869.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

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THE algebraical theory of the Transformation of Elliptic Functions was established by Jacobi in a remarkably simple and elegant form, but it has not hitherto been developed with much completeness or success. The cases $n=3$ and $n=5$ are worked out very completely in the *Fundamenta Nova* (1829); viz. considering the equation

$$\frac{M d\mathrm{y}}{\sqrt{1-\mathrm{y}^2 \,.\, 1-\lambda^2\mathrm{y}^2}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2 \,.\, 1-k^2\mathrm{x}^2}},$$

($k=u^4$, $\lambda=v^4$; say this is the $Mk\lambda$- or Muv-form), Jacobi finds, in the two cases respectively, a modular equation between the fourth roots u, v, say the uv-modular equation, and, as rational functions of u, v, the value of M and the values of the coefficients of the several powers of x in the numerator and denominator of the fraction which gives the value of y; but there is no attempt at a like development of the general case. I shall have occasion to speak of other researches by Jacobi, Brioschi and myself; but I will first mention that my original idea in the present memoir was to develop the following mode of treatment of the theory:

In place of the $Mk\lambda$-form, using the $\rho\alpha\beta$-form

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}}$$

(I write for greater convenience 2α, 2β in place of the α of Jacobi and Brioschi and the β of Brioschi), we can, by expanding each side in a series, integrating, and reverting the resulting series for y, obtain y in the form

$$y=\rho x\,(1+\Pi_1 x^2+\Pi_2 x^4+\ldots),$$

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where $\Pi_1, \Pi_2, \Pi_3, \ldots$ denote given functions of ρ, α, β. Taking n odd and $=2s+1$ we assume for y an expression

$$y=\frac{x(A_s+A_{s-1}x^2+\ldots+A_1x^{2s-2}+x^{2s})}{1+A_1x^2+\ldots+A_{s-1}x^{2s-2}+A_sx^{2s}},$$

where the last coefficient A_s is at once seen to be $=\rho$. Comparing with the series-value $y=\rho x(1+\Pi_1x^2+\Pi_2x^4+\ldots)$, we have an infinite series of equations. The first of these is, in fact, $A_s=\rho$; the next $(s-1)$ equations give linearly $A_1, A_2, \ldots, A_{s-1}$ in terms of the coefficients Π; that is, of ρ, α, β: the two which follow serve in effect to determine ρ, β as functions of α: and then, ρ and β having these values, all the remaining equations will be satisfied identically.

The process is an eminently practical one, so far as regards the determination of the coefficients $A_1, A_2, \ldots, A_{s-1}$ as functions of ρ, α, β; it is less so, and requires eliminations more or less complicated, as regards the determination of the relations between ρ, α, β. As to this, it may be remarked that the problem is not so much the determination of the equation between ρ and α (or say the $\rho\alpha$-multiplier equation, or simply the $\rho\alpha$-equation), and of the equation between β, α (or say the $\alpha\beta$-modular equation, or simply the $\alpha\beta$-equation), as it is to determine the complete system of relations between ρ, α, β; treating these as coordinates, we have what may be called the multiplier-modular-curve, or say the MM-curve, and the relations in question are those which determine this curve.

In the absence of special exceptions, it follows from general principles that the coefficients $A_1, A_2, \ldots, A_{s-1}$, *quà* rational functions of ρ, α, β, must also be rational functions of α, β or of α, ρ; and I think it may be assumed that this is the case; the method, however, affords but little assistance towards thus expressing them.

In connexion with the foregoing theory, I consider the solutions of the problem of transformation given by Jacobi's partial differential equation ("Suite de Notices sur les Fonctions elliptiques," *Crelle*, t. IV. (1829), pp. 185—193), and by what I call the Jacobi-Brioschi differential equations. The first and third of these were obtained by Jacobi in the memoir*, "De functionibus ellipticis Commentatio," *Crelle*, t. IV. (1829), pp. 371—390 (see p. 376); but the second equation, which completes the system, was, I believe, first given by Brioschi in the second appendix to his translation of my *Elliptic Functions: Trattato elementare delle Funzioni ellitiche:* Milan, 1880. I had, strangely enough, overlooked the great importance of these equations. I shall have occasion also to refer to results, and further develop the theory contained in my memoir, "On the Transformation of Elliptic Functions," *Phil. Trans.*, t. CLXIV. (1874), pp. 397—456, [578], and the addition thereto, *Phil. Trans.*, t. CLXXXIX. (1878), pp. 419—424, [692].

I remark that, while I have only worked out the formulæ for the cases $n=3$ and $n=5$, and a few formulæ for the case $n=7$, the memoir is intended to be a contribution to the general theory of the $\rho\alpha\beta$-transformation; I hope to be able to complete the theory for the case $n=7$.

[* *Ges. Werke*, bd. I., pp. 295—318; in particular, p. 303.]

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Comparison of the $Mk\lambda$- and $\rho\alpha\beta$-Forms. The Modular and Multiplier Equations.

Art. Nos. 1 to 12.

1. The equation

$$\frac{Mdy}{\sqrt{1-\mathrm{y}^2 . 1-\lambda^2\mathrm{y}^2}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2 . 1-k^2\mathrm{x}^2}},$$

if we write therein

$$\mathrm{x}=\frac{x}{\sqrt{k}},\ =\frac{x}{u^2};\quad \mathrm{y}=\frac{y}{\sqrt{\lambda}},\ =\frac{y}{v^2},$$

becomes

$$\frac{Mdy}{v^2\sqrt{1-(v^4+v^{-4})\,y^2+y^4}}=\frac{dx}{u^2\sqrt{1-(u^4+u^{-4})\,x^2+x^4}};$$

viz. this is

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$$

if only

$$2\alpha=u^4+\frac{1}{u^4},\quad 2\beta=v^4+\frac{1}{v^4},\quad \rho=\frac{v^2}{u^2M}.$$

2. We have a uv-modular equation, and, as shown in my Transformation Memoir*, p. 450, this may be converted into a u^4v^4-modular equation; in particular, $n=3$, the equation is

$$y^4+6x^2y^2+x^4-4xy\,(4x^2y^2-3x^2-3y^2+4)=0,$$

where x, y denote u^4, v^4 respectively; say the equation is

$$F(x,\ y),\ =x^4+x^3\,(-16y^3+12y)+x^2\,(6y^2)+x\,(12y^3-16y)+y^4,\ =0.$$

From the equation $F(x,\ y)=0$, we derive

$$x^{-2}F(x,\ y)\,.\,x^{-2}F(x,\ y^{-1})=0;$$

say this is

$$(Ax^2+Bx+C+Dx^{-1}+Ex^{-2})\,(A'x^2+B'x+C'+D'x^{-1}+E'x^{-2})=0,$$

viz. the equation is

$$AA'x^4+(AB'+A'B)\,x^3+\ldots+EE'x^{-4}=0,$$

where, by reason of the symmetry of $F(x,\ y)$, the coefficients AA', EE' of x^4, x^{-4}, those of x^3, x^{-3}, &c., have equal values; the form thus is

$$\mathfrak{A}\,(x^4+x^{-4})+\mathfrak{B}\,(x^3+x^{-3})+\mathfrak{C}\,(x^2+x^{-2})+\mathfrak{D}\,(x+x^{-1})+\mathfrak{E}=0,$$

where x^4+x^{-4}, x^3+x^{-3}, x^2+x^{-2}, are given functions of $x+x^{-1}$, $=2\alpha$; viz. we have

$$\begin{aligned} x\ +x^{-1} &= 2\alpha,\\ x^2+x^{-2} &= 4\alpha^2-2,\\ x^3+x^{-3} &= 8\alpha^3-6\alpha,\\ x^4+x^{-4} &= 16\alpha^4-16\alpha^2+2.\end{aligned}$$

[* This Collection, vol. IX., p. 170.]

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The coefficients $\mathfrak{A}$, $\mathfrak{B}$,... are in like manner expressible as functions of $y+y^{-1}$, $=2\beta$; thus we have $\mathfrak{A}=1$,

$$\mathfrak{B} = AB' + A'B$$
$$= -16(y^3+y^{-3}) + 12(y+y^{-1}),\ = -16(8\beta^3-6\beta)+12\,.\,2\beta;$$

or, finally, $\mathfrak{B} = -128\beta^3+120\beta$; and so for the other coefficients. The numerical coefficients contain, all of them, the factor 16; and, throwing this out, we obtain, for $n=3$, the $\alpha\beta$-modular equation in the form

	α^4	α^3	α^2	α	1
β^4					+ 1
β^3		− 64		+ 60	
β^2			− 186		+ 192
β		+ 60		− 64	
1	+ 1		+ 192		− 192
	+ 1	− 4	+ 6	− 4	+ 1

= 0,

where observe that the form is symmetrical as regards α, β; and, further, that the sums of the numerical coefficients in the lines or columns are the binomial coefficients 1, −4, +6, −4, +1. Observe, further, that the sums in the direction of the sinister diagonal are −64, −64, +320, −192; viz. dividing by −64, it thus appears that, writing $\beta=\alpha$, the equation becomes

$$\alpha^6+\alpha^4-5\alpha^2+3=0;$$

that is, $(\alpha^2-1)^2(\alpha^2+3)=0$.

Again, writing $\beta=-\alpha$, then dividing by 16, the equation becomes

$$4\alpha^6-19\alpha^4+28\alpha^2-12=0;$$

that is,

$$(4\alpha^2-3)(\alpha^2-2)^2=0.$$

3. So also, for $n=5$, we have the u^4v^4-modular equation in the form

$$\left.\begin{array}{l} x^6+655x^4y^2+655x^2y^4+y^6-640x^2y^2-640x^4y^4 \\ \quad +xy\,(-256+320x^2+320y^2-70x^4-660x^2y^2-70y^4 \\ \qquad +320x^4y^2+320x^2y^4-256x^4y^4) \end{array}\right\}=0;$$

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and in precisely the same manner, we obtain the $\alpha\beta$-modular equation; viz. casting out a factor 64, this is

	β^6	β^5	β^4	β^3	β^2	β	1	
α^6							+ 1	
α^5		− 4096		+ 6400		− 2310		
α^4			+ 69120		− 172785		+ 103680	
α^3		+ 6400		− 133140		+ 126720		= 0,
α^2			− 172785		+ 276480		− 103680	
α		− 2310		+ 126720		− 124416		
1	+ 1		+ 103680		− 103680			
	+ 1	− 6	+ 15	− 20	+ 15	− 6	+ 1	

where the form is symmetrical as regards α, β; the sums of the numerical coefficients in the lines or columns are $1, -6, +15, -20, +15, -6, +1$. The sums in the direction of the sinister diagonal all divide by -4096; viz. throwing out this factor, we have, for $\beta=\alpha$, the equation

$$\alpha^{10}-20\alpha^8+118\alpha^6-180\alpha^4+81\alpha^2=0;$$

that is,

$$\alpha^2(\alpha^2-1)^2(\alpha^2-9)^2=0.$$

If $\beta=-\alpha$, the coefficients divide by 64; and throwing out this factor, the equation is

$$64\alpha^{10}+880\alpha^8-3247\alpha^6+3600\alpha^4-1296\alpha^2=0;$$

that is,

$$\alpha^2(\alpha^4+16\alpha^2-16)(8\alpha^2-9)^2=0.$$

4. We have a Mu-multiplier equation of the form $F\left(\frac{1}{M},\ 2u^8-1\right)=0$ (see Memoir*, pp. 420—422), but we cannot, by the preceding formulæ, deduce thence a $\rho\alpha$-multiplier equation; in fact, writing therein $\frac{1}{M}=\frac{u^2\rho}{v^2}$, the resulting equation is $F\left(\frac{u^2\rho}{v^2},\ 1-2u^8\right)=0$, which is a $\rho\alpha$-multiplier equation only on the assumption that $1-2u^8$, u^2 and v^2 are therein regarded as given functions of α. But it is very

[* This Collection, vol. x., pp. 334, 335.]

