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ON A THEOREM RELATING TO HOMOGRAPHIC FIGURES.

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THE following theorem is not new, but I do not remember where to find it:

Given any two homographic figures; there exists in the first figure a point S, which is the focus of an (ellipse or hyperbola, say an) ellipse Σ; and in the second figure a point S', which is the focus of a (hyperbola or ellipse, say a) hyperbola Σ', such that the points S, S' and the conics Σ, Σ' correspond to each other, and the conics Σ, Σ' are so related that the foci and vertices of the one may be superimposed upon the vertices and foci of the other. Moreover the perpendiculars from S, S' upon corresponding tangents of Σ, Σ' will be equal.

Write

$$x : y : 1 = \alpha\xi + \beta\eta + \gamma : \alpha'\xi + \beta'\eta + \gamma' : \alpha''\xi + \beta''\eta + \gamma'',$$

then l, m, λ, μ may be so determined that $(x-l)+i(y-m)$ shall vanish with $(\xi-\lambda)+i(\eta-\mu)$, for if we write

$$\begin{aligned} J &= \alpha - l\alpha'', & J' &= \alpha' - m\alpha'', \\ K &= \beta - l\beta'', & K' &= \beta' - m\beta'', \\ L &= \gamma - l\gamma'', & L' &= \gamma' - m\gamma'', \end{aligned}$$

we have

$$(x-l)+i(y-m) = \frac{(J+iJ')\{(\xi-\lambda)+i(\eta-\mu)\}}{\alpha''\xi+\beta''\eta+\gamma''};$$

and therefore

$$\begin{aligned} K + iK' &= i(J+iJ'), \\ L + iL' &= (J+iJ')(\lambda+i\mu), \end{aligned}$$

that is

$$K = -J, \quad K' = J,$$

or

$$l\beta'' + m\alpha'' = \alpha' + \beta,$$

$$-l\alpha' + m\beta'' = -\alpha + \beta',$$

whence also

$$l(\alpha''^2 + \beta''^2) = \alpha'\beta'' - \alpha''\beta' + \alpha\alpha'' + \beta\beta'',$$

$$m(\alpha''^2 + \beta''^2) = \alpha''\beta - \alpha\beta'' + \alpha\alpha' + \beta\beta',$$

which determine l, m; and then λ, μ are given by

$$\lambda + i\mu = \frac{L + iL'}{J + iJ'},$$

but more simply as follows, viz. remarking that if $c = \beta'\gamma'' - \beta''\gamma'$, &c., so that

$$\xi : \eta : 1 = ax + a'y + a'' : bx + b'y + b'' : cx + c'y + c'',$$

then we have

$$\lambda(c^2 + c'^2) = bc' - b'c + ac + a'c',$$

$$\mu(c^2 + c'^2) = ca' - c'a + bc + b'c';$$

the equations $x = l$, $y = m$ give the point S and $\xi = \lambda$, $\eta = \mu$ the point S'. The values of J, J' are

$$J = \frac{-c\alpha'' - c'\beta''}{\alpha''^2 + \beta''^2}, \quad J' = \frac{-c'\alpha'' + c\beta''}{\alpha''^2 + \beta''^2};$$

and therefore

$$J^2 + J'^2 = \frac{c^2 + c'^2}{\alpha''^2 + \beta''^2} = \frac{k^2}{K^2},$$

if

$$k = \sqrt{(c^2 + c'^2)}, \quad K = \sqrt{(\alpha''^2 + \beta''^2)}:$$

we have therefore

$$\sqrt{\{(x-l)^2 + (y-m)^2\}} = \frac{k}{K}\,\frac{\sqrt{\{(\xi-\lambda)^2 + (\eta-\mu)^2\}}}{\alpha''\xi + \beta''\eta + \gamma''}. \qquad (*)$$

Consider now the expression

$$(x-l)\cos\vartheta + (y-m)\sin\vartheta - \varpi,$$

which made equal to 0 would be the equation of a line the perpendicular distance of which from S is ϖ, and which distance is inclined at an angle ϑ to the axis of x; then

$$\begin{aligned}(x-l)\cos\vartheta + (y-m)\sin\vartheta &= \tfrac{1}{2}\left[(\cos\vartheta - i\sin\vartheta)\{(x-l) + i(y-m)\}\right.\\ &\qquad \left.+ (\cos\vartheta + i\sin\vartheta)\{(x-l) - i(y-m)\}\right],\\ &= \frac{1}{2(\alpha''\xi + \beta''\eta + \gamma'')}\left[(\cos\vartheta - i\sin\vartheta)\{(\xi-\lambda) + i(\eta-\mu)\}(J + iJ')\right.\\ &\qquad \left.+ (\cos\vartheta + i\sin\vartheta)\{(\xi-\lambda) - i(\eta-\mu)\}(J - iJ')\right]\end{aligned}$$

$$= \frac{k}{2(\alpha''\xi+\beta''\eta+\gamma'')K}\left[\{\cos(\vartheta-\vartheta_0)-i\sin(\vartheta-\vartheta_0)\}\{(\xi-\lambda)+i(\eta-\mu)\}\right.$$
$$\left.+\ \{\cos(\vartheta-\vartheta_0)+i\sin(\vartheta-\vartheta_0)\}\{(\xi-\lambda)-i(\eta-\mu)\}\right]$$
$$= \frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\{(\xi-\lambda)\cos(\vartheta-\vartheta_0)+(\eta-\mu)\sin(\vartheta-\vartheta_0)\},$$

