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ON A PARTICULAR CASE OF CASTILLON'S PROBLEM.

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THE problem referred to as Castillon's problem, being itself a particular case of the problem of the in-and-circumscribed triangle, is as follows: viz. "In a circle to inscribe a triangle the sides of which pass through three given points," and the reciprocal problem of course presents itself "about a circle to circumscribe a triangle, the angles of which lie in given lines." If in Castillon's problem the three given points are the angles of a triangle circumscribed about the circle, or if in the reciprocal problem the three given lines are the sides of a triangle circumscribed about the circle, we have a circle and an inscribed and circumscribed triangle, such that the sides of the inscribed triangle pass through the angles of the circumscribed triangle, and the problem arises "given one of the triangles to determine the other triangle." The problem, so far as I am aware, was first proposed by Clausen, who has given, *Crelle*, t. IV. (1829), p. 391, a very elegant solution, which I propose to reproduce here.

Let the angle of a point be defined as the inclination to a fixed radius, of the line from the centre through the given point.

Let α, β, γ, be the angles of the points of contact of the sides of the circumscribed triangle.

α', β', γ', the angles of the angular points of the circumscribed triangle.

a, b, c, the angles of the angular points of the inscribed triangle.

a', b', c', the angles of the intersections of the perpendiculars from the centre on the sides of the inscribed triangle.

Therefore

$$\begin{aligned} 2a' &= b + c, & 2\alpha' &= \beta + \gamma, \\ 2b' &= c + a, & 2\beta' &= \gamma + \alpha, \\ 2c' &= a + b, & 2\gamma' &= \alpha + \beta. \end{aligned}$$

Hence observing that the distance of one of the angles of the circumscribed triangle is $\sec\frac{1}{2}(\beta-\gamma)$, and that the projection of this line perpendicular to the corresponding side of the inscribed triangle is equal to $\sec\frac{1}{2}(\beta-\gamma)\cos(\alpha'-a')$, which is equal to the projection of the radius perpendicular to the same side, or to $\cos\frac{1}{2}(b-c)$, we have

$$\cos(\alpha'-a')=\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(b-c),$$
$$\cos(\beta'-b')=\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(c-a),$$
$$\cos(\gamma'-c')=\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(a-b).$$

Write

$$b-c=4f,\quad b+c-2\alpha=4x,$$
$$c-a=4g,\quad c+a-2\beta=4y,$$
$$a-b=4h,\quad a+b-2\gamma=4z;$$

therefore

$$\cos(y+z+g-h)=\cos 2f\cos(f+y-z),$$
$$\cos(z+x+h-f)=\cos 2g\cos(g+z-x),$$
$$\cos(x+y+f-g)=\cos 2h\cos(h+x-y),$$

or, since $f+g+h=0$,

$$\cos\{(y-h)+(z+g)\}=\cos 2f\cos\{(y-h)-(z+g)\},$$
$$\cos\{(z-f)+(x+h)\}=\cos 2g\cos\{(z-f)-(x+h)\},$$
$$\cos\{(x-g)+(y+f)\}=\cos 2h\cos\{(x-g)-(y+f)\};$$

or

$$\tan^2 f=\tan(y-h)\tan(z+g),$$
$$\tan^2 g=\tan(z-f)\tan(x+h),$$
$$\tan^2 h=\tan(x-g)\tan(y+f).$$

Put

$$x+h=\eta,\quad x-g=\zeta_1,$$
$$y+f=\zeta,\quad y-h=\xi_1,$$
$$z+g=\xi,\quad z-f=\eta_1,$$

$$\tan f=l,\quad \tan\xi=\mathrm{x},\quad \tan\xi_1=\mathrm{x}_1,$$
$$\tan g=m,\quad \tan\eta=\mathrm{y},\quad \tan\eta_1=\mathrm{y}_1,$$
$$\tan h=n,\quad \tan\zeta=\mathrm{z},\quad \tan\zeta_1=\mathrm{z}_1.$$