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remarkable that the $\rho\alpha$-equation, in fact, is $F(\rho, \alpha) = 0$. To prove this, assume that the equation

$$\frac{dy}{\sqrt{1 - 2\beta y^2 + y^4}} = \frac{\rho\, dx}{\sqrt{1 - 2\alpha x^2 + x^4}}$$

has a $\rho\alpha$-multiplier equation $F(\rho, \alpha) = 0$. Starting from the equation

$$\frac{M d\mathrm{y}}{\sqrt{1 - \mathrm{y}^2 . 1 - \lambda^2\mathrm{y}^2}} = \frac{d\mathrm{x}}{\sqrt{1 - \mathrm{x}^2 . 1 - k^2\mathrm{x}^2}},$$

we may, by effecting on each side a quadric transformation, convert this into

$$\frac{dy}{\sqrt{1 - 2(2v^8 - 1)y^2 + y^4}} = \frac{M^{-1} dx}{\sqrt{1 - 2(2u^8 - 1)x^2 + x^4}};$$

and this being so, we have, between M^{-1} and $2u^8 - 1$, the relation

$$F\left(\frac{1}{M},\ 2u^8 - 1\right) = 0;$$

or, conversely, if this be the form of the Mu-multiplier equation, then the $\rho\alpha$-multiplier equation is $F(\rho, \alpha) = 0$.

5. The quadric transformations are

$$x = \frac{\sqrt{1 - \mathrm{x}^2}}{\mathrm{x}\sqrt{1 - k^2\mathrm{x}^2}}, \quad y = \frac{\sqrt{1 - \mathrm{y}^2}}{\mathrm{y}\sqrt{1 - \lambda^2\mathrm{y}^2}}.$$

We have then only to show that

$$\frac{dx}{\sqrt{1 - 2(2u^8 - 1)x^2 + x^4}} = \frac{d\mathrm{x}}{\sqrt{1 - \mathrm{x}^2 . 1 - k^2\mathrm{x}^2}};$$

for then, in like manner,

$$\frac{dy}{\sqrt{1 - 2(2v^8 - 1)y^2 + y^4}} = \frac{d\mathrm{y}}{\sqrt{1 - \mathrm{y}^2 . 1 - \lambda^2\mathrm{y}^2}},$$

and we pass from the assumed differential relation between x, y to the above-mentioned differential equation between x, y.

6. For the quadric transformation between x, x, write

$$\theta^{\frac{1}{2}} = k - ik', \quad \theta^{-\frac{1}{2}} = k + ik',$$

$\left(\text{whence also } \theta = \dfrac{k - ik'}{k + ik'}\right)$, and therefore

$$\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} = 2k, \quad \theta + \theta^{-1} = 2k^2 - 2k'^2 = 2(2k^2 - 1), \quad = 2(2u^8 - 1);$$

we have

$$1 - \theta x^2 = 1 - \theta\frac{1 - \mathrm{x}^2}{\mathrm{x}^2(1 - k^2\mathrm{x}^2)}, \quad = \frac{1}{\mathrm{x}^2(1 - k^2\mathrm{x}^2)}\{-\theta + (\theta + 1)\mathrm{x}^2 - k^2\mathrm{x}^4\},$$

$$= \frac{-\theta}{\mathrm{x}^2(1 - k^2\mathrm{x}^2)}(1 - \theta^{-\frac{1}{2}}k\mathrm{x}^2)^2;$$

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and similarly,

$$1-\theta^{-1}x^2=\frac{-\theta^{-1}}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}(1-\theta^{\frac{1}{2}}k\mathrm{x}^2)^2.$$

Consequently,

$$(1-\theta x^2)(1-\theta^{-1}x^2)=1-2(2u^8-1)x^2+x^4=\frac{1}{\mathrm{x}^4(1-k^2\mathrm{x}^2)^2}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4)^2;$$

or say

$$\sqrt{1-2(2u^8-1)x^2+x^4}=\frac{1}{\mathrm{x}^2(1-k^2\mathrm{x}^2)}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4).$$

Moreover,

$$dx=\frac{d\mathrm{x}}{\mathrm{x}^2(1-\mathrm{x}^2)^{\frac{1}{2}}(1-k^2\mathrm{x}^2)^{\frac{3}{2}}}(1-2k^2\mathrm{x}^2+k^2\mathrm{x}^4),$$

and thence the required equation

$$\frac{dx}{\sqrt{1-2(1-2u^8)x^2+x^4}}=\frac{d\mathrm{x}}{\sqrt{1-\mathrm{x}^2\,.\,1-k^2\mathrm{x}^2}};$$

this completes the proof.

7. Thus, referring to the *Mu*-equations given in the place referred to, we obtain the following $\rho\alpha$-multiplier equations. When $n=3$, we have

$$\rho^4-6\rho^2-8\alpha\rho-3=0.$$

This may be written in the forms

$$\begin{aligned}8\alpha\rho\quad &=\rho^4-6\rho^2-3,\\ 8(\alpha+1)\rho&=(\rho-1)^3(\rho+3),\\ 8(\alpha-1)\rho&=(\rho+1)^3(\rho-3).\end{aligned}$$

Next, for $n=5$, we have $\rho^6-10\rho^5+35\rho^4-60\rho^3+55\rho^2+(38-64\alpha^2)\rho+5=0.$

This may be written in the two forms

$$64\alpha^2\rho\quad=(\rho^2-4\rho-1)^2(\rho^2-2\rho+5)$$

and

$$64(\alpha^2-1)\rho=(\rho-1)^5(\rho-5).$$

And, for $n=7$, we have

$$\rho^8-28\rho^6-112\alpha\rho^5-210\rho^4-224\alpha\rho^3+(-1484+1344\alpha^2)\rho^2+(-560\alpha+512\alpha^3)\rho+7=0.$$

8. The relation between ρ and β, or say the $\rho\beta$-multiplier equation, may be obtained by a known property of elliptic functions; viz. writing $\rho\sigma=\pm n$ (the sign is $-$ for $n=3$, $n=7$, or generally for any prime value $4p+3$: and it is $+$ for $n=5$ and generally for any prime value $=4p+1$), then we have between σ, β the same relation as between ρ, α. Thus, if $n=3$, $\sigma=-\dfrac{3}{\rho}$, for ρ, α writing σ, β, the equation is $\sigma^4-6\sigma^2-8\beta\sigma-3=0$; or, as this may be written,

$$\rho^4+8\beta\rho^3+18\rho^2-27=0;$$

and so for the other cases; but it is perhaps more convenient to retain the σ; thus, if $n=5$, $\sigma=\dfrac{5}{\rho}$, we have

$$\sigma^6-10\sigma^5+35\sigma^4-60\sigma^3+55\sigma^2+(38-64\beta^2)\sigma+5=0.$$

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9. We are hence able to express β as a rational function of ρ, α. We, in fact, have

$$8\alpha = \frac{1}{\sqrt{\rho}}(\rho^2 - 4\rho - 1)\sqrt{\rho^2 - 2\rho + 5}, \quad 8\beta = -\frac{1}{\sqrt{\sigma}}(\sigma^2 - 4\sigma - 1)\sqrt{\sigma^2 - 2\sigma + 5},$$

(the signs must be opposite), and then for σ, substituting its value $= \frac{1}{5}\rho$, and observing that $\sigma^2 - 2\sigma + 5$ is thus $= \frac{5}{\rho^2}(\rho^2 - 2\rho + 5)$, we find

$$\frac{\beta}{\alpha} = \frac{\rho^2 + 20\rho - 25}{\rho^2(\rho^2 - 4\rho - 1)},$$

which is the required formula.

Observe that, for $\rho = \sigma = \sqrt{5}$, the formulæ with the sign $-$, as above, give $\beta = -\alpha$, whereas with the sign $+$ they would have given $\beta = \alpha$. For the value in question, $\rho = \sqrt{5}$, the equation

$$64\alpha^2 = \frac{1}{\rho}(\rho^2 - 4\rho - 1)^2(\rho^2 - 2\rho + 5),$$

gives

$$64\alpha^2 = \frac{1}{\sqrt{5}}16(1 - \sqrt{5})^2(10 - 2\sqrt{5});$$

that is,

$$\alpha^2 = \frac{1}{\sqrt{5}}(3 - \sqrt{5})(5 - \sqrt{5}), \quad = (3 - \sqrt{5})(\sqrt{5} - 1);$$

that is, $\alpha^2 = -8 + 4\sqrt{5}$, or $\alpha^4 + 16\alpha^2 - 16 = 0$; it appears, *ante* No. 3, that this value belongs to the case $\beta = -\alpha$ and not to $\beta = \alpha$.

10. But there is another way of arriving at a formula containing β. Starting from Jacobi's equation

$$nM^2 = \frac{\lambda\lambda'^2}{kk'^2} \cdot \frac{dk}{d\lambda},$$

and introducing for λ, λ', k, k', M their values in terms of u, v, we have

$$\frac{nv^4}{u^4\rho^2} = \frac{v^4(1 - v^8)\,u^3du}{u^4(1 - u^8)\,v^3dv};$$

that is,

$$\frac{dv}{du} = \frac{\rho^2}{n}\frac{u^3(1 - v^8)}{v^3(1 - u^8)};$$

but, from the values of α, β, we find

$$\frac{dv}{du} = \frac{v^5(1 - u^8)}{u^5(1 - v^8)}\frac{d\beta}{d\alpha},$$

and, combining these results,

$$\frac{d\beta}{d\alpha} = \frac{\rho^2}{n}\frac{u^8}{v^8} \cdot \frac{(1 - v^8)^2}{(1 - u^8)^2}, \quad = \frac{\rho^2}{n}\frac{(v^4 - v^{-4})^2}{(u^4 - u^{-4})^2};$$

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that is,

$$\frac{d\beta}{d\alpha}=\frac{\rho^2}{n}\frac{\beta^2-1}{\alpha^2-1}.$$

We have, consequently,

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2 d\alpha}{n(\alpha^2-1)},$$

and therefore

$$\tfrac{1}{2}\log\frac{\beta-1}{\beta+1}=\frac{1}{n}\int\frac{\rho^2 d\alpha}{\alpha^2-1},$$

where ρ^2 must be regarded as a function of α, or α of ρ; and from the form of the equation, it appears that the integral must be expressible as the logarithm of an algebraic function of ρ, α.

