270.

ON THE DOUBLE TANGENTS OF A CURVE OF THE FOURTH ORDER.

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THE present memoir is intended to be supplementary to that "On the Double Tangents of a Plane Curve (*Phil. Trans.*, vol. CXLIX. (1859), pp. 193—212) [260]." I take the opportunity of correcting an error which I have there fallen into, and which is rather a misleading one, viz. the emanants U_1, U_2, .. were numerically determined in such manner as to become equal to U on putting (x_1, y_1, z_1) equal to (x, y, z); the numerical determination should have been (and in the latter part of the memoir is assumed to be) such as to render H_1, H_2, &c. equal to H, on making the substitution in question; that is, in the place of the formulæ

$$U_1 = \frac{1}{n}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\,U,$$

$$U_2 = \frac{1}{n(n-1)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)^2\,U, \text{ \&c.},$$

$$\vdots$$

there ought to have been

$$U_1 = \frac{1}{(n-2)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\,U,$$

$$U_2 = \frac{1}{(n-2)(n-3)}(x_1\partial_x + y_1\partial_y + z_1\partial_z)^2\,U, \text{ \&c.}$$

$$\vdots$$

[this error is corrected *ante* p. 189].

The points of contact of the double tangents of the curve of the fourth order or quartic $U=0$, are given as the intersections of the curve with a curve of the fourteenth order $\Pi=0$; the last-mentioned curve is not absolutely determinate, since instead of $\Pi=0$, we may, it is clear, write $\Pi+MU=0$, where M is an arbitrary function of the tenth order. I have in the memoir spoken of Hesse's original form (say $\Pi_1 = 0$) of the curve of the fourteenth order obtained by him in 1850, and of his transformed form (say $\Pi_2 = 0$) obtained in 1856. The method in the memoir itself (Mr Salmon's method) gives, in the case in question of a quartic curve, a third form, say $\Pi_3 = 0$. It appears by his paper "On the Determination of the Points of Contact of Double Tangents to an Algebraic Curve (*Quart. Math. Journ.* vol. III. p. 317 (1859))," that Mr Salmon has verified by algebraic transformations the equivalence of the last-mentioned form with those of Hesse; but the process is not given. The object of the present memoir is to demonstrate the equivalence in question, viz. that of the equation $\Pi_3 = 0$ with the one or other of the equations $\Pi_1 = 0$, $\Pi_2 = 0$, in virtue of the equation $U = 0$. The transformation depends, 1st, on a theorem used by Hesse for the deduction of his second form $\Pi_2 = 0$ from the original form $\Pi_1 = 0$, which theorem is given in his paper "Transformation der Gleichung der Curven 14ten Grades welche eine gegebene Curve 4ten Grades in den Berührungspuncten ihrer Doppeltangenten schneiden," *Crelle,* t. LII. pp. 97—103 (1856), containing the transformation in question; I prove this theorem in a different and (as it appears to me) more simple manner; 2nd, on a theorem relating to a cubic curve proved incidentally in my memoir "On the Conic of Five-pointic Contact at any point of a Plane Curve (*Phil. Trans.*, vol. CXLIX. (1859), see p. 385 [261])," the cubic curve being in the present case any first emanent of the given quartic curve: the demonstration occupies only a single paragraph, and it is here reproduced; and I reproduce also Hesse's demonstration of the equivalence of the two forms $\Pi_1 = 0$ and $\Pi_2 = 0$.

Let $U = (*\between x, y, z)^4$ be a quartic function of (x, y, z); (a, b, c, f, g, h) its second differential coefficients; $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})$ the reciprocal system

$$(bc - f^2,\ ca - g^2,\ ab - h^2,\ gh - af,\ hf - bg,\ fg - ch);$$

and let H be the Hessian of U, or determinant $abc - af^2 - bg^2 - ch^2 + 2fgh$ (H is of course a sextic function of x, y, z); (a', b', c', f', g', h') the second differential coefficients of H; $(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}')$ the reciprocal system

$$(b'c' - f'^2,\ c'a' - g'^2,\ a'b' - h'^2,\ g'h' - a'f',\ h'f' - b'g',\ f'g' - c'h').$$

Then $U = 0$ being the equation of a quartic curve, the equation of the curve of the fourteenth order which by its intersections determines the points of contact of the double tangents of the quartic curve, may be taken to be (Hesse's original form)

$$\Pi_1 = (\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H}\between \partial_x H, \partial_y H, \partial_z H)^2 - 3H(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H}\between \partial_x, \partial_y, \partial_z)^2 H = 0\ [1].$$