where J, J' have been replaced by

$$J=\frac{k}{K}\cos\vartheta_0,\quad J'=\frac{k}{K}\sin\vartheta_0.$$

And putting besides

$$\vartheta-\vartheta_0=\theta,$$

we have more simply

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\{(\xi-\lambda)\cos\theta+(\eta-\mu)\sin\theta\},$$

whence

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi$$
$$=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\left\{(\xi-\lambda)\cos\theta+(\eta-\mu)\sin\theta-\frac{\varpi K}{k}(\alpha''\xi+\beta''\eta+\gamma'')\right\}$$
$$=\frac{k}{(\alpha''\xi+\beta''\eta+\gamma'')K}\left\{(\xi-\lambda)\left(\cos\theta-\frac{\varpi K\alpha''}{k}\right)+(\eta-\mu)\left(\sin\theta-\frac{\varpi K\beta''}{k}\right)-\frac{\varpi K}{k}(\alpha''\lambda+\beta''\mu+\gamma'')\right\}.$$

Write now

$$\alpha''=K\cos j,\ \beta''=K\sin j;\ \mathrm{a}=\frac{k}{K^2},\ \mathrm{b}=\frac{1}{K}(\alpha''\lambda+\beta''\mu+\gamma'');$$

we have

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\left\{\frac{\mathrm{a}}{\mathrm{b}}\left(\cos\theta-\frac{\varpi\cos j}{\mathrm{a}}\right)(\xi-\lambda)+\frac{\mathrm{a}}{\mathrm{b}}\left(\sin\theta-\frac{\varpi\sin j}{\mathrm{a}}\right)(\eta-\mu)-\varpi\right\}$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\{(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi\},$$

that is

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta\quad-\varpi\qquad(*)$$
$$=\frac{K\mathrm{b}}{\alpha''\xi+\beta''\eta+\gamma''}\{(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi\},$$

where it will be noticed that

$$(x-l)\cos\vartheta+(y-m)\sin\vartheta-\varpi=0,$$

and

$$(\xi-\lambda)\cos\phi+(\eta-\mu)\sin\phi-\varpi=0,$$

are lines in the first figure and in the second figure, corresponding to each other and at the same distance ϖ from the points S, S' respectively. Call these lines T and T'.

But the relations between θ, ϕ, ϖ are given by

$$\varpi \cos j = \mathrm{a} \cos \theta - \mathrm{b} \cos \phi,$$

$$\varpi \sin j = \mathrm{a} \sin \theta - \mathrm{b} \sin \phi,$$

or, by changing the fixed axes from which θ, ϕ are respectively measured,

$$\varpi = \mathrm{a} \cos \theta - \mathrm{b} \cos \phi,$$

$$0 = \mathrm{a} \sin \theta - \mathrm{b} \sin \phi.$$

Now in these equations ϖ is the perpendicular distance from S upon the line $(x-l)\cos\vartheta + (y-m)\sin\vartheta - \varpi = 0$ and θ (which only differs from ϑ by a constant angle) is the inclination of this perpendicular to a certain fixed line; ϖ is also the perpendicular distance of the line $(\xi-\lambda)\cos\phi + (\eta-\mu)\sin\phi - \varpi = 0$ from the point S', and ϕ is the inclination of this perpendicular to a fixed line. Eliminating successively ϕ and θ, we have

$$\varpi = \mathrm{a} \cos \theta - \sqrt{(\mathrm{b}^2 - \mathrm{a}^2 \sin^2 \theta)},$$

$$\varpi = -\mathrm{b} \cos \phi + \sqrt{(\mathrm{a}^2 - \mathrm{b}^2 \sin^2 \phi)},$$

or, as these equations may also be written,

$$\varpi^2 - 2\varpi \mathrm{a} \cos \theta + \mathrm{a}^2 - \mathrm{b}^2 = 0,$$

$$\varpi^2 + 2\varpi \mathrm{b} \cos \phi + \mathrm{b}^2 - \mathrm{a}^2 = 0.$$

Suppose $\mathrm{a} > \mathrm{b}$, the former equation shows that the line T is a tangent to a certain hyperbola, and the latter equation shows that the line T' is a tangent to a certain ellipse, and it is easily seen that, taking for the transverse axes the lines from which the angles θ and ϕ are respectively measured, the equation of the hyperbola is

$$\frac{x^2}{\mathrm{b}^2} - \frac{y^2}{\mathrm{a}^2 - \mathrm{b}^2} = 1,$$

and that of the ellipse is

$$\frac{\xi^2}{\mathrm{a}^2} + \frac{y^2}{\mathrm{a}^2 - \mathrm{b}^2} = 1,$$

which are the conics Σ, Σ' referred to in the enunciation. And if the second conic is superimposed upon the first in such manner that the coordinates ξ, η may belong to the same axes with x, y; then the two conics will have the assumed relation, viz. the foci of either conic will coincide with the vertices of the other conic.