883.

ANALYTICAL FORMULÆ IN REGARD TO AN OCTAD OF POINTS.

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THE term "tetrad" is used in two distinct but not inconsistent senses, viz. a tetrad denotes any four points; and it also denotes the four vertices of a self-conjugate tetrahedron in regard to a quadric surface. In fact, given any four points, there exists a triply infinite series of quadric surfaces such that, in regard to any one of them, the four points form a self-conjugate tetrahedron.

Two or more tetrads, in regard to one and the same quadric surface, are called similar tetrads.

The eight points of intersection of any three quadric surfaces are an octad; and we have the theorem that any seven points determine the octad, viz. the quadric surfaces which pass through any seven given points pass also through a uniquely determinate eighth point.

We have the further theorem that any two similar tetrads form an octad.

In particular, the vertices of the tetrahedron

$$(x = 0,\ y = 0,\ z = 0,\ w = 0),$$

or, say the points 1000, 0100, 0010, 0001 are a tetrad in regard to the quadric surface $x^2 + y^2 + z^2 + w^2 = 0$. The points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2), (x_3, y_3, z_3, w_3), (x_4, y_4, z_4, w_4), or say the points 1, 2, 3, 4, will be a tetrad in regard to the same quadric surface, if only

$$x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2 = 0, \text{ \&c., (six equations).}$$

Hence, these equations being satisfied, the two tetrads form an octad; the equation of a general quadric surface through the points of the first tetrad is

$$fyz + gzx + hxy + lxw + myw + nzw = 0,$$

and we can from three such equations eliminate any two terms; the two terms eliminated may be such as xy, xz, (12 forms) or such as yz, xw, (3 forms); and the equation may thus be presented either in the form

$$fyz + (lx + my + nz)w = 0,$$

or in the form

$$gzx + hxy + myw + nzw = 0.$$

The absolute magnitudes of the coordinates

$$(x_1, y_1, z_1, w_1), \text{ \&c.},$$

of these points are properly indeterminate; but we may if we please fix the absolute magnitudes by assuming

$$x_1^2 + y_1^2 + z_1^2 + w_1^2 = M, \text{ \&c.},$$

(four equations, the same quantity M in each of them), and we have as before

$$x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2 = 0, \text{ \&c., (six equations)},$$

viz. the coordinates are taken to satisfy these 10 equations; and this being so, it is to be shown that there exist quadric surfaces as above. The 10 equations may be expressed in a correlative form in like manner with the ordinary six equations for the transformation of the rectangular coordinates (x, y, z), viz. the new form is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = M, \text{ \&c., (four equations)},$$

and

$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0, \text{ \&c., (six equations)},$$

and the coordinates thus also satisfy these 10 equations.

Thus the equation $fyz + (lx + my + nz)w = 0$, if we determine the coefficients in such wise that the surface passes through the points 1, 2, 3, becomes

$$\begin{vmatrix} yz\ , & xw\ , & yw\ , & zw \\ y_1z_1, & x_1w_1, & y_1w_1, & z_1w_1 \\ y_2z_2, & x_2w_2, & y_2w_2, & z_2w_2 \\ y_3z_3, & x_3w_3, & y_3w_3, & z_3w_3 \end{vmatrix} = 0,$$

and this equation must therefore be satisfied on writing therein (x_4, y_4, z_4, w_4) for (x, y, z, w); viz. the equation thus obtained, det. $(1, 2, 3, 4) = 0$, must be satisfied in virtue of the equations which connect the coordinates of the four points. But, instead of effecting this verification, it is better to exhibit the equation in (x, y, z, w) in a form wherein the coordinates of the four points enter symmetrically; and the new equation, *quà* transformation of the original form, is satisfied for $(x, y, z, w) = (x_1, y_1, z_1, w_1)$: and then of course, by reason of the symmetry, it is also satisfied for

$$(x, y, z, w) = (x_4, y_4, z_4, w_4).$$

The transformation may be effected directly; but I prefer to write down the final result, and afterwards verify it.

The form is

$$\begin{aligned} yz\,.\,Mw_1w_2w_3w_4 + xw\,[&- x_1y_1z_1\,.\,w_2w_3w_4 - x_2y_2z_2\,.\,w_3w_4w_1 - x_3y_3z_3\,.\,w_4w_1w_2 - x_4y_4z_4\,.\,w_1w_2w_3] \\ + yw\,[&\ \ x_1^2w_1\,.\,z_2z_3z_4 + x_2^2w_2\,.\,z_3z_4z_1 + x_3^2w_3\,.\,z_4z_1z_2 + x_4^2w_4\,.\,z_1z_2z_3] \\ + zw\,[&\ \ x_1^2w_1\,.\,y_2y_3y_4 + x_2^2w_2\,.\,y_3y_4y_1 + x_3^2w_3\,.\,y_4y_1y_2 + x_4^2w_4\,.\,y_1y_2y_3] = 0. \end{aligned}$$