Or it may be taken to be (Hesse's transformed form)

$$\Pi_2 = 5(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H}\between \partial_x H, \partial_y H, \partial_z H)^2 - 3(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}'\between \partial_x U, \partial_y U, \partial_z U)^2 = 0.$$

And moreover, if $U_1 = \tfrac{1}{2}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\,U$, and if H_1 be the Hessian of U_1, and $(a'', b'', c'', f'', g'', h'')$ the second differential coefficients of $H - 3H_1$, where in the differentiations (x_1, y_1, z_1) are treated as constants but after the differentiations are

[1] In quoting this formula in my former memoir, the numerical factor 3 is by mistake omitted. [This correction should have been made *ante* p. 187.]

effected they are replaced by (x, y, z), and if $(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}'')$ be the reciprocal system

$$(b''c''-f''^2,\ c''a''-g''^2,\ a''b''-h''^2,\ g''h''-a''f'',\ h''f''-b''g'',\ f''g''-c''h''),$$

then the equations of the curve of the fourteenth order may be taken to be (Salmon's form)

$$\Pi_3=(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}''\between \partial_x U, \partial_y U, \partial_z U)^2=0.$$

I have preferred to write the three equations in the foregoing forms; but it is clear that the terms

$$(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x, \partial_y, \partial_z)^2 H \qquad ; \quad (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}'\between \partial_x, \partial_y, \partial_z)^2 U$$

might also have been written

$$(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between a', b', c', 2f', 2g', 2h')\ ;\ (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}'\between a, b, c, 2f, 2g, 2h).$$

As already noticed, it has been shown by Hesse (and his demonstration is to be here reproduced) that the two forms $\Pi_1=0$ and $\Pi_2=0$ are equivalent to each other. And the object of the memoir is to show that the third form $\Pi_3=0$ is equivalent to the other two. The equivalences in question subsist in virtue of the equation $U=0$, that is, the functions Π_1, Π_2, Π_3 are not identical, but differ from each other by multiples of U.

Demonstration of Hesse's Theorem.

Let (a, b, c, f, g, h), $(a'\ b', c', f', g', h')$ be any systems of coefficients of a ternary quadratic function; $(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H})$, $(\text{A}', \text{B}', \text{C}'\ \text{F}', \text{G}', \text{H}')$ the reciprocal systems as above, (x, y, z) arbitrary quantities. Consider the function

$$\square=(a, b, c, f, g, h\between x, y, z)^2\cdot(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}'\between a, b, c, 2f, 2g, 2h)$$
$$-(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}'\between ax+hy+gz, hx+by+fz, gx+fy+cz)^2.$$

The term involving A' is

$$a(a, b, c, f, g, h\between x, y, z)^2-(ax+hy+gz)^2,$$

which is

$$=(ab-h^2)y^2+(ac-g^2)z^2+2(af-gh)yz,$$
$$=Cy^2+Bz^2-2Fyz;$$

and the term involving $2F'$ is

$$f(a, b, c, f, g, h\between x, y, z)^2-(hx+by+fz)(gx+fy+cz),$$

which is

$$=(af-gh)x^2+(f^2-bc)yz+(fg-ch)zx+(hf-bg)xy,$$
$$=-Fx^2-Ayz+Hzx+Gxy;$$

and the entire expression for $\square$ is thus

$$\begin{aligned}
&A'(Cy^2+Bz^2-2Fyz)\\
+\;&B'(Az^2+Cx^2-2Gzx)\\
+\;&C'(Bx^2+Ay^2-2Hxy)\\
+\;&2F'(-Fx^2-Ayz+Hzx+Gxy)\\
+\;&2G'(-Gy^2-Bzx+Fxy+Hyz)\\
+\;&2H'(-Hz^2-Cxy+Gyz+Fzx);
\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}
\square = (&BC'+B'C-2FF',\ CA'+C'A-2GG',\ AB'+A'B-2HH',\\
&GH'+G'H-AF'-A'F,\ HF'+H'F-BG'-B'G,\ FG'+F'G-CH'-C'H \between x, y, z)^2,
\end{aligned}$$

which is really the fundamental theorem. It is however used as follows; viz. the right-hand side being symmetrical in regard to the two systems

$$(a, b, c, f, g, h),\ (a', b', c', f' g', h'),$$

the left-hand side, which is not in form symmetrical as regards the two systems, must be so in reality; or if $\square'$ is what $\square$ becomes by interchanging the two systems, then $\square'=\square$; or substituting for $\square$ and $\square'$ their values, we have

$$\begin{aligned}
&(a, b, c, f, g, h \between x, y, z)^2 . (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between a, b, c, 2f, 2g, 2h)\\
&\qquad - (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between ax+hy+gz,\ hx+by+fz,\ gx+fy+cz)^2\\
&= (a', b', c', f', g', h' \between x, y, z)^2 . (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between a', b', c', 2f', 2g', 2h')\\
&\qquad - (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between a'x+h'y+g'z,\ h'x+b'y+f'z,\ g'x+f'y+c'z)^2,
\end{aligned}$$

which is Hesse's theorem.