We have then

$$\tan^2 f=\tan\xi\tan\xi_1,$$
$$\tan^2 g=\tan\eta\tan\eta_1,$$
$$\tan^2 h=\tan\zeta\tan\zeta_1,$$

or, what is the same thing,

$$l^2=\mathrm{xx}_1,\quad m^2=\mathrm{yy}_1,\quad n^2=\mathrm{zz}_1.$$

But to obtain equations involving only one of the sets (x, y, z), (x_1, y_1, z_1), it is proper to write

$$\tan^2 f = \tan(\zeta + g)\tan\xi = \tan\xi_1 \tan(\zeta_1 - h),$$
$$\tan^2 g = \tan(\xi + h)\tan\eta = \tan\eta_1 \tan(\xi_1 - f),$$
$$\tan^2 h = \tan(\eta + f)\tan\zeta = \tan\zeta_1 \tan(\eta_1 - g);$$

taking the first set of equations, we have

$$l^2 = \frac{\mathrm{x}(\mathrm{z} + m)}{1 - m\mathrm{z}},$$
$$m^2 = \frac{\mathrm{y}(\mathrm{x} + n)}{1 - n\mathrm{x}},$$
$$n^2 = \frac{\mathrm{z}(\mathrm{y} + l)}{1 - l\mathrm{y}};$$

therefore

$$\mathrm{z} = \frac{l^2 - m\mathrm{x}}{\mathrm{x} + l^2 m}, \quad \mathrm{y} = \frac{m^2(1 - n\mathrm{x})}{\mathrm{x} + n};$$

therefore

$$\mathrm{y} + l \;:\; 1 - l\mathrm{y} = m^2 + ln + (l - m^2 n)\,\mathrm{x} \;:\; n - lm^2 + (1 + lm^2 n)\,\mathrm{x};$$

therefore

$$n^2\{n - lm^2 + (1 + lm^2 n)\,\mathrm{x}\}(l^2 m + \mathrm{x}) = (l^2 - m\mathrm{x})\{m^2 + ln + (l - m^2 n)\,\mathrm{x}\};$$

therefore

$$\begin{aligned}&(l^2 m^2 + l^3 n - l^2 mn^3 + l^3 m^3 n^2)\\ &+ \mathrm{x}\,(-n^3 + lm^2 n^2 - l^2 mn^2 - l^3 m^3 n^3 - m^3 - lmn + l^3 - l^2 m^2 n)\\ &+ \mathrm{x}^2(-lm + m^3 n - n^2 - lm^2 n^3) = 0.\end{aligned}$$

Now $l + m + n = lmn$, and by means of this relation we find

$$\text{Coef. } \mathrm{x}^\circ = l\{lm^2 + l^2 n - n^2(l + m + n) + m(l + m + n)^2\},$$
$$\text{Coef. } \mathrm{x} = -n^3 + mn(l + m + n) - ln(l + m + n) - (l + m + n)^3 - m^3 - lmn + l^3 - lm(l + m + n),$$
$$\text{Coef. } \mathrm{x}^2 = \frac{1}{l}\{-l^2 m + m^2(l + m + n) - ln^2 - n(l + m + n)^2\};$$

or, reducing,

$$\begin{aligned}\frac{1}{l}\text{ coef. } \mathrm{x}^\circ &= (m^3 + 2m^2 n - n^3) + l(3m^2 + 2mn - n^2) + l^2(m + n)\\ &= (m + n)\{(m^2 + mn - n^2) + l(3m - n) + l^2\}\\ &= (m + n)\{(l + m)^2 + (l + m)(l + n) - (l + n)^2\},\end{aligned}$$

$$\begin{aligned}-\tfrac{1}{2}\text{ coef. } \mathrm{x} &= m^3 + m^2 n + mn^2 + n^3 + 2l(m^2 + 2mn + n^2) + 2l^2(m + n)\\ &= (m + n)\{m^2 + n^2 + 2l(m + n) + 2l^2\}\\ &= (m + n)\{(l + m)^2 + (l + n)^2\},\end{aligned}$$