11. Thus, when $n=3$, we have

$$8\alpha=\rho^3-6\rho-\frac{3}{\rho};$$

whence

$$8\frac{d\alpha}{d\rho}=3\rho^2-6+\frac{3}{\rho^2},\quad=\frac{3}{\rho^2}(\rho^2-1)^2,$$

and thence easily

$$\tfrac{1}{2}\log\frac{\beta-1}{\beta+1}=\int\frac{8\rho^2 d\rho}{\rho^2-1\,.\,\rho^2-9},\quad=-\int\frac{d\rho}{\rho^2-1}+9\int\frac{d\rho}{\rho^2-9},$$

$$=-\tfrac{1}{2}\log\frac{\rho-1}{\rho+1}+\tfrac{3}{2}\log\frac{\rho-3}{\rho+3};$$

that is,

$$\frac{\beta-1}{\beta+1}=\frac{(\rho-3)^3(\rho+1)}{(\rho+3)^3(\rho-1)},$$

as may be at once verified.

12. In the case $n=5$, I verify the equation under the form

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{5}\cdot\frac{d\alpha}{\alpha^2-1}.$$

From the equations

$$64(\alpha^2-1)=\frac{1}{\rho}(\rho-1)^5(\rho-5),\ \text{and}\ 8\alpha=\frac{1}{\sqrt{\rho}}(\rho^2-4\rho-1)\sqrt{\rho^2+2\rho-5},$$

we have

$$\frac{128\alpha\,d\alpha}{\alpha^2-1}=\frac{5(\rho^2-4\rho-1)\,d\rho}{\rho(\rho-1)(\rho-5)},$$

and thence

$$\frac{16d\alpha}{\alpha^2-1}=\frac{5d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}}.$$

Similarly, observing the $-$ sign of 8β, we have

$$\frac{16d\beta}{\beta^2-1}=\frac{-5d\sigma}{(\sigma-1)(\sigma-5)\sqrt{\sigma(\sigma^2-2\sigma+5)}},$$

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whence, substituting for σ its value $=\frac{5}{\rho}$, we have

$$\frac{16d\beta}{\beta^2-1}=\frac{\rho^2\,d\rho}{(\rho-1)(\rho-5)\sqrt{\rho(\rho^2-2\rho+5)}},\quad =\frac{\rho^2}{5}\cdot\frac{16d\alpha}{\alpha^2-1},$$

which is right.

Connexion of the $Mk\lambda$- and $\rho\alpha\beta$-Theories. Order of Modular Equation.
Art. Nos. 13 to 18.

13. In the Transformation Memoir [578], starting from the equation

$$\frac{1-\mathrm{y}}{1+\mathrm{y}}=\frac{1-\mathrm{x}}{1+\mathrm{x}}\left(\frac{P-Q\mathrm{x}}{P+Q\mathrm{x}}\right)^2,$$

I sought to determine the coefficients of P, Q by the consideration that the relation between x, y remains unaltered when x, y are changed into $\frac{1}{k\mathrm{x}}$, $\frac{1}{\lambda\mathrm{y}}$ respectively. This comes to saying that, when for x, y we write $\frac{x}{u^2}$, $\frac{y}{v^2}$ respectively, the relation between x and y presents itself in the form

$$y=\frac{x(A_s+A_{s-1}x^2+\ldots+A_0x^{2s})}{A_0+A_1x^2+\ldots+A_sx^{2s}},$$

where $s=\frac{1}{2}(n-1)$, as before. For instance, when $n=7$, $P=\alpha+\gamma\mathrm{x}^2$, $Q=\beta+\delta\mathrm{x}^2$.

If, solving for y, we then for x, y write $\frac{x}{u^2}$, $\frac{y}{v^2}$, we find

$$y=\frac{v^2u^{-2}x\{(\alpha^2+2\alpha\beta)+(2\alpha\gamma+\beta^2+2\alpha\delta+2\beta\gamma)x^2u^{-4}+(\gamma^2+2\beta\delta+2\gamma\delta)x^4u^{-8}+\delta^2x^6u^{-12}\}}{\alpha^2+(2\alpha\gamma+\beta^2+2\alpha\beta)x^2u^{-4}+(\gamma^2+2\beta\delta+2\alpha\delta+2\beta\gamma)x^4u^{-8}+(\delta^2+2\gamma\delta)x^6u^{-12}};$$

and comparing this with

$$y=\frac{x(A_3+A_2x^2+A_1x^4+A_0x^6)}{A_0+A_1x^2+A_2x^4+A_3x^6},$$

we have for each of the coefficients A two different expressions. Equating these and making a slight change of form, we obtain the relations between u, v, α, β, γ, δ used in the Memoir: thus,

$$A_0=\alpha^2=v^2u^{-14}\delta^2,\ A_1=v^2u^{-10}(\gamma^2+2\beta\delta+2\gamma\delta)=u^{-4}(2\alpha\gamma+\beta^2+2\alpha\beta),\ \&\text{c.};$$

in the Memoir, $k\,(=u^4)$ is used instead of u, and $\Omega\,(=v^2u^{-2})$ instead of v, and the equations thus are

$$k^3\alpha^2=\Omega\delta^2,$$
$$k(2\alpha\gamma+2\alpha\beta+\beta^2)=\Omega(\gamma^2+2\gamma\delta+2\beta\delta),$$
$$\gamma^2+2\beta\gamma+2\alpha\delta+2\beta\delta=\Omega k(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),$$
$$\delta^2+2\gamma\delta=\Omega k^3(\alpha^2+2\alpha\beta);$$

viz. these are the equations [*Coll. Math. Papers*, vol. IX., p. 119]. The idea in the present Memoir is that of considering the coefficients A in the stead of α, β, ...

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14. We have here, and in general for any odd value of n, equations of the form

$$(\Omega =) \frac{U}{U'} = \frac{V}{V'} = \ldots,$$

where $U, V, \ldots, U', V', \ldots$ are quadric functions of the coefficients $\alpha, \beta, \gamma, \ldots$; and these equations serve to establish between Ω and k a relation called the Ωk-modular equation, and which in regard to Ω is of the same degree as the uv-modular equation is in regard to v. Leaving out the equation $(\Omega =)$, we have

$$\begin{vmatrix} U, & V, & W, \ldots \\ U', & V', & W', \ldots \end{vmatrix} = 0;$$

and to each system of values of $\alpha, \beta, \gamma, \delta, \ldots$ (or say of their ratios) given by these equations, there corresponds a single value of Ω; the number of values of Ω, or the degree in Ω, of the Ωk-equation is thus found as $= (n+1)\, 2^{\frac{1}{2}(n-3)}$. This is far too high; for $n = 3, 5, 7, \ldots$, the degrees are 4, 12, 32, ...; those of the proper Ωk-equations are 4, 6, 8, ...

15. I showed, or endeavoured to show, that in the case $n = 5$, the extraneous factor was $(\Omega - 1)^6$, ($\Omega - 1 = 0$, the Ωk-modular equation belonging to $n = 1$, for which the transformation is the trivial one $y = x$), and that in the case $n = 7$, the extraneous factor was $\{(\Omega, 1)^4\}^6$, ($(\Omega, 1)^4 = 0$, the Ωk-modular equation for the case $n = 3$); generally the extraneous factors seem to depend on the Ωk-functions for the values $n-4$, $n-8$, &c. The ground for this is that, in the assumed formula for any given value n, we may take P, Q to contain a common factor $1 \pm k\mathrm{x}^2$ $\Big($observe that, to a factor près, this is unaltered by the change x into $\frac{1}{k\mathrm{x}}$, viz. it becomes $\frac{1}{k\mathrm{x}^2}(1 \pm k\mathrm{x}^2)$, a condition which is necessary$\Big)$, and we thereby reduce the equation to

$$\frac{1-\mathrm{y}}{1+\mathrm{y}} = \frac{1-\mathrm{x}}{1+\mathrm{x}} \left(\frac{P' - Q'\mathrm{x}}{P' + Q'\mathrm{x}}\right)^2,$$

in which equation the degrees of the numerator and the denominator are each diminished by 4, and the equation thus belongs to the value $n - 4$.

16. I remark here that, in the case of n an odd prime, the degree of the modular equation is $= n + 1$; but for any other odd value, the degree is

$$\sigma'(n), = n\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\ldots,$$

where $a, b, \ldots$ are all the unequal prime factors of n; thus, if $n = a^\alpha$, the degree is

$$a^\alpha\left(1 + \frac{1}{a}\right), = a^{\alpha-1}(a+1).$$

In the case of a number $n = abc\ldots$, without any squared factor, the degree is

$$abc\ldots\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right)\ldots, = (a+1)(b+1)(c+1)\ldots,$$

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the sum of the factors of n. We have

$$\sigma'(n) = \text{coeff. } x^n \text{ in } \Sigma\phi(x^N),$$

where

$$\phi x = x + 3x^3 + 5x^5 + \ldots, = \frac{x(1+x^2)}{(1-x^2)^2},$$

and the summation extends to all odd values of N having no squared factor; thus,

$$\begin{array}{lcccccccccccccccc}
\phi(x) & = x & + & 3x^3 & + & 5x^5 & + & 7x^7 & + & 9x^9 & + & 11x^{11} & + & 13x^{13} & + & 15x^{15}\ldots \\
\phi(x^3) & = & & 1x^3 & & & & & + & 3x^9 & & & & & + & 5x^{15}, \\
\phi(x^5) & = & & & & 1x^5 & & & & & & & & & + & 3x^{15}, \\
\phi(x^7) & = & & & & & & 1x^7 & & & & & & & & \\
\phi(x^{11}) & = & & & & & & & & & & 1x^{11} & & & & \\
\phi(x^{13}) & = & & & & & & & & & & & & 1x^{13} & & \\
\phi(x^{15}) & = & & & & & & & & & & & & & & 1x^{15}, \\
\ldots\ldots & & & & & & & & & & & & & & & \\
\ldots\ldots & & & & & & & & & & & & & & & \\
\hline
\Sigma\phi(x^N) & = x & + & 4x^3 & + & 6x^5 & + & 8x^7 & + & 12x^9 & + & 12x^{11} & + & 14x^{13} & + & 24x^{15}\ldots
\end{array}$$

17. Supposing that the reduction is completely accounted for as above, then, to obtain the numerical relations, the numbers 1, 4, 12, 32, ..., $(n+1)\,2^{\frac{1}{2}(n-3)}$ have to be expressed linearly in terms of 1, 4, 6, 8, ..., $\sigma'(n)$, viz. $(n+1)\,2^{\frac{1}{2}(n-3)}$ as a linear function of $\sigma'(n)$, $\sigma'(n-4)$, $\sigma'(n-8)$, ..., and we have

$$\begin{aligned}
1 &= 1, \\
4 &= 4, \\
12 &= 6 + 6\,.\ 1, \\
32 &= 8 + 6\,.\ 4, \\
80 &= 12 + 6\,.\ 6 + 32\,.\ 1, \\
192 &= 12 + 6\,.\ 8 + 33\,.\ 4, \\
448 &= 14 + 6\,.\,12 + 33\,.\ 6 + 164\,.\ 1, \\
1024 &= 24 + 6\,.\,12 + 33\,.\ 8 + 166\,.\ 4, \\
2304 &= 18 + 6\,.\,14 + 33\,.\,12 + 166\,.\ 6 + 810\,.\,1, \\
5120 &= 20 + 6\,.\,24 + 33\,.\,12 + 166\,.\ 8 + 817\,.\,4, \\
11264 &= 32 + 6\,.\,18 + 33\,.\,14 + 166\,.\,12 + 817\,.\,6 + 3768\,.\,1, \\
24576 &= 24 + 6\,.\,20 + 33\,.\,24 + 166\,.\,12 + 817\,.\,8 + 3778\,.\,4, \\
&\ldots\ldots \\
&\ldots\ldots
\end{aligned}$$

but it is of course very doubtful whether these relations have any value in regard to the present theory.

