# 880.

## NOTE ON THE RELATION BETWEEN THE DISTANCE OF FIVE POINTS IN SPACE.

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IN Lagrange's paper "Solutions analytiques de quelques problèmes sur les pyramides triangulaires" (*Berlin Memoirs*, 1773; *Œuvres*, t. III., see p. 677), there is contained a formula for the relation between the distances from each other of five points in space; viz. this is

$$\begin{aligned}4\Delta^2 f = {} & \alpha(a+f-g)^2+\alpha'(a'+f-g')^2+\alpha''(a''+f-g'')^2 \\ & +2\beta(a'+f-g')(a''+f-g'')+2\beta'(a+f-g)(a''+f-g'') \\ & +2\beta''(a+f-g)(a'+f-g'),\end{aligned}$$

or, in a slightly altered notation, say

$$\Pi=-4\Delta^2 f+(\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \beta_2,\ \beta_3 \between a_1+f-g_1,\ a_2+f-g_2,\ a_3+f-g_3)^2=0,$$

where, if the points are called 1, 2, 3, 4, 5, then

$c_1, c_2, c_3$ are the squared distances 23, 31, 12,

$a_1, a_2, a_3$ „ 41, 42, 43,

$g_1, g_2, g_3$ „ 51, 52, 53,

and

$f$ is the squared distance 45.

The values of $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, in terms of these squared distances, are

$$\begin{aligned}\alpha_1 &= -(a_2-a_3)^2+2c_1(a_2+a_3)-c_1^2,\\ \alpha_2 &= -(a_3-a_1)^2+2c_2(a_3+a_1)-c_2^2,\\ \alpha_3 &= -(a_1-a_2)^2+2c_3(a_1+a_2)-c_3^2,\\ \beta_1 &= -(a_3-a_1)(a_1-a_2)+2c_1a_1-c_2(a_1+a_2)-c_3(a_3+a_1)+c_2c_3,\\ \beta_2 &= -(a_1-a_2)(a_2-a_3)+2c_2a_2-c_3(a_2+a_3)-c_1(a_1+a_2)+c_3c_1,\\ \beta_3 &= -(a_2-a_3)(a_3-a_1)+2c_3a_3-c_1(a_3+a_1)-c_2(a_2+a_3)+c_1c_2.\end{aligned}$$

After some reductions, the value of $4\Delta^2$, in terms of these squared distances, is found to be

$$\begin{aligned} 4\Delta^2 = \ & c_1(a_3-a_1)(a_1-a_2) + a_1(-c_1^2 + c_1c_2 + c_1c_3) - c_1c_2c_3 \\ & + c_2(a_1-a_2)(a_2-a_3) + a_2(-c_2^2 + c_2c_3 + c_2c_1) \\ & + c_3(a_2-a_3)(a_3-a_1) + a_3(-c_3^2 + c_3c_1 + c_3c_2), \end{aligned}$$

which is, in fact,

$$8\Delta^2 = \begin{vmatrix} . & 1, & 1, & 1, & 1 \\ 1, & . & c_3, & c_2, & a_1 \\ 1, & c_3, & . & c_1, & a_2 \\ 1, & c_2, & c_1, & . & a_3 \\ 1, & a_1, & a_2, & a_3, & . \end{vmatrix},$$

where observe that each term of the determinant contains the factor 2.

By the formula in my paper "On a theorem in the Geometry of Position," *Camb. Math. Jour.*, t. II. (1841), pp. 267—271, [1], (introducing into it the present notation), the relation between the distances of the five points is given in the form

$$\Omega = \begin{vmatrix} . & 1, & 1, & 1, & 1, & 1 \\ 1, & . & c_3, & c_2, & a_1, & g_1 \\ 1, & c_3, & . & c_1, & a_2, & g_2 \\ 1, & c_2, & c_1, & . & a_3, & g_3 \\ 1, & a_1, & a_2, & a_3, & . & f \\ 1, & g_1, & g_2, & g_3, & f, & . \end{vmatrix} = 0.$$

The equations $\Pi = 0$, $\Omega = 0$ should therefore agree with each other; we have in $\Omega$ the term

$$-f^2 \begin{vmatrix} 1, & c_3, & . & c_1 \\ 1, & c_2, & c_1, & . \\ . & 1, & 1, & 1 \\ 1, & . & c_3, & c_2 \end{vmatrix},$$

which is

$$= -f^2(c_1^2 + c_2^2 + c_3^2 - 2c_2c_3 - 2c_3c_1 - 2c_1c_2);$$

and similarly in $\Pi$ we have the term

$$f^2(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3),$$

which is easily shown to be

$$= -f^2(c_1^2 + c_2^2 + c_3^2 - 2c_2c_3 - 2c_3c_1 - 2c_1c_2);$$

and it thus appears that we have identically $\Pi = \Omega$.

It is to be remarked that, in Lagrange's form, the points 4 and 5 are regarded as determined each of them by means of its squared distances from the vertices of the triangle 123, and that the formula gives (by a quadratic equation) the squared distance 45; but that nevertheless the two points 4 and 5 do not present themselves symmetrically in the formula; in fact, $\Delta^2$ and the coefficients $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ of the formula relate all of them to the tetrahedron 4123; as noticed in the paper, $\Delta = 6 \times$ volume of tetrahedron: $\sqrt{(\alpha_1)}$, $\sqrt{(\alpha_2)}$, $\sqrt{(\alpha_3)}$ are the doubled areas of the faces 423, 431, 412 respectively, and the doubled area of the face 123 is

$$= \sqrt{(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3)};$$

it may be added that

$$\beta_1 \div \sqrt{(\alpha_2\alpha_3)},\ \ \beta_2 \div \sqrt{(\alpha_3\alpha_1)},\ \ \beta_3 \div \sqrt{(\alpha_1\alpha_2)},$$

are equal to the cosines of the dihedral angles at the edges 41, 42, 43 respectively.