If in particular (a, b, c, f, g, h) are the second differential coefficients of a function $u=(*\between x, y, z)^p$, and (a', b', c', f', g', h') the second differential coefficients of a function $u'=(*\between x, y, z)^{p'}$, then the equation becomes

$$\begin{aligned}
&p(p-1)\,u . (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x, \partial_y, \partial_z)^2 u - (p-1)^2 (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x u, \partial_y u, \partial_z u)^2\\
&= p'(p'-1)\,u' . (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 u' - (p'-1)^2 (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x u', \partial_y u', \partial_z u')^2;
\end{aligned}$$

and if for u, u' we take the quartic function U and the sextic function H, its Hessian, we have

$$\begin{aligned}
&12U . (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x, \partial_y, \partial_z)^2 U - 9 (\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x U, \partial_y U, \partial_z U)^2\\
&= 30H . (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 H - 25 (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x H, \partial_y H, \partial_z H)^2;
\end{aligned}$$

and if in this identical equation we write $U=0$, then from the resulting equation and the equation

$$\Pi_1 = -3H (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 H + (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x H, \partial_y H, \partial_z H)^2$$

we may eliminate any one of the three terms

$$(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x U, \partial_y U, \partial_z U)^2,$$

$$H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 H,$$

$$(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x H, \partial_y H, \partial_z H)^2;$$

and in particular if the second term be eliminated, we obtain the equation

$$\Pi_2 = 5(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x H, \partial_y H, \partial_z H)^2 - 3(\text{A}', \text{B}', \text{C}', \text{F}', \text{G}', \text{H}' \between \partial_x U, \partial_y U, \partial_z U)^2,$$

and the equivalence of the two forms $\Pi_1 = 0$ and $\Pi_2 = 0$ is thus established.

But Hesse's theorem leads also to the demonstration of the equivalence of the third form $\Pi_3 = 0$. To use it for this purpose, I remark that if $(a'', b'', c'', f'', g'', h'')$ are the second differential coefficients of $H - 3H_1$, where after the differentiations (x_1, y_1, z_1) are to be replaced by (x, y, z), then the theorem gives

$$\begin{aligned} 12U.(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}'' \between \partial_x, \partial_y, \partial_z)^2 U - 9(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = (a'', b'', c'', f'', g'', h'' \between x, y, z)^2.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 (H - 3H_1) \\ - (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between a''x + h''y + g''z, h''x + b''y + f''z, g''x + f''y + c''z)^2. \end{aligned}$$

But on putting (x, y, z) for (x_1, y_1, z_1) we have (since H is a homogeneous function of the order 6, and H_1 before the change is a homogeneous function of the order 3 in (x, y, z)) $a''x + h''y + g''z = 5\partial_x H - 3.2\partial_x H_1 = 5\partial_x H - 3\partial_x H$ (since, on making the substitution, $H_1 = H$, but $\partial_x H_1 = \frac{1}{2}\partial_x H) = 2\partial_x H$; and thus

$$(a''x + h''y + g''z, h''x + b''y + f''z, g''x + f''y + c''z) = (2\partial_x H, 2\partial_y H, 2\partial_x H);$$

and similarly, on making the substitution,

$$(a'', b'', c'', f'', g'', h'' \between x, y, z)^2 = 6.5H - 3.3.2H_1 = (30 - 18)H = 12H.$$

Hence writing therein $U = 0$, the foregoing equation becomes

$$\begin{aligned} -9(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = 12H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 (H - 3H_1) \\ -4 (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x U, \partial_y U, \partial_z U)^2, \end{aligned}$$

which may also be written

$$\begin{aligned} -9(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}'' \between \partial_x U, \partial_y U, \partial_z U)^2 \\ = 12H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 H \\ -36H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x, \partial_y, \partial_z)^2 H_1 \\ -4 (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H} \between \partial_x U, \partial_y U, \partial_z U)^2, \end{aligned}$$

where (x_1, y_1, z_1) are ultimately to be replaced by (x, y, z). The second line in fact vanishes, which I show as follows:

Demonstration of my Theorem for a Cubic Curve.