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18. In the same way that, by assuming a common factor, $1+k\text{x}^2$, in the values of P and Q, we pass from the case n to the case $n-4$, so, by assuming a common factor, $1+x^2$, in the numerator and denominator of the expression for y in terms of x and the coefficients B, we pass from the case n to the case $n-2$. Contrariwise, in the solutions given by the Jacobi-Brioschi differential equations and by the Jacobi partial differential equation, the solution for a given value of n does not thus contain the solution for an inferior value of n; see *post* Nos. 36 and 43.

I pass now to the theory before referred to.

The Development $y=\rho x(1+\Pi_1x^2+\Pi_2x^4+\ldots)$. Art. Nos. 19 and 20.

19. Starting from the equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho\, dx}{\sqrt{1-2\alpha x^2+x^4}},$$

and writing for shortness

$$\begin{array}{ll}
R_1=\tfrac{1}{3}\,\alpha, & S_1=\tfrac{1}{3}\,\beta,\\
R_2=\tfrac{1}{5}\,(\tfrac{3}{2}\,\alpha^2-\tfrac{1}{2}), & S_2=\tfrac{1}{5}\,(\tfrac{3}{2}\,\beta^2-\tfrac{1}{2}),\\
R_3=\tfrac{1}{7}\,(\tfrac{5}{2}\,\alpha^3-\tfrac{3}{2}\,\alpha), & S_3=\tfrac{1}{7}\,(\tfrac{5}{2}\,\beta^3-\tfrac{3}{2}\,\beta),\\
R_4=\tfrac{1}{9}\,(\tfrac{35}{8}\,\alpha^4-\tfrac{15}{4}\,\alpha^2+\tfrac{3}{8}), & S_4=\tfrac{1}{9}\,(\tfrac{35}{8}\,\beta^4-\tfrac{15}{4}\,\beta^2+\tfrac{3}{8}),\\
\ldots\ldots\ldots & \ldots\ldots\ldots\\
\ldots\ldots\ldots & \ldots\ldots\ldots
\end{array}$$

(viz. save as to the exterior factors $\frac{1}{3}$, $\frac{1}{5}$, ..., $3R_1$, $5R_2$, ... are the Legendrian functions of α, and $3S_1$, $5S_2$, ... the Legendrian functions of β), we have

$$dy\,(1+3S_1y^2+5S_2y^4+\ldots)=\rho\, dx\,(1+3R_1x^2+5R_2x^4+\ldots),$$

whence integrating, so that y may vanish with x, we have

$$y+S_1y^3+S_2y^5+\ldots=\rho\,(x+R_1x^3+R_2x^5+\ldots),$$

say this is

$$=u.$$

20. We then have $y=u-fy$, where $fy=S_1y^3+S_2y^5+\ldots$; and thence, expanding by Lagrange's theorem,

$$y=u-fu+\frac{1}{2}(f^2u)'-\frac{1}{2\,.\,3}(f^3u)''+\frac{1}{2\,.\,3\,.\,4}(f^4u)'''-\ldots,$$

we have

$$fu\;=S_1u^3+S_2u^5+S_3u^7+S_4u^9+\ldots,$$

and thence

$$\begin{aligned}
f^2u &= S_1^2u^6+2S_1S_2u^8+(2S_1S_3+S_2^2)\,u^{10}+\ldots,\\
f^3u &= S_1^3u^9+3S_1^2S_2u^{11}+\ldots,\\
f^4u &= S_1^4u^{12}+\ldots;
\end{aligned}$$

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consequently,

$$\begin{aligned} y = \; & u, \\ & + u^3 (-S_1), \\ & + u^5 (-S_2 + 3S_1^2), \\ & + u^7 (-S_3 + 8S_1 S_2 - 12 S_1^3), \\ & + u^9 (-S_4 + 10 S_1 S_3 + 5 S_2^2 - 55 S_1^2 S_2 + 55 S_1^4) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and writing herein

$$\begin{aligned} u &= \rho \; \{x + R_1 x^3 + R_2 x^5 + R_3 x^7 + R_4 x^9 + \ldots\}, \\ u^3 &= \rho^3 \{x^3 + 3R_1 x^5 + (3R_2 + 3R_1^2) x^7 + (3R_3 + 6R_1 R_2 + R_1^3) x^9 + \ldots\}, \\ u^5 &= \rho^5 \{x^5 + 5R_1 x^7 + (5R_2 + 10 R_1^2) x^9 + \ldots\}, \\ u^7 &= \rho^7 \{x^7 + 7R_1 x^9 + \ldots\}, \\ u^9 &= \rho^9 \{x^9 + \ldots\}, \end{aligned}$$

we have the required series

$$y = \rho x \{1 + \Pi_1 x^2 + \Pi_2 x^4 + \Pi_3 x^6 + \ldots\},$$

where the values of the coefficients are

$$\begin{aligned} \Pi_1 &= R_1 + (-S_1) \rho^2, \\ \Pi_2 &= R_2 + (-S_1)\, 3R_1 \rho^2 + (-S_2 + 3S_1^2) \rho^4, \\ \Pi_3 &= R_3 + (-S_1)(3R_2 + 3R_1^2) \rho^2 + (-S_2 + 3S_1^2)\, 5R_1 \rho^4 + (-S_3 + 8S_1 S_2 - 12 S_1^2) \rho^6, \\ \Pi_4 &= R_4 + (-S_1)(3R_3 + 6R_1 R_2 + R_1^3) \rho^2 + (-S_2 + 3S_1^2)(5R_2 + 10R_1^2) \rho^4 \\ & \quad + (-S_3 + 8S_1 S_2 - 12 S_1^3)\, 7R_1 \rho^6 + (-S_4 + 10 S_1 S_3 + 5S_2^2 - 55 S_1^2 S_2 + 55 S_1^4) \rho^8, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and so on, as far as we please.

The Cubic Transformation, $n = 3$. Art. Nos. 21 to 28.

21. We have here

$$\frac{\rho + x^2}{1 + \rho x^2} = \rho (1 + \Pi_1 x^2 + \Pi_2 x^4 + \ldots);$$

whence, developing the left-hand side and equating coefficients,

$$\rho \Pi_1 = -\rho^2 + 1, \quad \rho \Pi_2 = \rho^3 - \rho, \quad \rho \Pi_3 = -\rho^4 + \rho^2, \ldots$$

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It will be convenient to write

$$\begin{aligned}
\Theta_1 &= \rho\Pi_1 + \rho^2 - 1, && = -S_1\rho^3 + \rho^2 + R_1\rho - 1,\\
\Theta_2 &= \Pi_2 - \rho^2 + 1, && = (-S_2 + 3S_1^2)\rho^4 - (3R_1S_1 + 1)\rho^2 + R_2 + 1,\\
\Theta_3 &= \Pi_3 + \rho^3 - \rho, && = (-S_3 + 8S_1S_2 - 12S_1^3)\rho^6\\
& && \quad + (-5R_1S_2 + 15R_1S_1^2)\rho^4\\
& && \quad + \rho^3\\
& && \quad + (-3R_2S_1 - 3R_1^2S_1)\rho^2\\
& && \quad - \rho\\
& && \quad + R_3,
\end{aligned}$$

....

....

where observe the difference of form in the function Θ_1, and in the subsequent functions Θ_2, Θ_3, ... In these last, a factor ρ is thrown out.

22. The two equations $\Theta_1 = 0$ and $\Theta_2 = 0$ serve to determine ρ, β in terms of α; the subsequent equations $\Theta_3 = 0$, $\Theta_4 = 0$, ... will then be, all of them, satisfied identically. This implies that Θ_3, Θ_4, ... are each of them a linear function of Θ_1, Θ_2. The *à posteriori* verification and determination of the factors is by no means easy; I have effected it only for Θ_3; we have

$$7\Theta_3 = (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)\,\Theta_1 + (-S_1\rho^2 - 10\rho + 25R_1)\,\Theta_2,$$

or, at full length,

$$7\left|\begin{array}{r} (-S_3 + 8S_1S_2 - 12S_1^3)\rho^6\\ + (-5R_1S_2 + 15R_1S_1^2)\rho^4\\ +\rho^3\\ + (-3R_2S_1 - 3R_1^2S_1)\rho^2\\ -\rho\\ +R_3 \end{array}\right|$$

$$\begin{aligned}
&= (\rho^3 - 3S_1\rho^2 - 2\rho + 27R_1)(-S_1\rho^3 + \rho^2 + R_1\rho - 1)\\
&\quad + (-S_1\rho^2 - 10\rho + 25R_1)((-S_2 + 3S_1^2)\rho^4 - (3R_1S_1 + 1)\rho^2 + R_2 + 1);
\end{aligned}$$

in verifying which we must, of course, take account of the relations between the expressions R and those between the expressions S; we have

$$\alpha = 3R_1 \text{ and thence } 10R_2 = 27R_1^2 - 1, \quad 14R_3 = 135R_1^3 - 9R_1;$$

similarly,

$$10S_2 = 27S_1^2 - 1, \quad 14S_3 = 135S_1^3 - 9S_1.$$

Equating the coefficients of ρ^6, we have

$$-7S_3 + 56S_1S_2 - 84S_1^3 = -S_1 + S_1S_2 - 3S_1^2;$$

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viz. multiplying by 2, this is

$$-14S_3+110S_1S_2-162S_1^3+2S_1=0;$$

or, finally, it is

$$(-135S_1^3+9S_1)+(297S_1^3-11S_1)-162S_1^3+2S_1=0,$$

an identity, as it should be. The identity of the coefficients of ρ^5, ρ^4, ρ^3, ρ^2, ρ, 1 may be verified in like manner.

23. Considering α as known, the values of ρ and β are determined by the foregoing equations $\Theta_1=0$, $\Theta_2=0$; that is,

$$-S_1\rho^3+\rho^2+R_1\rho-1=0,$$

$$(-S_2+3S_1^2)\rho^4-(3R_1S_1+1)\rho^2+R_2+1=0,$$

where, of course, the R's and S's are regarded as given functions of α and β respectively.

It is to be observed that the equations are satisfied by $\rho^2=1$, $\alpha=\beta$; viz. we have the transformation $y=\dfrac{x(\pm 1+x^2)}{1\pm x^2}$; that is, $y=\pm x$, which is the transformation of the first order, $n=1$. The two equations represent surfaces of the orders 4 and 6 respectively, and they have thus a complete intersection of the order 24. As part of this, we have, as just shown, each of the two lines $(\rho=1,\ \alpha=\beta)$ and $(\rho=-1,\ \alpha=\beta)$; but there is a more considerable reduction of order to be accounted for, the proper MM-curve being, as will appear, a unicursal curve of the order $=6$.