$$\begin{aligned}l\text{ coef. } \mathrm{x}^2 &= (m^3 - 2mn^2 - n^3) + l(m^2 - 2mn - 3n^2) - l^2(m + n)\\ &= (m + n)\{(m^2 - mn - n^2) + l(m - 3n) - l^2\}\\ &= (m + n)\{(l + m)^2 - (l + m)(l + n) - (l + n)^2\};\end{aligned}$$

and the equation becomes

$$\begin{aligned}&l^2\,\{(l+m)^2+(l+m)(l+n)-(l+n)^2\}\\ &-2l\mathrm{x}\,\{(l+m)^2+(l+n)^2\}\\ &+\ \mathrm{x}^2\,\{(l+m)^2-(l+m)(l+n)-(l+n)^2\}=0,\end{aligned}$$

which may be written

$$\left[\frac{\mathrm{x}}{l}\,\{(l+m)^2-(l+m)(l+n)-(l+n)^2\}-\{(l+m)^2+(l+n)^2\}\right]^2=5\,(l+m)^2(l+n)^2,$$

and we have therefore

$$\frac{\mathrm{x}}{l}=\frac{(l+m)^2+\sqrt{(5)}\,(l+m)(l+n)+(l+n)^2}{(l+m)^2-(l+m)(l+n)-(l+n)^2},$$

or, reducing and observing also that $l^2=\mathrm{xx}_1$, we have

$$\frac{\mathrm{x}}{l}=\frac{l}{\mathrm{x}_1}=\frac{l+m\ +\frac{1}{2}\,\{1+\sqrt{(5)}\}\,(l+n)}{(l+m)-\frac{1}{2}\,\{1+\sqrt{(5)}\}\,(l+n)}.$$

The values of $(l,\ m,\ n)$ are

$$l=\tan\tfrac{1}{4}(b-c),\quad m=\tan\tfrac{1}{4}(c-a),\quad n=\tan\tfrac{1}{4}(a-b),$$

and those of $(\mathrm{x},\ \mathrm{y},\ \mathrm{z})$, $(\mathrm{x}_1,\ \mathrm{y}_1,\ \mathrm{z}_1)$ are

$$\begin{aligned}&\mathrm{x}\ =\tan\tfrac{1}{4}(b+c-2\gamma),\quad \mathrm{y}=\tan\tfrac{1}{4}(c+a-2\alpha),\quad \mathrm{z}\ =\tan\tfrac{1}{4}(a+b-2\beta),\\ &\mathrm{x}_1=\tan\tfrac{1}{4}(b+c-2\beta),\quad \mathrm{y}_1=\tan\tfrac{1}{4}(c+a-2\beta),\quad \mathrm{z}_1=\tan\tfrac{1}{4}(a+b-2\alpha),\end{aligned}$$

and the foregoing result shows that the values of

$$\frac{\mathrm{x}}{l}\ \text{or}\ \frac{l}{\mathrm{x}_1};\qquad \frac{\mathrm{y}}{m}\ \text{or}\ \frac{m}{\mathrm{y}_1};\qquad \frac{\mathrm{z}}{n}\ \text{or}\ \frac{n}{\mathrm{z}_1}$$

are

$$\frac{\sin\frac{1}{2}(a-b)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(c-a)}{\sin\frac{1}{2}(a-b)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(c-a)},$$

$$\frac{\sin\frac{1}{2}(b-c)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(b-c)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(a-b)},$$

$$\frac{\sin\frac{1}{2}(c-a)+\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(c-a)-\frac{1}{2}\{1+\sqrt{(5)}\}\sin\frac{1}{2}(b-c)}.$$

Write for a moment $\dfrac{\mathrm{x}}{l}=\dfrac{l}{\mathrm{x}_1}=k$; therefore

$$\frac{\mathrm{x}-\mathrm{x}_1}{1+\mathrm{xx}_1}=\frac{lk-\dfrac{l}{k}}{1+l^2}=\frac{l\left(k-\dfrac{1}{k}\right)}{1+l^2}=\frac{2l}{1+l^2}\div\frac{2k}{k^2-1}=\sin\tfrac{1}{2}(b-c)\,.\,\frac{k^2-1}{2k}$$