We have to show that the equation is satisfied by writing therein

$$(x,\ y,\ z,\ w) = (x_1,\ y_1,\ z_1,\ w_1).$$

Calling the resulting value Ω_1, we have

$$\begin{aligned} \Omega_1 \div w_1 = {} & My_1z_1w_2w_3w_4 - x_1^2y_1z_1w_2w_3w_4 - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ & + x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4) + y_1z_1[x_2^2w_2(z_3z_4 + y_3y_4) + x_3^2w_3(z_4z_2 + y_4y_2) + x_4^2w_4(z_2z_3 + y_2y_3)], \end{aligned}$$

which is

$$\begin{aligned} = {} & y_1z_1w_2w_3w_4(x_2^2 + x_3^2 + x_4^2) - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ & + x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4) - y_1z_1[x_2^2w_2(x_3x_4 + w_3w_4) + x_3^2w_3(x_2x_4 + w_2w_4) + x_4^2w_4(x_2x_3 + w_2w_3)]. \end{aligned}$$

The expression contains terms in $w_2w_3w_4$ which destroy each other; omitting them, we have

$$\begin{aligned} \Omega_1 \div w_1 = {} & - x_1(x_2y_2z_2w_3w_4w_1 + x_3y_3z_3w_4w_1w_2 + x_4y_4z_4w_1w_2w_3) \\ & + x_1^2w_1(y_1z_2z_3z_4 + z_1y_2y_3y_4) \\ & - x_2x_3x_4y_1z_1(x_2w_2 + x_3w_3 + x_4w_4), \end{aligned}$$

where the last line is

$$= x_2x_3x_4y_1z_1\,.\,x_1w_1.$$

Hence the whole divides by x_1w_1, or we have

$$\begin{aligned} \Omega_1 \div x_1w_1^2 = {} & - x_2y_2z_2\,.\,w_3w_4 - x_3y_3z_3\,.\,w_4w_2 - x_4y_4z_4\,.\,w_2w_3 \\ & + x_1y_1z_2z_3z_4 + x_1z_1y_2y_3y_4 + y_1z_1x_2x_3x_4, \end{aligned}$$

viz. this is

$$\begin{aligned} = {} & x_2x_3x_4y_1z_1 + y_2y_3y_4z_1x_1 + z_2z_3z_4x_1y_1 \\ & + x_2y_2z_2(x_3x_4 + y_3y_4 + z_3z_4) \\ & + x_3y_3z_3(x_2x_4 + y_2y_4 + z_2z_4) \\ & + x_4y_4z_4(x_2x_3 + y_2y_3 + z_2z_3); \end{aligned}$$

or, finally, it is

$$\begin{aligned} = {} & x_2x_3x_4(y_1z_1 + y_2z_2 + y_3z_3 + y_4z_4) \\ & + y_2y_3y_4(z_1x_1 + z_2x_2 + z_3x_3 + z_4x_4) \\ & + z_2z_3z_4(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4), \end{aligned}$$

which is $= 0$; that is, we have $\Omega_1 = 0$, the required result.

The foregoing equation of the quadric surface can be by mere interchanges of the letters exhibited in 12 different forms.

We may, in like manner, first establish and then verify the equation

$$\begin{aligned} & zx\,(-z_1^2w_1y_2y_3y_4 - z_2^2w_2y_3y_4y_1 - z_3^2w_3y_4y_1y_2 - z_4^2w_4y_1y_2y_3) \\ + & xy\,(\ \ y_1^2w_1z_2z_3z_4 + y_2^2w_2z_3z_4z_1 + y_3^2w_3z_4z_1z_2 + y_4^2w_4z_1z_2z_3) \\ + & yw\,(-y_1^2x_1z_2z_3z_4 - y_2^2x_2z_3z_4z_1 - y_3^2x_3z_4z_1z_2 - y_4^2x_4z_1z_2z_3) \\ + & zw\,(\ \ z_1^2x_1y_2y_3y_4 + z_2^2x_2y_3y_4y_1 + z_3^2x_3y_4y_1y_2 + z_4^2x_4y_1y_2y_3) = 0\,; \end{aligned}$$

this can be by a cyclical interchange of the letters (x, y, z) exhibited in 3 different forms.

Of course, the fifteen equations belong each of them to a quadric surface through the 8 points. Any three of the fifteen equations, say $U=0$, $V=0$, $W=0$, may be used for determining the octad; the equation of any other quadric through the octad is then $\alpha U + \beta V + \gamma W = 0$.