24. Multiplying the second equation by $10\rho^2$, and for $10R_2$ and $10S_2$ writing their values $27R_1^2-1$ and $27S_1^2-1$ respectively, we have

$$(3S_1^2+1)\rho^6-(30R_1S_1+10)\rho^4+(27R_1^2+9)\rho^2=0;$$

and if herein we substitute for $S_1\rho^3$ its value from the first equation, $=\rho^2+R_1\rho-1$, we have

$$3(\rho^2+R_1\rho-1)^2+\rho^6-30R_1\rho(\rho^2+R_1\rho-1)-10\rho^4+(27R_1^2+9)\rho^2=0;$$

that is,

$$\rho^6-7\rho^4-24R_1\rho^3+3\rho^2+24R_1\rho+3=0;$$

viz. this is

$$(\rho^2-1)(\rho^4-6\rho^2-24R_1\rho-3)=0,$$

containing, as it ought to do, the factor ρ^2-1. Throwing this out, and repeating the first equation, we have

$$-S_1\rho^3+\ \rho^2+\ \ R_1\rho-1=0,$$

$$\rho^4\qquad -6\rho^2-24R_1\rho-3=0,$$

which two equations may be replaced by

$$\rho^4-24S_1\rho^3+18\rho^2\qquad\qquad -27=0,$$

$$\rho^4\qquad\qquad -\ 6\rho^2-24R_1\rho-\ \ 3=0,$$

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which are the $\rho\beta$- and $\rho\alpha$-equations respectively. Recollecting that R_1 and S_1 denote $\frac{1}{3}\alpha$ and $\frac{1}{3}\beta$, they agree with the results obtained in No. 7. The $\alpha\beta$-modular equation is obtained by the elimination of ρ from these two equations, and may be at once written down in the form, Det. $=0$, where the determinant is of the order 8, but contains S_1 and R_1, that is, β and α, each of them, in the fourth order only: the form is thus the same with that of the $\alpha\beta$-equation obtained in No. 2; but the identification would be a work of some labour.

25. The equations may be written

$$24S_1\rho^3 = \rho^4 + 18\rho^2 - 27,$$

$$24R_1\rho = \rho^4 - \ \ 6\rho^2 - \ \ 3,$$

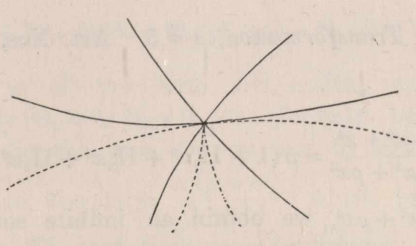
and, treating R_1, S_1, ρ as coordinates, it hence appears that the MM-curve is (as mentioned above) a unicursal curve of the order 6; in fact, we have R_1, S_1, each of them given as a rational function of ρ; and cutting the curve by an arbitrary plane $AR_1 + BS_1 + C\rho + D = 0$, the substitution of the values of R_1, S_1 in this equation gives for ρ an equation of the order 6.

26. The same conclusion may be obtained from the foregoing system of a cubic and a quartic equation in ρ. Considering R_1, S_1, ρ as coordinates, they represent, each of them, a surface of the order 4, and the complete intersection is of the order 16; but this is made up of a line in the plane infinity counting 10 times, and of the MM-curve, which is thus of the order $16 - 10, = 6$. In fact, introducing, for homogeneity, a fourth coordinate θ, the two equations are

$$-S_1\rho^3 + \ \rho^2\theta^2 + \ \ \ R_1\rho\theta^2 - \ \theta^4 = 0,$$

$$\rho^4 \qquad -6\rho^2\theta^2 - 24R_1\rho\theta^2 - 3\theta^4 = 0,$$

and the line $\rho = 0$, $\theta = 0$ is thus a triple line on each of these surfaces; viz. cutting them by an arbitrary plane, we have for the first surface an ordinary triple point, as shown by the continuous lines of the annexed figure, and for the second surface a triple point = cusp + two nodes, as shown by the dotted lines of the figure. There is, moreover, as shown in the figure, a contact of two branches, and the number of intersections is thus $=10$.



27. If we assume $\rho\sigma = -3$, that is, $\rho = -\dfrac{3}{\sigma}$, and substitute this value in the equation for S_1, the two equations become

$$24S_1\sigma = \sigma^4 - 6\sigma^2 - 3,$$

$$24R_1\rho = \rho^4 - 6\rho^2 - 3;$$

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viz. β is the same function of $\sigma\left(=-\frac{3}{\rho}\right)$ that α is of ρ. This accords with the theorem in Elliptic Functions that a combination of two transformations leads to a multiplication.

28. We have

$$24(R_1+\tfrac{1}{3})\rho=\rho^4-6\rho^2+8\rho-3,\ =(\rho-1)^3(\rho+3),$$

or, what is the same thing,

$$24(R_1+\tfrac{1}{3})\ \ =(\rho-1)^2(\rho+\sigma+2);$$

and, in like manner,

$$24(R_1-\tfrac{1}{3})\rho=\rho^4-6\rho^2-8\rho-3,\ =(\rho+1)^3(\rho-3),$$

and, consequently,

$$24(R_1-\tfrac{1}{3})\ \ =(\rho+1)^2.(\rho+\sigma-2);$$

with the like equations between S_1, σ, ρ. It will be recollected that

$$R_1=\tfrac{1}{3}\alpha,\ =\tfrac{1}{6}\left(u^4+\frac{1}{u^4}\right);$$

hence

$$24(R_1\pm\tfrac{1}{3})=4\left(u^4+\frac{1}{u^4}\pm 2\right),\ =4\left(u^2\pm\frac{1}{u^2}\right)^2.$$

The formulæ just obtained are useful for obtaining the uv-modular equation from the foregoing equations; or say

$$4\left(v^4+\frac{1}{v^4}\right)\sigma=\sigma^4-6\sigma^2-3,$$

$$4\left(u^4+\frac{1}{u^4}\right)\rho=\rho^4-6\rho^2-3,$$

where $\rho\sigma=-3$, and we have to eliminate ρ and σ; the elimination gives

$$\frac{v^2}{u^2}-\frac{u^2}{v^2}+2vu-\frac{2}{uv}=0,$$

that is,

$$v^4+2v^3u^3-2vu-u^4=0.$$

The Quintic Transformation, $n=5$. Art. Nos. 29 to 32.

29. We have here

$$\frac{\rho+A_1x^2+x^4}{1+A_1x^2+\rho x^4}=\rho(1+\Pi_1x^2+\Pi_2x^4+\Pi_3x^6+\ldots),$$

and multiplying by $1+A_1x^2+\rho x^4$, we obtain an infinite series of equations, the first three of which are

$$A_1=\rho\Pi_1+A_1\rho,$$
$$1=\rho\Pi_2+A_1\rho\Pi_1,$$
$$0=\rho\Pi_3+A_1\rho\Pi_2+\rho^2\Pi_1,$$

....

....

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The first of these gives

$$A_1 = \frac{\rho\Pi_1}{-\rho+1}, \ = \frac{\Theta_1 - \rho^2 + 1}{-\rho+1};$$

and the other two equations then determine the MM-curve. These being satisfied, the remaining equations will be satisfied identically. It is proper to introduce Θ_1, Θ_2, Θ_3 into the equations instead of Π_1, Π_2, Π_3. We have first

$$1 = \rho\Pi_2 + \rho\Pi_1 \frac{\Theta_1 - \rho^2 + 1}{-\rho+1} + \rho^2,$$

that is,

$$0 = \rho(\Theta_2 + \rho^2 - 1) - \frac{(\Theta_1 - \rho^2 + 1)^2}{\rho - 1} + \rho^2 - 1;$$

viz. this is

$$\rho(\rho-1)(\Theta_2 + \rho^2 - 1) - (\Theta_1 - \rho^2 + 1)^2 + (\rho^2 - 1)(\rho - 1) = 0;$$

or, finally, this is

$$\rho(\rho-1)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0.$$

Secondly, we have

$$0 = \Pi_3 + \frac{\rho\Pi_1\Pi_2}{-\rho+1} + \rho\Pi_1 = 0;$$

that is,

$$\Theta_3 - \rho^3 + \rho + \frac{(\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1)}{-\rho+1} + \Theta_1 - \rho^2 + 1 = 0;$$

viz. this is

$$(\Theta_3 + \Theta_1 - \rho^3 - \rho^2 + \rho + 1)(-\rho+1) + (\Theta_1 - \rho^2 + 1)(\Theta_2 + \rho^2 - 1) = 0;$$

or, finally, it is

$$\Theta_3(-\rho+1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0.$$

30. We have thus the two equations

$$(\rho^2 - \rho)\Theta_2 - \Theta_1^2 + 2\Theta_1(\rho^2 - 1) = 0,$$

$$\Theta_3(-\rho+1) + \Theta_1\Theta_2 + \Theta_1(\rho^2 - \rho) - \Theta_2(\rho^2 - 1) = 0;$$

and recollecting that Θ_3 is of the form $L\Theta_1 + M\Theta_2$, we see that each of these equations is satisfied if only $\Theta_1 = 0$, $\Theta_2 = 0$ (the formulæ belonging to the cubic transformation). This ought to be the case, for we can, by writing $A_1 = \rho + 1$, reduce the expression $\frac{x(\rho + A_1x^2 + x^4)}{1 + A_1x^2 + \rho x^4}$ to the form $\frac{x(\rho + x^2)}{1 + \rho x^2}$, which belongs to the cubic transformation (see *ante* No. 17). The equations may be written

$$\rho\Theta_2 = -(2\rho + 2)\Theta_1 + \frac{\Theta_1^2}{\rho - 1},$$

$$\rho\Theta_3 = \quad (3\rho^2 + 4\rho + 2)\Theta_1 - (3\rho + 3)\frac{\Theta_1^2}{\rho - 1} + \frac{\Theta_1^3}{(\rho-1)^2}.$$

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31. The investigation may be presented in a slightly different form by introducing the functions Θ at an earlier stage; viz. writing

$$\rho\Pi_1 = \Theta_1 - \rho^2 + 1,\ \rho\Pi_2 = \rho\Theta_2 + \rho^3 - \rho, \ldots,$$

we have

$$\frac{\rho + A_1x^2 + x^4}{1 + A_1x^2 + \rho x^4} = \rho + (\Theta_1 - \rho^2 + 1)\,x^2 + (\rho\Theta_2 + \rho^3 - \rho)\,x^4 + \ldots$$

$$= \frac{\rho + x^2}{1 + \rho x^2} + \Theta_1 x^2 + \rho\Theta_2 x^4 + \rho\Theta_3 x^6 + \ldots$$

Transposing, reducing, and dividing by x^2, we have

$$\frac{(1 - x^2)\,[\rho^2 - 1 + A_1(-\rho + 1)]}{(1 + \rho x^2)(1 + A_1x^2 + \rho x^4)} = \Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots,$$

whence clearly $\rho^2 - 1 + A_1(-\rho + 1) = \Theta_1$, giving for A_1 the before-mentioned value; and we then have

$$1 + A_1x^2 + \rho x^4 = 1 + (\rho + 1)\,x^2 + \rho x^4 - \frac{\Theta_1 x^2}{\rho - 1},\ = (1 + x^2)(1 + \rho x^2) - \frac{\Theta_1 x^2}{\rho - 1}.$$

The equation thus becomes

$$\frac{(1 - x^2)\,\Theta_1}{(1 + x^2)(1 + \rho x^2)^2\left(1 - \dfrac{\Theta_1 x^2}{\rho - 1\,.\,1 + x^2\,.\,1 + \rho x^2}\right)} = \Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots,$$

and expanding the left-hand side, first in the form

$$\frac{(1 - x^2)\,\Theta_1}{(1 + x^2)(1 + \rho x^2)^2} + \frac{(1 - x^2)\,x^2\Theta_1^2}{(\rho - 1)(1 + x^2)^2(1 + \rho x^2)^3} + \frac{(1 - x^2)\,x^4\Theta_1^3}{(\rho - 1)(1 + x^2)^3(1 + \rho x^3)^4} + \ldots$$

and then each of these terms separately in powers of x^2, and comparing with $\Theta_1 + \rho\Theta_2 x^2 + \rho\Theta_3 x^4 + \ldots$, we have the two equations in the last-mentioned form, and an infinite series of other equations, which will be satisfied identically.