$$=\sin\tfrac{1}{2}(b-c)\,.\,\frac{P^2-Q^2}{2PQ}\ \text{if}\ k=\frac{P}{Q};$$

and taking P, Q for the numerator and the denominator respectively of the foregoing fractional expression for k, we find

$$P^2 - Q^2 = 2\{1+\surd(5)\} \sin\tfrac{1}{2}(a-b) \sin\tfrac{1}{2}(c-a),$$

$$\begin{aligned} 2PQ &= 2\left[\sin^2\tfrac{1}{2}(a-b) - \tfrac{1}{4}\{1+\surd(5)\}^2 \sin^2\tfrac{1}{2}(c-a)\right] \\ &= -\{1+\surd(5)\}\left[\tfrac{1}{2}\{1-\surd(5)\}\sin^2\tfrac{1}{2}(a-b) + \tfrac{1}{2}\{1+\surd(5)\}\sin^2\tfrac{1}{2}(c-a)\right]; \end{aligned}$$

also

$$\frac{\mathrm{x}-\mathrm{x}_1}{1+\mathrm{x}\mathrm{x}_1} = \tan\tfrac{1}{2}(\beta-\gamma),$$

we have therefore

$$\tan\tfrac{1}{2}(\beta-\gamma) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\surd(5)\}\sin^2\frac{1}{2}(a-b) + \frac{1}{2}\{1+\surd(5)\}\sin^2\frac{1}{2}(c-a)},$$

$$\tan\tfrac{1}{2}(\gamma-\alpha) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\surd(5)\}\sin^2\frac{1}{2}(b-c) + \frac{1}{2}\{1+\surd(5)\}\sin^2\frac{1}{2}(a-b)},$$

$$\tan\tfrac{1}{2}(\alpha-\beta) = \frac{-2\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(a-b)}{\frac{1}{2}\{1-\surd(5)\}\sin^2\frac{1}{2}(c-a) + \frac{1}{2}\{1+\surd(5)\}\sin^2\frac{1}{2}(b-c)},$$

equations which determine the circumscribed triangle when the inscribed triangle is given.

The more general problem, for a conic and an inscribed and circumscribed triangle such that the sides of the inscribed triangle pass through the angles of the circumscribed triangle, "given one of the triangles to determine the other" is solved by Möbius, *Crelle*, t. v. (1830), p. 103, by means of his Barycentric Calculus, which is in fact the method of trilinear coordinates. The solution is in effect as follows:

Let $\xi=0$, $\eta=0$, $\zeta=0$ be the equations of the sides of the inscribed triangle, ξ, η, ζ may be considered as containing each of them an implicit constant factor, and the equation of the conic may be taken to be

$$\eta\zeta + \zeta\xi + \xi\eta = 0,$$

moreover, if $x=0$, $y=0$, $z=0$ be the equations of the sides of the circumscribed triangle, then considering x, y, z as also containing each of them an implicit constant factor, the equation of the conic may be taken to be

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0.$$

Suppose now the sides of the inscribed triangle pass through the angles of the circumscribed triangle, we have equations such as

$$\xi = b'y + cz, \quad \eta = c'z + ax, \quad \zeta = a'x + by;$$

substituting these values we must have identically

$$\eta\zeta + \zeta\xi + \xi\eta + m(x^2 + y^2 + z^2 - 2yz - 2zx - 2xy) = 0,$$

or

$$\begin{aligned} aa' + m &= 0, \quad bc + bc' + b'c = 2m, \\ bb' + m &= 0, \quad ca + ca' + c'a = 2m, \\ cc' + m &= 0, \quad ab + ab' + a'b = 2m, \end{aligned}$$

and, substituting for a', b', c' their values,

$$(m-bc)^2 = mb^2, \quad (m-ca)^2 = mc^2, \quad (m-ab)^2 = ma^2,$$

or, if $m = n^2$,

$$n^2 - bc = nb, \quad n^2 - ca = nc, \quad n^2 - ab = na,$$

{the signs on the second side must be all $+$ or all $-$ and the $-$ sign may be omitted without loss of generality}.

