# 882.

## A CORRESPONDENCE OF CONFOCAL CARTESIANS WITH THE RIGHT LINES OF A HYPERBOLOID.

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TAKE $\alpha$, $\beta$, $\gamma$ arbitrary, $A$, $B$, $C = \beta - \gamma$, $\gamma - \alpha$, $\alpha - \beta$ (so that $A + B + C = 0$), and writing $\rho$, $\sigma$, $\tau$ for rectangular coordinates, consider the hyperboloid

$$A\rho^2 + B\sigma^2 + C\tau^2 + ABC = 0.$$

Let $\rho_0$, $\sigma_0$, $\tau_0$ be the coordinates of a point on the surface $(A\rho_0^2 + B\sigma_0^2 + C\tau_0^2 + ABC = 0)$. The equations of a line through this point are $\rho, \sigma, \tau = \rho_0 + f\Omega,\ \sigma_0 + g\Omega,\ \tau_0 + h\Omega$ ($\Omega$ indeterminate); and if this lies on the surface, we have

$$A\rho_0 f + B\sigma_0 g + C\tau_0 h = 0,$$
$$Af^2 + Bg^2 + Ch^2 = 0,$$

which equations determine the ratios $f : g : h$; the equations give

$$(A\rho_0 f + B\sigma_0 g)^2 = C\tau_0^2 . Ch^2, \quad = - C\tau_0^2 (Af^2 + Bg^2);$$

that is,

$$(A^2\rho_0^2 + AC\tau_0^2) f^2 + 2AB\rho_0\sigma_0 fg + (B^2\sigma_0^2 + BC\tau_0^2) g^2 = 0,$$

whence

$$\{(B^2\sigma_0^2 + BC\tau_0^2) g + AB\rho_0\sigma_0 f\}^2$$
$$= \{A^2B^2\rho_0^2\sigma_0^2 - (A^2\rho_0^2 + AC\tau_0^2)(B^2\sigma_0^2 + BC\tau_0^2)\} f^2,$$
$$= - ABC (A\rho_0^2 + B\sigma_0^2 + C\tau_0^2) \tau_0^2 f^2,$$
$$= A^2B^2C^2\tau_0^2 f^2;$$

that is,

$$\{(B\sigma_0^2 + C\tau_0^2) g + A\rho_0\sigma_0 f\}^2 = A^2C^2\tau_0^2 f^2,$$

or say

$$(B\sigma_0^2 + C\tau_0^2) g + A (\rho_0\sigma_0 \pm C\tau_0) f = 0,$$

which equation, together with $A\rho_0 f + B\sigma_0 g + C\tau_0 h = 0$, determines the ratios $f : g : h$. We have thus the two lines through the point $(\rho_0, \sigma_0, \tau_0)$.

But the equations of the line may be conveniently represented in a different form; writing the equation first obtained in the form

$$\sigma_0 (B\sigma_0 g + A\rho_0 f) + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

this is

$$-\sigma_0 C\tau_0 h + C\tau_0^2 g \pm AC\tau_0 f = 0,$$

viz.

$$-h\sigma_0 + g\tau_0 \pm Af = 0;$$

and we have the like equations

$$-f\tau_0 + h\rho_0 \pm Bg = 0,$$
$$-g\rho_0 + f\sigma_0 \pm Ch = 0,$$

where the sign is the same in each of the three equations.

The equations of the line on the surface may be written

$$\begin{array}{cccc} . & h\sigma & -g\tau & -h\sigma_0 + g\tau_0 = 0, \\ -h\rho & . & +f\tau & -f\tau_0 + h\rho_0 = 0, \\ g\rho & -f\sigma & . & -g\rho_0 + f\sigma_0 = 0, \\ (h\sigma_0 - g\tau_0)\rho & +(f\tau_0 - h\rho_0)\sigma & +(g\rho_0 - f\sigma_0)\tau & . \quad = 0; \end{array}$$

and hence from the foregoing three equations, taking the sign $-$, we have

$$\begin{array}{cccc} . & h\sigma - & g\tau & +Af = 0, \\ -\ h\rho & . & +\ f\tau & +Bg = 0, \\ g\rho - & f\sigma & . & +Ch = 0, \\ -Af\rho & -Bg\sigma & -Ch\tau & . \quad = 0, \end{array}$$

where $Af^2 + Bg^2 + Ch^2 = 0$, for the equations of a line on the surface.