32. The successive coefficients might be called $\Phi_2, \Phi_3, \ldots$; say

$$\Phi_2 = (\rho^2 - \rho)\,\Theta_2 - \Theta_1^2 + 2\,(\rho^2 - 1)\,\Theta_1,$$

$$\Phi_3 = (-\rho + 1)\,\Theta_3 + \Theta_1\Theta_2 + (\rho^2 - \rho)\,\Theta_1 - (\rho^2 - 1)\,\Theta_2,$$

and similarly for $\Phi_4, \ldots$; and it would then be proper to show *à posteriori* that each of the equations $\Phi_4 = 0$, $\Phi_5 = 0, \ldots$ is satisfied identically in virtue of the two equations $\Phi_2 = 0$, $\Phi_3 = 0$, or, what is the same thing, that the functions $\Phi_4, \Phi_5, \ldots$ are each of them a linear function (with coefficients which are functions of ρ) of the two functions Φ_2 and Φ_3. I do not attempt to do this, nor even to discuss the MM-curve by means of the equations $\Phi_2 = 0$, $\Phi_3 = 0$; but I will obtain equivalent results, and complete the solution by means of the Jacobi-Brioschi equations, in effect reproducing the investigation contained in the third appendix of the *Funzioni ellittiche*.

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The General Transformation, $n=2s+1$. Art. No. 33.

33. The equation here is

$$\frac{\rho + A_{s-1}x^2 + \dots}{1 + A_1 x^2 + \dots} = \rho\,(1 + \Pi_1 x^2 + \dots).$$

The general theory is sufficiently illustrated by the preceding particular cases, and I wish at present only to notice the equation obtained by comparing the coefficients of x^2; viz. this is $A_{s-1} - \rho A_1 = \rho \Pi_1$, or, substituting for Π_1 its value,

$$A_{s-1} - \rho A_1 = \tfrac{1}{3}(\alpha\rho - \beta\rho^3).$$

The Jacobi-Brioschi Equations. Art. Nos. 34 to 42.

34. These were obtained for the differential equation

$$\frac{dx}{\sqrt{a'x^4 + b'x^3 + c'x^2 + d'x + e'}} = \frac{dy}{\sqrt{ay^4 + by^3 + cy^2 + dy + e}};$$

viz. if this be satisfied by $y = U \div V$, where U, V are rational and integral functions of x of the degrees n and $n-1$ respectively, then, writing for shortness

$$\phi = a'x^4 + b'x^3 + c'x^2 + d'x + e',$$

and using accents to denote differentiation in regard to x, the numerator and denominator U, V satisfy the equations

$$\begin{aligned} (VV'' - V'^2)\,\phi + \tfrac{1}{2}\,VV'.\phi' \quad &+ aU^2 + \quad \tfrac{1}{2}bUV + pV^2 = 0,\\ -(VU'' + V''U - 2V'U')\,\phi - \tfrac{1}{2}(VU' + V'U)\,\phi &+ \tfrac{1}{2}\,bU^2 + (c-2p)\,UV + \tfrac{1}{2}dV^2 = 0,\\ (UU'' - U'^2)\,\phi + \tfrac{1}{2}\,UU'.\phi' \quad &+ pU^2 + \quad \tfrac{1}{2}dUV + eV^2 = 0, \end{aligned}$$

where p is a function $= \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, with coefficients a, b, c, the values of which have to be determined. By way of verification, observe that, multiplying by U^2, UV, V^2, and adding, we obtain

$$-(VU' - V'U)^2\,\phi + aU^4 + bU^3V + cU^2V^2 + dUV^3 + eV^4 = 0;$$

that is,

$$-\frac{1}{V^2}(VU' - V'U)^2\,(a'x^4 + b'x^3 + c'x^2 + d'x + e') + ay^4 + by^3 + cy^2 + dy + e = 0,$$

the result obtained by substituting for y its value, $= U \div V$, in the differential equation.

35. Considering the foregoing special form

$$\frac{dx}{\sqrt{1 - 2\alpha x^2 + x^4}} = \frac{dy}{\rho\sqrt{1 - 2\beta y^2 + y^4}},$$

so that a, b, c, d, e have the values ρ^2, 0, $-2\beta\rho^2$, 0, ρ^2 and ϕ is $= 1 - 2\alpha x^2 + x^4$, the equations are

$$\begin{aligned} (VV'' - V'^2)\,\phi + \quad &\tfrac{1}{2}\,VV'.\phi' + \rho^2U^2 + \quad pV^2 = 0,\\ -(VU'' + V''U - 2V'U')\,\phi + &\tfrac{1}{2}(VU' + V'U)\,\phi' - (2\beta\rho^2 + 2p)\,UV \quad = 0,\\ (UU'' - U'^2)\,\phi + \quad &\tfrac{1}{2}\,UU'.\phi' + pU^2 + \quad \rho^2V^2 = 0, \end{aligned}$$

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where, writing as before, $n = 2s+1$, and assuming that the last coefficient, $A_{\frac{1}{2}(n-1)}$ or A_s, is $=\rho$, we have

$$U = x(\rho + A_{s-1}x^2 + A_{s-2}x^4 + \ldots + A_1 x^{2s-2} + x^{2s}),$$
$$V = 1 + A_1 x^2 + A_2 x^4 + \ldots + A_{s-1} x^{2s-2} + \rho x^{2s},$$

and where, as is easily shown, p has the value $= -\{2A_1 + (2s+1)x^2\}$. In comparing with Brioschi, it will be recollected that 2α, 2β are written in place of his α, β.

36. The equations contain n, and they are not satisfied by values of U, V belonging to any inferior value of n; U, V may each of them be multiplied by any common constant factor at pleasure, but not by a common variable factor P; viz. it is assumed that the fraction $U \div V$ is in its least terms, and consequently that (save as to a constant factor) U, V are determinate functions. It is easy to verify that the equations (being verified by U, V) are not verified by PU, PV, but it is interesting to show *à priori* why this is so. The equations are obtained as follows. Consider the differential equation in the form $\frac{dx}{\sqrt{X}} = \frac{dy}{\sqrt{Y}}$, and suppose that an integral equation is given in the form $F=0$ (F a rational and integral function of x, y); we thence deduce a relation $Ldx + Mdy = 0$ between the differentials, and this must agree with the given differential equation; that is, we have $L\sqrt{X} + M\sqrt{Y} = 0$, or, rationalizing, $L^2X - M^2Y = 0$; viz. this last equation must agree with the equation $F=0$, or, what is the same thing, $L^2X - M^2Y$ must contain F as a factor; say we have

$$L^2X - M^2Y = F \,.\, G,$$

where G is a function of x, y. In the particular case where the integral is of the form

$$y = U \div V,$$

we have

$$F = Vy - U,$$

and we have therefore

$$L^2X - M^2Y = G\,(Vy - U);$$

and it is by means of this identity that the equations are obtained. But suppose that there is a common factor P, or that we have $y = PU \div PV$; then, if we write $F = PVy - PVU$, $= P\,(Vy - U)$, there is no necessity that $L^2X - M^2Y$ should contain as a factor this expression of F, and it will not in fact contain it; all that is necessary is that $L^2X - M^2Y$ shall contain the factor $Vy - U$; and thus the equations obtained for U, V do not apply to PU, PV. We might, of course, introduce an arbitrary *constant* factor Θ; contrast herewith the solution by means of the Jacobi partial differential equation, *post* No. 43, where Θ is not arbitrary but has a determinate value.

37. In virtue of the assumed forms of U, V, the first and the third equations give each of them the same relations between the coefficients A; and only one of these equations, say the first, need be attended to. It will be observed that this equation

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does not contain β; it consequently serves to determine the coefficients A in terms of ρ, α, and to establish a relation between ρ, α, that is, the multiplier equation. We can from this, as will be explained, deduce the equation between ρ, β; the theory thus depends entirely upon the first equation; say this is

$$(VV'' - V'^2)(1 - 2\alpha x^2 + x^4) + VV'(-2\alpha x + 2x^3) + \rho^2 U^2 - \{2A_1 + (2s+1)x^2\} V^2 = 0.$$

38. We have $V = 1 + A_1 x^2 + A_2 x^4 + \ldots$, but the equation contains the quadric functions $VV'' - V'^2$, VV', and V^2; it is convenient to write

$$VV'' - V'^2 = K_1 + K_2 x^2 + K_3 x^4 + \ldots,$$
$$V^2 = L_0 + L_1 x^2 + L_2 x^4 + \ldots;$$

whence of course

$$VV' = 2L_1 x + 4L_2 x^3 + \ldots,$$

and we have

$K_1=$	$K_2=$	$K_3=$	$K_4=$	$K_5=$	$K_6=$	$K_7=$	$K_8=$	. . .
$2A_1$	$12A_2$	$30A_3$	$56A_4$	$90A_5$	$132A_6$	$182A_7$	$240A_8$	
	$-2A_1^2$	$-2A_1A_2$	$+8A_1A_3$	$+26A_1A_4$	$+52A_1A_5$	$+86A_1A_6$	$+128A_1A_7$	,
			$-\ 4A_2^2$	$-\ 6A_2A_3$	$+\ 4A_2A_4$	$+22A_2A_5$	$+\ 48A_2A_6$	
					$-\ 6A_3^2$	$-10A_3A_4$	$0A_3A_5$	
							$-\ 8A_4^2$	

$L_0=$	$L_1=$	$L_2=$	$L_3=$	$L_4=$	. . .
1	$2A_1$	$2A_2$	$2A_3$	$2A_4$	
		$+\ A_2^2$	$+2A_1A_2$	$+2A_1A_3$	.
				$+\ A_2^2$	

The coefficients of U^2 are at once obtained; say we have $U^2 = \Lambda_0 x^2 + \Lambda_1 x^4 + \Lambda_2 x^6 + \ldots$,

