\lambda \, (F - D) + \ \mu (B - C) = 0,$$

which is one of the equations.

Cambridge, 21 *March* 1887.