876.

ON SYSTEMS OF RAYS.

[From the *Messenger of Mathematics,* vol. XVII. (1888), pp. 73—78.]

SIR W. R. HAMILTON'S Memoir, "Theory of Systems of Rays" (I do not at present consider the three Supplements), dated June, 1827, is printed *Trans. R. I. Acad.*, vol. XV. (1828), pp. 69—174. There is one page of Introduction, and pp. 70 to 80, an elaborate Contents. Part First: On ordinary systems of Reflected Rays. Part Second: On ordinary systems of Refracted Rays. Part Third: On extraordinary systems and systems of Rays in general. But only the First Part was published. This is considered under the headings: (I) Analytic Expressions of the Law of Ordinary Reflexion. (II) Theory of Focal Mirrors. (III) Surfaces of Constant Action. (IV) Classification of Systems of Rays. (V) On the Pencils of a Reflected System. (VI) On the Developable Pencils, the Two Foci of a Ray and the Caustic Curves and Surfaces. (VII) Lines of Reflexion on a Mirror. (VIII) On Osculating Focal Mirrors. (IX) On Thin and Undevelopable Pencils. (X) On the Axes of a Reflected System. (XI) Images Formed by Mirrors. (XII) Aberrations. (XIII) Density. And we have, p. 174, a "Conclusion to the First Part," wherein this first part is described as "an attempt to establish general principles respecting the system of rays produced by the ordinary reflexion of light at any mirror or combination of mirrors shaped and placed in any manner whatever; and to show that the mathematical properties of such a system may all be deduced by analytic methods from the form of ONE CHARACTERISTIC FUNCTION: as in the application of Analysis to Geometry, the properties of a plane curve or of a curve surface may all be deduced by uniform methods from the form of the function which characterises its equation."

The foregoing headings (I) to (V) may be regarded as containing the general theory, and the remaining ones (VI) to (XIII) as containing applications and developments.

I remark on the theory as follows:

Considering a congruence or doubly infinite system of lines (or say of rays), suppose that for any particular ray the cosine-inclinations are α, β, γ $(\alpha^2+\beta^2+\gamma^2=1)$, and that the coordinates of a point on the ray are (x, y, z). We may look at the system in two ways:

1°. The rays are considered as emanating from the points of a surface: here, if (x, y, z) are considered as belonging to a point on the surface, then z is a given function of (x, y) (or more generally x, y, z are given functions of two arbitrary parameters u, v); and to determine the congruence, we must have α, β, γ each of them a given function of (x, y) or of (u, v), such that identically $\alpha^2+\beta^2+\gamma^2=1$, but with no other condition as to the form of the functions.

2°. The rays may be considered irrespectively of any surface: here (α, β, γ) are each of them a given function of (x, y, z), such that always $\alpha^2+\beta^2+\gamma^2=1$; but there are further conditions, viz. it is assumed that we have one and the same ray, whatever is the point (x, y, z) on the ray; or, what is the same thing (using ρ to denote an arbitrary distance), that α, β, γ regarded as functions of x, y, z remain unaltered by the change of x, y, z into $x+\rho\alpha$, $y+\rho\beta$, $z+\rho\gamma$ respectively; this implies that we have

$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\alpha,\quad (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\beta,\quad (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\gamma,$$

each $=0$; and, conversely, it may be shown that when these relations are satisfied then α, β, γ remain unaltered by the change in question.

The equation $\alpha^2+\beta^2+\gamma^2=1$, gives

$$\alpha\delta_x\alpha+\beta\delta_x\beta+\gamma\delta_x\gamma,\quad \alpha\delta_y\alpha+\beta\delta_y\beta+\gamma\delta_y\gamma,\quad \alpha\delta_z\alpha+\beta\delta_z\beta+\gamma\delta_z\gamma,$$

each $=0$; and combining with the last-mentioned system of equations, it follows that

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta,$$

are proportional to α, β, γ; or say $=k\alpha, k\beta, k\gamma$ respectively.

Hamilton considers whether the rays are such that they are cut at right angles by a surface; supposing this is so (say in this case the rays are a rectangular, or orthotomic system, or that they are the normals of a surface), then if x, y, z refer to the point on the surface, we must have

$$\alpha\,dx+\beta\,dy+\gamma\,dz=0;$$

this implies

$$\alpha(\delta_z\beta-\delta_y\gamma)+\beta(\delta_x\gamma-\delta_z\alpha)+\gamma(\delta_y\alpha-\delta_x\beta)=0,$$

a condition which, by what precedes, is $k(\alpha^2+\beta^2+\gamma^2)=0$; viz. we must have $k=0$, and therefore