$\Lambda_0=$	$\Lambda_1=$	$\Lambda_2=$	$\Lambda_3=$	$\Lambda_4=$	. . .
ρ^2	$2\rho A_{s-1}$	$2\rho A_{s-2}$	$2\rho A_{s-3}$	$2\rho A_{s-4}$	
		$+A^2_{s-1}$	$+2A_{s-1}A_{s-2}$	$+2A_{s-1}A_{s-3}$	.
				$+\ A^2_{s-2}$	

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Substituting in the equation and equating to zero the coefficients of the several powers of x^2, we find

$$\begin{aligned}
&K_1 \qquad -2A_1L_0 &&= 0,\\
&K_2 \qquad -2A_1L_1 + (-2s-1)L_0 - 2\alpha(K_1 + \ L_1) + \rho^2\Lambda_0 &&= 0,\\
&K_3 + K_1 - 2A_1L_2 + (-2s+1)L_1 - 2\alpha(K_2 + 2L_2) + \rho^2\Lambda_1 &&= 0,\\
&K_4 + K_2 - 2A_1L_3 + (-2s+3)L_2 - 2\alpha(K_3 + 3L_3) + \rho^2\Lambda_2 &&= 0,\\
&K_5 + K_3 - 2A_1L_4 + (-2s+5)L_3 - 2\alpha(K_4 + 4L_4) + \rho^2\Lambda_3 &&= 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

The number of equations is $=2(s+1)$, for the equation contains terms in $x^0, x^2, x^4, \dots, x^{4s+2}$; but the first equation, and also the last and last but one equations, are in fact identities; there remain thus $2(s+1)-3, =2s-1$ equations; but these are equivalent to s independent equations, serving to determine the $s-1$ coefficients $A_1, A_2, \dots, A_{s-1}$, and to determine the relation between ρ and α. In writing down the equations for a determinate value of s, the coefficients A_0, A_s must be taken to be $=0$ and ρ respectively; and coefficients with a negative suffix or a suffix greater than s, must be taken to be each $=0$.

39. Thus, $(n=3)$ $s=1$, we have the $2(s+1), =4$ equations:

$$\begin{aligned}
&2\rho \qquad -2\rho\,.\,1 &&= 0,\\
&-2\rho^2 \qquad -2\rho\,.\,2\rho + (-3)\,1 \ -2\alpha(\ \ 2\rho + 2\rho) \ + \rho^2\,.\,\rho^2 &&= 0,\\
&0 + 2\rho - 2\rho\,.\,\rho^2 + (-1)\,2\rho - 2\alpha(-2\rho^2 + 2\rho^2) + \rho^2\,.\,2\rho &&= 0,\\
&0 - 2\rho^2 - 2\rho\,.\,0 \ + (+1)\ \rho^2 - 2\alpha(\ \ 0 + 3\,.\,0) + \rho^2\,.\,1 &&= 0,
\end{aligned}$$

where the first, third and fourth equations are each of them an identity; the second equation is $-2\rho^2 - 4\rho^2 - 3 - 8\alpha\rho + \rho^4 = 0$; viz. in accordance with what precedes, writing $\alpha = 3R_1$, this is the foregoing equation

$$\rho^4 - 6\rho^2 - 24R_1\rho - 3 = 0.$$

To complete the solution, we use the theorem in elliptic functions referred to *ante* (No. 8); viz. writing $\rho\sigma = -3$, then we have β the same function of σ that α is of ρ; whence

$$\sigma^4 - 6\sigma^2 - 24S_1\sigma - 3 = 0,$$

and we thus have two equations giving the MM-curve.

40. In the case, $(n=5)$ $s=2$, we have the $2(s+1), =6$ equations:

$$\begin{aligned}
&2A_1 \qquad -2A_1\,.\,1 &&= 0,\\
&12\rho - 2A_1^2 - 2A_1(2A_1) \qquad -5\,.\,1 \qquad -2\alpha\,.\,2A_1 \qquad + \rho^2\,.\,\rho^2 &&= 0,\\
&-2\rho A_1 + 2A_1 - 2A_1(2\rho + A_1^2) \quad -3\,.\,2A_1 \quad -2\alpha\{12\rho - 2A_1^2 + 2(2\rho + A_1^2)\} + \rho^2\,.\,2\rho A_1 &&= 0,\\
&-4\rho^2 + 12\rho \quad -2A_1^2 - 2A_1\,.\,2A_1\rho - 1(2\rho + A_1^2) - 2\alpha\{-2\rho A_1 + 3\,.\,2A_1\rho\} + \rho^2(2\rho + A_1^2) &&= 0,\\
&0 \quad -2\rho A_1 \quad -2A_1\,.\,\rho^2 \qquad +1\,.\,2A_1\rho \qquad -2\alpha\{-4\rho^2 \ + 4\,.\,\rho^2\} \qquad + \rho^2\,.\,2A_1 &&= 0,\\
&0 \quad -4\rho^2 \quad -2A_1\,.\,0 \qquad +3\,.\,\rho^2 \qquad -2\alpha\{\ 0 \ +5\,.\,0\} \qquad + \rho^2\,.\,1 &&= 0,
\end{aligned}$$

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where the first, fifth and sixth equations are each of them an identity. The remaining equations are

$$(\rho^2-2\rho+5)(\rho^2+2\rho-1)-6A_1^2-8A_1\alpha=0,$$
$$2\rho^3A_1-6\rho A_1-32\rho\alpha-2A_1^3-4A_1=0,$$
$$2\rho(\rho^2-2\rho+5)+10\rho-4A_1^2\rho-8\alpha A_1\rho+3A_1^2=0.$$

41. Writing the first and third of these in the forms

$$\begin{aligned}-6A_1^2 \qquad\qquad\qquad &-8A_1\alpha \;+(\rho^2-2\rho+5)(\rho^2+2\rho-1)=0,\\ A_1^2(\rho^2-4\rho+3)&-8A_1\alpha\rho+(\rho^2-2\rho+5)\,2\rho \qquad\qquad\quad =0,\end{aligned}$$

they determine A_1^2, $8A_1\alpha$ in terms of ρ; viz. we find

$$A_1^2=(\rho^2-2\rho+5)\ \rho,$$
$$8A_1\alpha=(\rho^2-2\rho+5)(\rho^2-4\rho-1);$$

and then, writing the second equation in the form

$$(\rho^3-3\rho-2)A_1^2-16\rho\alpha A_1-A_1^4=0,$$

and substituting these values of A_1^2 and $8A_1\alpha$, and omitting the factor $\rho^2-2\rho-5$, we have the identity

$$\rho(\rho^3-3\rho-2)-2\rho(\rho^2-4\rho-1)-\rho^2(\rho^2-2\rho+5)=0;$$

viz. the second equation is then also satisfied.

Forming the square of $8A_1\alpha$, and for A_1^2 substituting its value, then omitting a factor $\rho^2-2\rho+5$, we find

$$64\rho\alpha^2=(\rho^2-2\rho+5)(\rho^2-4\rho-1)^2,$$
$$=\rho^6-10\rho^5+35\rho^4-60\rho^3+55\rho^2+38\rho+5;$$

or, as this may also be written,

$$64\rho\ (\alpha^2-1)=(\rho-1)^5(\rho-5),$$

and we then have also, as before,

$$64\sigma(\beta^2-1)=(\sigma-1)^5(\sigma-5),$$

which two equations determine the MM-curve.

The coefficient A_1 is given by the foregoing equation for $8A_1\alpha$, say the value is

$$A_1=\frac{1}{8\alpha}(\rho^2-2\rho+5)(\rho^2-4\rho-1).$$

The value $A_1=\dfrac{\rho\Pi_1}{-\rho+1}$, obtained in No. 29, on substituting for Π_1 its value, is

$$A_1=\frac{\frac{1}{3}(\beta\rho^3-\alpha\rho)}{\rho-1},$$

and these two values are, in fact, equivalent in virtue of the value of β obtained in No. 9.

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42. I consider the case $n=7$, in order to show the form of the equations which have to be solved; these equations are

$$2A_1 - 2A_1 . 1 \qquad = 0,$$

$$12A_2 - 2A_1^2 - 2A_1 . 2A_1 - 7 . 1 - 2\alpha(2A_1 + 1 . 2A_1) + \rho^2 . \rho^2 \qquad = 0,$$

$$30\rho - 2A_1A_2 + 2A_1 - 2A_1(2A_2 + A_1^2) - 5 . 2A_1$$
$$- 2\alpha(12A_2 - 2A_1^2 + 2(2A_2 + A_1^2)) + \rho^2 . 2\rho A_2 = 0,$$

$$8A_1\rho - 4A_2^2 + 12A_2 - 2A_1^2 - 2A_1(2\rho + 2A_1A_2) - 3(2A_2 + A_1^2)$$
$$- 2\alpha(30\rho - 2A_1A_2 + 3(2\rho + 2A_1A_2)) + \rho^2(2\rho A_1 + A_2^2) = 0,$$

$$- 6A_2\rho + 30\rho - 2A_1A_2 - 2A_1(2A_1\rho + A_2^2) - 1(2\rho + 2A_1A_2)$$
$$- 2\alpha(8A_1\rho - 4A_2^2 + 4(2A_1\rho + A_2^2)) + \rho^2(2\rho + 2A_1A_2) = 0,$$

$$- 6\rho^2 \quad + 8A_1\rho - 4A_2^2 - 2A_1(2A_2\rho) + 1(2A_1\rho + A_2^2)$$
$$- 2\alpha(- 6A_2\rho + 5(2A_2\rho)) + \rho^2(2A_2 + A_1^2) = 0,$$

$$0 \quad - 6A_2\rho - 2A_1 . \rho^2 + 3(2A_2\rho) - 2\alpha(- 6\rho^2 + 6 . \rho^2) + \rho^2 . 2A_1 \qquad = 0,$$

$$0 \quad - 6\rho^2 + 5 . \rho^2 - 2\alpha(0 + 7 . 0) + \rho^2 . 1 \qquad = 0;$$

viz. the first, seventh and eighth equations are satisfied identically, and there remain five equations connecting ρ, α, A_1, A_2.

These equations* should lead to the before-mentioned $\alpha\beta$-modular equation

$$\rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\rho^2 + (-560\alpha + 512\alpha^3)\rho + 7 = 0,$$

and to expressions for A_1, A_2 as rational functions of α, ρ: and they should be, all five of them, satisfied by these results; but I do not see how the results are to be worked out; there is, so far as appears, no clue to the discovery of the rational functions of α, ρ.

The Jacobi Partial Differential Equation. Art. Nos. 43 to 48.

43. Writing, as above, 2α in place of Jacobi's α, this is

$$(1 - 2\alpha x^2 + x^4)\frac{d^2z}{dx^2} + (n-1)(2\alpha x - 2x^3)\frac{dz}{dx} + n(n-1)x^2z - 4n(\alpha^2 - 1)\frac{dz}{d\alpha} = 0,$$

satisfied by the numerator and denominator U, V, each of them taken with the same proper value of the coefficient A_0, or, what is the same thing, by the values

$$U = \Theta x(A_s + A_{s-1}x^2 + A_{s-2}x^4 + \ldots + A_1x^{2s-2} + x^{2s}),$$

$$V = \Theta\ (1 \ + A_1x^2 \ + A_2x^4 \ + \ldots + A_{s-1}x^{2s-2} + A_sx^{2s}),$$

* [See this volume, p. 535.]