In like manner, taking the sign $+$, and for $f, g, h$ writing new values $f', g', h'$, we have

$$\begin{array}{cccc} . & h'\sigma - & g'\tau & -Af' = 0, \\ -\ h'\rho & . & +\ f'\tau & -Bg' = 0, \\ g'\rho - & f'\sigma & . & -Ch' = 0, \\ Af'\rho & +Bg'\sigma & +Ch'\tau & . \quad = 0, \end{array}$$

where $Af'^2 + Bg'^2 + Ch'^2 = 0$, for the equations of a line on the surface.

The two systems of equations evidently belong to the lines of the two different kinds respectively. Writing for shortness $P, Q, R = gh' + g'h,\ hf' + h'f,\ fg' + f'g$, the two lines in fact intersect in a point, the coordinates say $(\rho_0, \sigma_0, \tau_0)$ whereof are $= \Theta QR,\ \Theta RP,\ \Theta PQ$, where

$$\Theta = \frac{A}{g^2h'^2 - g'^2h^2} = \frac{B}{h^2f'^2 - h'^2f^2} = \frac{C}{f^2g'^2 - f'^2g^2},$$

the three expressions for $\Theta$ being equal to each other in virtue of the equations

$$Af^2 + Bg^2 + Ch^2 = 0, \quad Af'^2 + Bg'^2 + Ch'^2 = 0.$$

Take now, in a plane, $P$, $Q$, $R$ points on any line, say the axis of $x$, at distances $\alpha$, $\beta$, $\gamma$ from the origin, then for a point of the plane, coordinates $(x, y)$, if $\rho$, $\sigma$, $\tau$ be the distances of the point from these three points, or say foci, we have

$$\rho^2 = (x-\alpha)^2 + y^2,$$
$$\sigma^2 = (x-\beta)^2 + y^2,$$
$$\tau^2 = (x-\gamma)^2 + y^2;$$

and if as before $A, B, C = \beta - \gamma, \gamma - \alpha, \alpha - \beta$, we thence have

$$A\rho^2 + B\sigma^2 + C\tau^2 + ABC = 0.$$

A point, coordinates $(\rho, \sigma, \tau)$, of the hyperboloid thus corresponds to a point in the plane, distances $\rho$, $\sigma$, $\tau$ from the three foci $R$, $S$, $T$ respectively; and to any line

$$\begin{array}{rrrrl} . & h\sigma - & g\tau + & Af & = 0, \\ -h\rho & . & + f\tau + & Bg & = 0, \\ g\rho - & f\sigma & . & + Ch & = 0, \\ -Af\rho & -Bg\sigma & -Ch\tau & . & = 0, \end{array}$$

corresponds the Cartesian represented by these linear equations. Similarly, to the line represented by the other system of equations

$$\begin{array}{rrrrl} . & h'\sigma - & g'\tau - & Af' & = 0, \\ -h'\rho & . & + f'\tau - & Bg' & = 0, \\ g'\rho - & f'\sigma & . & - Ch' & = 0, \\ Af'\rho & + Bg'\sigma & + Ch'\tau & . & = 0, \end{array}$$

corresponds the Cartesian represented by these equations; the two curves intersect in the point $\rho_0, \sigma_0, \tau_0 = \Theta QR, \Theta RP, \Theta PQ$, corresponding to the intersection of the lines on the hyperboloid; and moreover, *quâ* confocal Cartesians, they intersect at right angles.