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta$$

each $=0$; that is, $\alpha\,dx+\beta\,dy+\gamma\,dz$ must be a complete differential, say it is $=dV$. And we then have $V=c$, the equation of a system of parallel surfaces each of them cutting the rays at right angles. Evidently $\alpha, \beta, \gamma=\delta_x V, \delta_y V, \delta_z V$ respectively, and the function V satisfies the partial differential equation

$$(\delta_x V)^2+(\delta_y V)^2+(\delta_z V)^2=1.$$

Hamilton in effect considers only systems of rays of the form in question, viz. those which are the normals of a surface (or, what is the same thing, the normals of a system of parallel surfaces), and it is such a system which is said to have the characteristic function V. It is shown that a system of rays originally of this kind remains a system of this kind after any number of reflexions (or ordinary refractions); in particular, if the rays originally emanate from a point, then, after any number of reflexions at mirrors of any form whatever, they are a system of rays cut at right angles by a surface. And moreover, there is given for the surface a simple construction, viz. starting from any surface which cuts the rays at right angles, and measuring off on the path of each ray (as reflected at the mirror or succession of mirrors) one and the same arbitrary distance, we have a set of points forming a surface which cuts at right angles the system of rays as reflected at the mirror or last of the mirrors.

The ray-systems considered by Hamilton are thus the normals of a surface $V-c=0$, and a large part of the properties of the system are thus included under the known theory of the normals of a surface; it may be remarked that the analytical formulæ are somewhat simplified by the circumstance that V instead of being (as usual) an arbitrary function of (x, y, z) is a function satisfying the partial differential equation $(\delta_x V)^2+(\delta_y V)^2+(\delta_z V)^2=1$. In particular, we have the theorem that each ray is intersected by two consecutive rays in foci which are the centres of curvature of the normal surface; the intersecting rays are rays proceeding from the curves of curvature of the normal surface, &c. There are other properties easily deducible from, but not actually included in, the theory of the normals; for instance, the intersecting rays aforesaid are rays proceeding from certain curves on the mirror, analogous to, but which obviously are not, the curves of curvature of the mirror. The natural mode of treatment of this part of the theory is to regard the rays as proceeding not from the normal surface, but from the mirror, and (by an investigation perfectly analogous to that for the normals of a surface) to enquire as to the intersection of the ray by rays proceeding from consecutive points of the mirror; it would thus appear that there are on the mirror two directions, such that proceeding along either of them to a consecutive point, the ray from the original point is intersected by the ray from the consecutive point, but that these directions are *not* in general at right angles, &c.

But in regard to such an investigation, the restriction introduced by the Hamiltonian theory is altogether unnecessary; there is no occasion to consider the rays which proceed from the several points of the mirror as being rays which are the normals of a surface, and the question is considered from the more general point of view as well by Malus in his *Théorie de la Double Refraction, &c.*, Paris, 1810, as more recently by Kummer in the Memoir "Allgemeine Theorie der gradlinigen Strahlensysteme," *Crelle*, t. LVII. (1860), pp. 189—230, viz. we have in Kummer a surface of any form whatever (defined according to the Gaussian theory, x, y, z given functions of the arbitrary parameters u, v), and from the several points thereof rays proceeding according to any law whatever, viz. the cosine-inclinations α, β, γ (or as Kummer writes them ξ, η, ζ) being given functions (such of course that $\alpha^2+\beta^2+\gamma^2=1$) of the same parameters u, v. It may be remarked: 1° that Kummer, while considering the simplifications of the general theory presenting themselves in the case where the rays are normals of the surface, does not specifically consider the case where, not being such normals, they are (as in the Hamiltonian theory) normals of a surface: 2° that some interesting investigations in regard to the shortest distances between consecutive rays, while naturally connecting themselves with the normals of the surface, or with that of the rays which are normals of another surface, do not properly belong to the "Allgemeine Theorie" of a congruence, which is independent of the notion of rectangularity.

It has been already remarked that the system may be looked at in the two ways 1° and 2°, and it is in the former of these that the question is considered by Kummer; it is interesting to work out part of the theory in the latter of the two ways. Taking X, Y, Z as current coordinates, we have, for a line through the point (x, y, z), the equations

$$X,\ Y,\ Z = x + \alpha\rho,\quad y + \beta\rho,\quad z + \gamma\rho;$$

α, β, γ are functions of (x, y, z), satisfying identically the equation $\alpha^2 + \beta^2 + \gamma^2 = 1$ (and therefore the derived equations in regard to x, y, z respectively); and also satisfying the equations

$$(\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\alpha = 0,\quad (\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\beta = 0,\quad (\alpha\delta_x + \beta\delta_y + \gamma\delta_z)\,\gamma = 0.$$

It should be remarked that, if these equations were not satisfied, then instead of a congruence there would be a complex, or triply infinite system of lines, viz. to the several points of space (x, y, z) there would correspond lines $X, Y, Z = x + \alpha\rho,\ y + \beta\rho,\ z + \gamma\rho$ as above, which lines would not reduce themselves to a doubly infinite system.