Let $U = (*\between x, y, z)^3$ be a cubic function; it may by a linear transformation of the coordinates be reduced to the canonical form $x^3 + y^3 + z^3 + 6lxyz$, and we then have

$$
\begin{aligned}
&(\mathrm{A, B, C, F, G, H}\between \partial_x, \partial_y, \partial_z)^2 H \div 6^5 \\
&= \quad (yz - l^2x^2) \,.\, -6l^2x \\
&\quad + (zx - l^2y^2) \,.\, -6l^2y \\
&\quad + (xy - l^2z^2) \,.\, -6l^2z \\
&\quad + 2\,(l^2yz - lx^2) \,.\, (1 + 2l^3)\, x \\
&\quad + 2\,(l^2zx - ly^2) \,.\, (1 + 2l^3)\, y \\
&\quad + 2\,(l^2xy - lz^2) \,.\, (1 + 2l^3)\, z \\
&= \quad -18l^2\, xyz \quad + 6l^4\, (x^3 + y^3 + z^3) \\
&\quad + 6l^2\,(1 + 2l^3)\, xyz - 2l\,(1 + 2l^3)\,(x^3 + y^3 + z^3) \\
&= (-12l^2 + 12l^5)\, xyz + (-2l + 2l^4)\,(x^3 + y^3 + z^3) \\
&= \quad 2\,(-l + l^4)\,(x^3 + y^3 + z^3 + 6lxyz);
\end{aligned}
$$

or since $-l + l^4$ is equal to the quartinvariant S, and the equation is an invariantive one, we have for any cubic function whatever

$$(\mathrm{A, B, C, F, G, H}\between \partial_x, \partial_y, \partial_z)^2 H \div 6^5 = 2S \,.\, U,$$

which is the theorem in question. There is a difference of notation, and consequently a different numerical factor, in the theorem as stated in the memoir on the conic of five-pointic contact, referred to above.

If, as above, U is a quartic function $(*\between x, y, z)^4$, and $U_1 = \frac{1}{2}(x_1\partial_x + y_1\partial_y + z_1\partial_z)\, U$, then U_1 is a cubic function; and we have

$$(\mathrm{A}_1, \mathrm{B}_1, \mathrm{C}_1, \mathrm{F}_1, \mathrm{G}_1, \mathrm{H}_1 \between \partial_x, \partial_y, \partial_z)^2 H_1 \div 6^5 = 2S_1 \,.\, U_1,$$

where it is to be noticed that S_1 denotes a quartic function in the coefficients of U_1, and consequently a quartic function in (x_1, y_1, z_1), the coefficients being quartic functions of the coefficients of U. On writing (x, y, z) in the place of (x_1, y_1, z_1), S_1 becomes a quartic function of (x, y, z), which is in fact a quarticovariant quartic of U.

If in the foregoing equation we write (x, y, z) in the place of (x_1, y_1, z_1), then U_1 becomes equal to $2U$; and consequently, if $U = 0$, the right-hand side of the equation vanishes. Moreover $(a_1, b_1, c_1, f_1, g_1, h_1)$ (the second differential coefficients of U_1) become equal to (a, b, c, f, g, h), and consequently the coefficients $(\text{A}_1, \text{B}_1, \text{C}_1, \text{F}_1, \text{G}_1, \text{H}_1)$ become equal to $(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H})$. Hence, assuming always that $U=0$, the equation becomes

$$(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x, \partial_y, \partial_z)^2 H_1 = 0,$$

where after the differentiations (x_1, y_1, z_1) are replaced by (x, y, z). This is the form which is required for the present purpose.

Returning to the foregoing expression of $-9(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}''\between \partial_x U, \partial_y U, \partial_z U)^2$, this now becomes

$$\begin{aligned} -9\Pi_3 &= -9(\text{A}'', \text{B}'', \text{C}'', \text{F}'', \text{G}'', \text{H}''\between \partial_x U, \partial_y U, \partial_z U)^2 \\ &= 4\{3H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x, \partial_y, \partial_z)^2 H - (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x U, \partial_y U, \partial_z U)^2\}, \end{aligned}$$

so that the equation $\Pi_3 = 0$ gives

$$\Pi_1 = (\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x U, \partial_y U, \partial_z U)^2 - 3H.(\text{A}, \text{B}, \text{C}, \text{F}, \text{G}, \text{H}\between \partial_x, \partial_y, \partial_z)^2 H = 0,$$

and the equivalence of the equations $\Pi_1 = 0$ and $\Pi_3 = 0$ is thus established.