# 879.

## NOTE ON THE DIFFERENTIAL EQUATION $\dfrac{dx}{\surd(1-x^2)}+\dfrac{dy}{\surd(1-y^2)}=0.$

[From the *Messenger of Mathematics*, vol. XVIII. (1889), p. 90.]

We have

$$\frac{\sin u-\sin v}{\cos u-\cos v}=-\cot\tfrac{1}{2}(u+v),\ =-\sqrt{\left\{\frac{1+\cos(u+v)}{1-\cos(u+v)}\right\}},$$

and thence, writing $\cos u=x$, $\sin u=\surd(1-x^2)=\surd(X)$, and similarly

$$\cos v=y,\ \sin v=\surd(1-y^2)=\surd(Y),$$

we have

$$\frac{\surd(X)-\surd(Y)}{x-y}=-\sqrt{\left\{\frac{1+xy-\surd(XY)}{1-xy+\surd(XY)}\right\}},$$

an identical equation which, in the form

$$\frac{2-x^2-y^2-2\surd(XY)}{(x-y)^2}=\frac{1+xy-\surd(XY)}{1-xy+\surd(XY)},$$

may be verified directly without any difficulty. The integral of the proposed differential equation can of course be taken to be $c=xy-\surd(XY)$; and we have thus another form of integral

$$\frac{\surd(X)-\surd(Y)}{x-y}=-\sqrt{\left(\frac{1+c}{1-c}\right)},\ \text{say}\ =\surd(C),$$

viz. we have the integral

$$\left\{\frac{\surd(X)-\surd(Y)}{x-y}\right\}^2=C,$$

which is what Lagrange's integral of the differential equation

$$\frac{dx}{\surd(X)}+\frac{dy}{\surd(Y)}=0$$

becomes when the quartic functions $X$, $Y$ reduce themselves to the quadric functions $1-x^2$ and $1-y^2$ respectively.