Suppose that the line through the point x, y, z is met by the line through a consecutive point $(x + dx,\ y + dy,\ z + dz)$; then, if X, Y, Z refer to the point of intersection of the two lines, we have

$$dx + \alpha d\rho + \rho d\alpha,\quad dy + \beta d\rho + \rho d\beta,\quad dz + \gamma d\rho + \rho d\gamma = 0;$$

and consequently

$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix} = 0$$

as a relation connecting the increments dx, dy, dz, in order that the lines may intersect, viz. this is a quadric relation $(*\between dx,\ dy,\ dz)^2 = 0$ between the increments. In the case of a complex, this equation represents a cone (passing evidently through the line $dx : dy : dz = \alpha : \beta : \gamma$), but in the case of a congruence the cone must break up into a pair of planes intersecting in the line in question $dx : dy : dz = \alpha : \beta : \gamma$. To verify *à posteriori* that this is so, observe that the differential equations satisfied by α, β, γ give, as above,

$$\delta_y\gamma - \delta_z\beta,\quad \delta_z\alpha - \delta_x\gamma,\quad \delta_x\beta - \delta_y\alpha,$$

proportional to α, β, γ, or say $= 2k\alpha,\ 2k\beta,\ 2k\gamma$; and it hence follows that the differentials $\delta\alpha$, $\delta\beta$, $\delta\gamma$ can be expressed in the forms

$$\begin{aligned} d\alpha &= a\,dx + h\,dy + g\,dz + k\,(\beta\,dz - \gamma\,dy), \\ d\beta &= h\,dx + b\,dy + f\,dz + k\,(\gamma\,dx - \alpha\,dz), \\ d\gamma &= g\,dx + f\,dy + c\,dz + k\,(\alpha\,dy - \beta\,dx), \end{aligned}$$

where

$$0 = a\alpha + h\beta + g\gamma,$$
$$0 = h\alpha + b\beta + f\gamma,$$
$$0 = g\alpha + f\beta + c\gamma.$$

The equation

$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix} = 0$$

thus assumes the form

$$(a, b, c, f, g, h\between dx, dy, dz\between \gamma\, dy - \beta\, dz,\ \alpha\, dz - \gamma\, dx,\ \beta\, dx - \alpha\, dy)$$
$$+ k\{(\gamma\, dy - \beta\, dz)^2 + (\alpha\, dz - \gamma\, dx)^2 + (\beta\, dx - \alpha\, dy)^2\} = 0.$$

Write for shortness

$$\gamma\, dy - \beta\, dz,\quad \alpha\, dz - \gamma\, dx,\quad \beta\, dx - \alpha\, dy = \xi,\ \eta,\ \zeta;$$

then putting for a moment $a\, dx + h\, dy + g\, dz = \Theta$, from this equation and $a\alpha + h\beta + g\gamma = 0$ we deduce

$$\alpha\Theta = -h(\beta\, dx - \alpha\, dy) + g(\alpha\, dz - \gamma\, dx),$$

that is, $= -h\zeta + g\eta$; or, putting for Θ its value and forming the analogous equations, we have

$$\alpha\ (a\, dx + h\, dy + g\, dz) = -h\zeta + g\eta,$$
$$\beta\, (h\, dx + b\, dy + f\, dz) = -f\xi + h\zeta,$$
$$\gamma\, (g\, dx + f\, dy + c\, dz) = -g\eta + f\xi,$$

and the quadric equation in (dx, dy, dz) thus becomes

$$\frac{\xi}{\alpha}(-h\zeta + g\eta) + \frac{\eta}{\beta}(-f\xi + h\zeta) + \frac{\zeta}{\gamma}(-g\eta + f\xi) + k(\xi^2 + \eta^2 + \zeta^2) = 0,$$

which, in virtue of the linear relation $\alpha\xi + \beta\eta + \gamma\zeta = 0$ connecting the (ξ, η, ζ), breaks up into a pair of planes, each passing through the line $\xi = 0$, $\eta = 0$, $\zeta = 0$, (that is, $dx : dy : dz = \alpha : \beta : \gamma$).

We obtain at once, as the condition that the two planes may be at right angles to each other, $k = 0$; that is,

$$\delta_y\gamma - \delta_z\beta,\quad \delta_z\alpha - \delta_x\gamma,\quad \delta_x\beta - \delta_y\alpha \text{ each } = 0;$$

or, what is the same thing, $\alpha\, dx + \beta\, dy + \gamma\, dz = dV$, a complete differential; and, as was seen, this is the condition in order that the lines may be the normals of a surface.

It thus appears that in a congruence each line is intersected by two consecutive lines, which determine respectively two planes through the line; and further, that if for every line of the congruence, the two planes are at right angles to each other, then the consecutive lines are the normals of a surface.