878.

NOTE ON THE ANHARMONIC RATIO EQUATION.

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GIVEN any four quantities α, β, γ, δ, if θ be one of the values of the anharmonic ratio, the other values are

$$\frac{1}{\theta},\quad -(1+\theta),\quad -\frac{1}{1+\theta},\quad -\frac{\theta}{1+\theta},\quad -\frac{1+\theta}{\theta};$$

and hence the equation having these six roots is

$$(x-\theta)\left(x-\frac{1}{\theta}\right)(x+1+\theta)\left(x+\frac{1}{1+\theta}\right)\left(x+\frac{\theta}{1+\theta}\right)\left(x+\frac{1+\theta}{\theta}\right)=0;$$

or, multiplying out, the equation, as is well known, takes the form

$$(x^2+x+1)^3-\frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2}\,x^2(x+1)^2=0.$$

But to effect the multiplication in the easiest manner we may proceed as follows: writing

$$a,\ b,\ c=(\alpha-\delta)(\beta-\gamma),\quad (\beta-\delta)(\gamma-\alpha),\quad (\gamma-\delta)(\alpha-\beta),$$

so that $a+b+c=0$, the equation is

$$\left(x-\frac{b}{c}\right)\left(x-\frac{c}{b}\right)\left(x-\frac{c}{a}\right)\left(x-\frac{a}{c}\right)\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)=0.$$

The product of the first pair of factors is

$$x^2+1-\left(\frac{b}{c}+\frac{c}{b}\right)x,\ =(x+1)^2-\frac{a^2}{bc}\,x;$$

thus the equation is

$$\left\{(x+1)^2 - \frac{a^2}{bc}x\right\}\left\{(x+1)^2 - \frac{b^2}{ca}x\right\}\left\{(x+1)^2 - \frac{c^2}{ab}x\right\} = 0\,;$$

that is,

$$(x+1)^6 - \left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right) x(x+1)^4 + \left(\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2}\right) x^2 (x+1)^2 - x^3 = 0\,;$$

and recollecting that $a+b+c=0$, and writing $q = bc + ca + ab$, $r = abc$, the equation becomes

$$(x+1)^6 - 3(x+1)^4 x + \left(3 + \frac{q^3}{r^2}\right)(x+1)^2 x^2 - x^3 = 0\,;$$

that is,

$$(x^2+x+1)^3 + \frac{q^3}{r^2}(x+1)^2 x^2 = 0.$$

But, writing $\theta = \frac{b}{a}$, we have

$$(\theta^2+\theta+1)^3 + \frac{q^3}{r^2}(\theta+1)^2 \theta^2 = 0\,;$$

or finally,

$$(x^2+x+1)^3 - \frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2} x^2 (x+1)^2 = 0,$$

the required result.