Hence

$$a = b = c = -\tfrac{1}{2}\{1+\sqrt{(5)}\}\,n, \quad \sqrt{(5)} \text{ being written for } \pm\sqrt{(5)},$$

$$a' = b' = c' = -\frac{n^2}{a} = -\tfrac{1}{2}\{1-\sqrt{(5)}\}\,n = -\nu a,$$

if for shortness

$$-\nu = \frac{1-\sqrt{(5)}}{1+\sqrt{(5)}}, \text{ i.e. } \nu = \frac{\sqrt{(5)}-1}{\sqrt{(5)}+1}, \text{ or } \nu^2 - 3\nu + 1 = 0,$$

and ν having this value, the equations give

$$\xi = \nu y - z, \quad \eta = \nu z - x, \quad \zeta = \nu x - y,$$

whence also

$$\begin{aligned} 4(2\nu-1)\,x &= \nu\xi + \eta + (3\nu-1)\,\zeta, \\ 4(2\nu-1)\,y &= (3\nu-1)\,\xi + \nu\eta + \zeta, \\ 4(2\nu-1)\,z &= \xi + (3\nu-1)\,\eta + \nu\zeta. \end{aligned}$$

Each side of the circumscribed triangle has on it four points, viz. two angles of the circumscribed triangle, a point of contact with the conic, and an intersection with the corresponding side of the inscribed triangle. Thus for the side $x=0$, the four points are given as the intersections of $x=0$, with

$$y=0, \quad z=0, \quad y-z=0, \quad \nu y + z = 0,$$

and the anharmonic ratio of the four points is therefore a given quantity.(*)

Again, each side of the inscribed triangle has on it four points, viz. two angles of the inscribed triangle, a point of intersection with the tangent at the opposite angle of the inscribed conic, and a point of intersection with the corresponding side of the circumscribed triangle.

Thus for the side $\xi = 0$, the four points are given as the points of intersection of $\xi = 0$ with the lines

$$\eta = 0, \quad \zeta = 0, \quad \eta + \zeta = 0, \quad \eta + (3\nu-1)\,\zeta = 0,$$

and the anharmonic ratio of the four points is therefore a given quantity.(*)

If we draw tangents at the angles of the inscribed triangle, we have a new triangle, the sides of which are $\eta+\zeta=0$, $\zeta+\xi=0$, $\eta+\xi=0$, and joining the angles of this triangle with the points of contact of the opposite sides (i.e. the angles of the inscribed triangle), we have three lines $\eta-\zeta=0$, $\zeta-\xi=0$, $\xi-\eta=0$ meeting in a point $\xi=\eta=\zeta$, which is obviously the same as the point $x=y=z$, which is the point of intersection of the lines joining the angles of the circumscribed triangle with the points of contact of the opposite sides. (*)

The coordinates of the points of contact of the sides of the circumscribed triangle are given by $(x=0,\ y-z=0)$, $(y=0,\ z-x=0)$, $(z=0,\ x-y=0)$, these points form therefore an inscribed triangle the sides of which are

$$y+z-x=0,\quad z+x-y=0,\quad x+y-z=0.$$

Again, the tangents at the angles of the inscribed triangle form a circumscribed triangle the sides of which are

$$\eta+\zeta=0,\quad \zeta+\xi=0,\quad \xi+\eta=0;$$

therefore

$$(\zeta+\xi)-\nu(\xi+\eta)=(1-\nu)\,\xi-\nu\eta+\zeta=2\nu x\,(1-\nu+\nu^2)\,y-(1-\nu+\nu^2)\,z=-\,2\nu\,(y+z-x),$$

and we thus have

$$\zeta+\xi-\nu(\xi+\eta)=-\,2\nu\,(y+z-x),$$
$$\xi+\eta-\nu(\eta+\zeta)=-\,2\nu\,(z+x-y),$$
$$\eta+\zeta-\nu(\zeta+\xi)=-\,2\nu\,(x+y-z),$$

equations which show that the sides of the second inscribed triangle pass through the angles of the second circumscribed triangle, and that the two systems are consequently reciprocal. (*)

The four theorems marked (*) are all of them contained in the paper by Möbius.