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where now $A_s = \rho$ as before: Θ has its proper value; viz. disregarding an arbitrary merely numerical factor which might of course be introduced, the value is

$$\Theta = \sqrt{\frac{1}{M}\frac{\lambda'}{k'}}, \quad = \sqrt{\frac{u^2\rho}{v^2}\frac{\sqrt{1-v^8}}{\sqrt{1-u^8}}}, \quad = \sqrt{\rho}\,\frac{\sqrt[4]{v^4-v^{-4}}}{\sqrt[4]{u^4-u^{-4}}},$$

or, what is the same thing,

$$\Theta = \sqrt{\rho}\,\sqrt[8]{\frac{\beta^2-1}{\alpha^2-1}}.$$

If for z we write $\Theta\zeta$, then the equation becomes

$$(1-2\alpha x^2+x^4)\frac{d^2\zeta}{dx^2}+(n-1)(2\alpha x-2x^3)\frac{d\zeta}{dx}+n(n-1)x^2\zeta-4n(\alpha^2-1)\left(\frac{d\zeta}{d\alpha}+\frac{1}{\Theta}\frac{d\Theta}{d\alpha}\zeta\right)=0,$$

satisfied by the foregoing values without the factor Θ or, attending only to the denominator, say by the value

$$V = 1 + A_1x^2 + A_2x^4 + \ldots + A_{s-1}x^{2s-2} + \rho x^{2s}.$$

44. To calculate the value of $\frac{1}{\Theta}\frac{d\Theta}{d\alpha}$, we have

$$\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = \frac{\frac{1}{2}}{\rho}\frac{d\rho}{d\alpha} + \frac{\frac{1}{4}\beta}{\beta^2-1}\frac{d\beta}{d\alpha} - \frac{\frac{1}{4}\alpha}{\alpha^2-1};$$

but it has been seen (No. 10) that we have

$$\frac{d\beta}{d\alpha} = \frac{\rho^2}{n}\frac{\beta^2-1}{\alpha^2-1},$$

and the formula thus becomes

$$\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = \frac{\frac{1}{2}}{\rho}\frac{d\rho}{d\alpha} + \frac{\frac{1}{4n}(\beta\rho^2-n\alpha)}{\alpha^2-1}.$$

We have, as the first of the equations obtained by substituting in the partial differential equation,

$$2A_1 - 4n(\alpha^2-1)\frac{1}{\Theta}\frac{d\Theta}{d\alpha} = 0,$$

and we have hence the value of the first coefficient,

$$A_1 = n(\alpha^2-1)\frac{1}{\rho}\frac{d\rho}{d\alpha} + \tfrac{1}{2}(\beta\rho^2 - n\alpha);$$

or we may, by means of this result, get rid of the term $\frac{1}{\Theta}\frac{d\Theta}{d\alpha}$ from the partial differential equation; viz. the equation may be written

$$(1-2\alpha x^2+x^4)\frac{d^2\zeta}{dx^2}+(n-1)(2\alpha x-2x^3)\frac{d\zeta}{dx}+\{n(n-1)x^2-2A_1\}\zeta-4n(\alpha^2-1)\frac{d\zeta}{d\alpha}=0.$$

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Before going further, I remark that the last of the equations obtained by the substitution gives the coefficient A_{s-1}; but this is also given in terms of A_1 by the formula No. 33, $A_{s-1} - \rho A_1 = \frac{1}{3}(\alpha\rho - \beta\rho^3)$; combining the two formulæ, we have

$$A_1 \;= \frac{1}{\rho} n(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \quad \tfrac{1}{2}n\alpha \qquad + \tfrac{1}{2}\beta\rho^2,$$

$$A_{s-1} = \;\; n(\alpha^2 - 1)\frac{d\rho}{d\alpha} + (-\tfrac{1}{2}n + \tfrac{1}{3})\,\alpha\rho + \tfrac{1}{6}\beta\rho^3.$$

45. In the case $n = 3$, we have $A_{s-1} = A_0 = 1$, $A_1 = \rho$; the two equations become

$$3(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{3}{2}\alpha\rho - \rho^2 + \tfrac{1}{2}\beta\rho^3 = 0,$$

$$3(\alpha^2 - 1)\frac{d\rho}{d\alpha} - 1 - \tfrac{7}{6}\alpha\rho + \tfrac{1}{6}\beta\rho^3 = 0,$$

each of which is easily verified.

I remark also that, in the same case, $(n = 3)$, we have

$$\rho^4\frac{\beta^2 - 1}{\alpha^2 - 1} = \left(\frac{\rho^2 - 9}{\rho^2 - 1}\right)^2, \text{ and thence } \Theta = \sqrt{\rho}\sqrt[4]{\frac{\beta^2 - 1}{\alpha^2 - 1}} = \sqrt[4]{\frac{\rho^2 - 9}{\rho^2 - 1}};$$

hence

$$\frac{1}{\Theta}\frac{d\Theta}{d\rho} = \frac{4\rho}{(\rho^2 - 1)(\rho^2 - 9)};$$

and writing the equation $A_1 - 2n(\alpha^2 - 1)\dfrac{1}{\Theta}\dfrac{d\Theta}{d\alpha} = 0$ in the form

$$\rho - 2n(\alpha^2 - 1)\frac{1}{\Theta}\frac{d\Theta}{d\rho}\frac{d\rho}{d\alpha} = 0,$$

we can verify this equation.

46. In the case $n = 5$, we have for A_1 two equations, each ultimately giving the foregoing value

$$A_1 = \frac{1}{8\alpha}(\rho^2 - 2\rho + 5)(\rho^2 - 4\rho - 1).$$

Moreover, the equation $\Theta = \sqrt{\rho}\sqrt[4]{\dfrac{\beta^2 - 1}{\alpha^2 - 1}}$ gives, without difficulty, $\Theta = \dfrac{1}{\sqrt{\rho}}\dfrac{\rho - 5}{\rho - 1}$.

47. In the case $n = 7$, the formulæ give the two coefficients A_1, A_2; viz. we have

$$A_1 = \frac{1}{\rho}7(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{7}{2}\alpha \quad + \tfrac{1}{2}\beta\rho^2,$$

$$A_2 = \;\; 7(\alpha^2 - 1)\frac{d\rho}{d\alpha} - \tfrac{19}{6}\alpha\rho + \tfrac{1}{6}\beta\rho^3,$$

where the value of $\dfrac{d\rho}{d\alpha}$ must of course be obtained from the before-mentioned $\rho\alpha$-equation (given in No. 7). I have not considered these results nor endeavoured to compare them with the results for this case, obtained in the Transformation Memoir and the addition thereto, [578, 692].

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48. Substituting the value $1 + A_1x^2 + A_2x^4 + \ldots + A_{s-1}x^{2s-2} + \rho x^{2s}$ in the last-mentioned form of the partial differential equation, we obtain

$$2A_1 = 2A_1,$$

$$12A_2 = - 4(n-2)\alpha A_1 + 2A_1^2 - n(n-1) + 4n(\alpha^2 - 1)\frac{dA_1}{d\alpha},$$

$$30A_3 = - 8(n-4)\alpha A_2 + 2A_1A_2 - (n-2)(n-3)A_1 + 4n(\alpha^2-1)\frac{dA_2}{d\alpha},$$

$$56A_4 = -12(n-6)\alpha A_3 + 2A_1A_3 - (n-4)(n-5)A_2 + 4n(\alpha^2-1)\frac{dA_3}{d\alpha},$$

....

....

The number of equations is of course finite and $= s + 2$, but the last equation is an identity. To obtain the last equation but one, it is convenient to write down the general equation; viz. this is

$$(2r+1)(2r+2)A_{r+1} = -4r(n-2r)\alpha A_r + 2A_1A_r - (n-2r+1)(n-2r+2)A_{r-1} + 4n(\alpha^2-1)\frac{dA_r}{d\alpha};$$

and then, writing herein $r = s$, we have

$$0 = -4s(n-2s)\alpha\rho + 2A_1\rho - (n-2s+1)(n-2s+2)A_{s-1} + 4n(\alpha^2-1)\frac{d\rho}{d\alpha};$$

viz. for n substituting its value $2s+1$, the equation is

$$0 = -2(n-1)\alpha\rho + 2A_1\rho - 6A_{s-1} + 4n(\alpha^2-1)\frac{d\rho}{d\alpha}.$$

Recapitulation of Formulæ for the Cases $n = 3$ and $n = 5$. Art. Nos. 49 and 50.

49. In conclusion, it will be convenient to collect the formulæ as follows:

$$n = 3, \qquad y = \frac{x(\rho + x^2)}{1 + \rho x^2}, \quad \Theta = \sqrt{\frac{\rho^2 - 9}{\rho^2 - 1}},$$

$$8\alpha\rho = \rho^4 - 6\rho^2 - 3,$$

$$8(\alpha+1)\rho = (\rho-1)^3(\rho+3), \quad 8(\alpha-1)\rho = (\rho+1)^3(\rho-3),$$

$$\sigma = -\frac{3}{\rho}, \qquad 8\beta\sigma = \sigma^4 - 6\sigma^2 - 3,$$

$$8(\beta+1)\sigma = (\sigma-1)^3(\sigma+3), \quad 8(\beta-1)\sigma = (\sigma-1)^3(\sigma+3);$$

$\alpha\beta$-equation, see No. 2.

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50. $n=5,\quad y=\dfrac{x(\rho+A_1x^2+x^4)}{1+A_1x^2+\rho x^4},\quad \Theta=\dfrac{1}{\sqrt{\rho}}\dfrac{\rho-5}{\rho-1},\quad A_1=\dfrac{1}{8\alpha}(\rho^2-2\rho+5)(\rho^2-4\rho-1),$

$$64\alpha^2\rho=(\rho^2-4\rho-1)^2(\rho^2-2\rho+5),$$

or say

$$8\alpha\sqrt{\rho}=(\rho^2-4\rho-1)\sqrt{\rho^2-2\rho+5}:$$

$$64(\alpha^2-1)\rho=(\rho-1)^5(\rho-5),$$

$$\sigma=\frac{5}{\rho},\qquad 64\beta^2\sigma=(\sigma^2-4\sigma-1)^2(\sigma^2-2\sigma+5),$$

$$-8\beta\sqrt{\sigma}=(\sigma^2-4\sigma-1)\sqrt{\sigma^2-2\sigma+5},$$

$$64(\beta^2-1)\sigma=(\sigma-1)^5(\sigma-5),$$

$$\frac{\beta}{\alpha}=-\frac{\rho^2+20\rho-5}{\rho^2(\rho^2-4\rho-1)};$$

$\alpha\beta$-equation, see No. 3.

The $\rho\alpha$-equations for the cases in question, $n=3$ and $n=5$, are the so-called Jacobian equations of the fourth and the sixth degrees, studied by Brioschi (in the third appendix above referred to) and by others: the foregoing $\alpha\beta$-equations have not (so far as I am aware) been previously obtained; as rationally connected with the $\rho\alpha$-equations, they must belong to the same class of equations.

Cambridge, England, December 18, 1886.

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