NOTES AND REFERENCES.

235. CONTAINS the demonstration and I think the first publication (1858) of Hermite's formula alluded to 135, for the reduction of an elliptic differential to the form

$$\frac{dr}{\sqrt{-J+zI-4z^3}},$$

which is in fact the Weierstrassian form, the theory of which has been of late so extensively developed.

241, 242. Figures of Poinsot's stellated Polyhedra are given, Fouché et Comberousse, *Traité de Geométrie Elementaire*, 8vo. Paris, 1866, and Dostor, "Propriétés générales des polyèdres reguliers etoilés," *Liouv.* t. V. (1879), pp. 209—226.

246. In connexion with this paper, On Contour and Slope Lines (1859), I refer to the earlier paper, Reech, "Démonstration d'une proprieté générale des surfaces fermées," *Jour. École Polyt.* Cah. 37 (1858), pp. 169—178: the contour lines are here considered with reference to a closed surface; the special object is the demonstration of the formula $I+S=M+2$, where I is the number of summits, S the number of imits (the letters I, S being thus interchanged) and M the number of cols. I refer also to the paper, Maxwell, "On Hills and Dales," *Phil. Mag.* t. XL. (1870), pp. 421—427, and *Works* (4to. Cambridge, 1890), t. II. No. XLIII.

259. It would have been proper to distinguish between *ab* and *ba*, and thus for instance to have presented the face-symbols of the polyhedron considered in the form *abcd, aefb, bfge, dcgh, adhe, ehgf* (read *ab, bc, cd, da*, &c.) so as to obtain therein each duad *once* in each of its two forms *ab* and *ba*, &c.: and the like as regards the vertex-symbols. And so as to the edge-symbols, instead of $ab=KL$, it would have been better to write *aKbL*, to be read, in like manner right-handedly, as a face- or vertex-symbol.

264. See the paper, Jenkins, "On Professor Cayley's Extension of Arbogast's Method of Derivations," *Amer. Math. Jour.* t. X. (1888), pp. 29—41.

268. In connexion herewith I refer to the memoirs Mc Clintock, "On the resolution of equations of the fifth degree," *Amer. Math. Jour.* t. VI. (1884), pp. 301—314, and "Analysis of Quintic Equations," *Amer. Math. Jour.* t. VIII. (1886), pp. 45—84: the author considers the *dexter resolvent* equation, which as he remarks is my equation

in ϕ, *ante*, p. 321; the *sinister resolvent* equation deduced from it by reversing the order of the coefficients (a, b, c, d, e, f), this is in fact the equation in χ obtained from my covariant equation, p. 323, for $\Phi = \frac{a}{U}(\phi x - \chi y)$ by writing therein $x = 0$ (viz. ψ, Mc Clintock $= \chi$, Cayley), but he afterwards modifies the form of this sinister resolvent; and a *central resolvent* equation for τ $\left(= \frac{\psi}{\phi}, \text{ Mc Clintock}\right) = \frac{\chi}{\phi}$, Cayley. We obtain this equation by writing in the equation for Φ, $x = \chi$, $y = \phi$, whence $\Phi = 0$, that is the equation becomes simply $AJ - 25D^2 = 0$, where in the covariants A, D, J the original variables x, y are to be replaced by χ, ϕ respectively: viz. the equation thus is

$$(a, b, c, d, e, f \between \chi, \phi)^5 . (J_0, J_1 \between \chi, \phi) - 25 \{(D_0, D_1, D_2, D_3 \between \chi, \phi)^3\}^2 = 0,$$

or what is the same thing

$$(a, b, c, d, e, f \between \tau, 1)^5 . (J_0, J_1 \between \tau, 1) - 25 \{(D_0, D_1, D_2, D_3 \between \tau, 1)^3\}^2 = 0,$$

which is the central resolvent equation for τ.

It is proper to remark that the foregoing expression $AJ - 25D^2$ for the last coefficient of the equation in Φ, which as appears p. 324 was not given in my original memoir, was in fact suggested to me by Mc Clintock's formula.

The equation in ϕ is

$$\phi^6 - 100 B\phi^4 + 2000 (3B^2 - 4H) \phi^2 - 800 A \sqrt{5Q'} \phi + AJ - 25D^2 = 0,$$

viz. getting rid of the radical, we have

$$\{\phi^6 - 100 B\phi^4 + 2000 (3B^2 - 4H) \phi^2 + AJ - 25D^2\}^2 - 320000 AQ'\phi^2 = 0,$$

a rational sextic equation in ϕ^2; and we infer that ϕ^2 is expressible as a rational function of τ. But the actual expression is obtained by Mc Clintock, and constitutes a very important and remarkable theorem; viz. we have

$$\frac{\phi^2}{500} = -\frac{(D_0, D_1, D_2, D_3 \between \tau, 1)^3}{(a, b, c, d, e, f \between \tau, 1)^5}, = -\frac{(J_0, J_1 \between \tau, 1)}{25 (D_0, D_1, D_2, D_3 \between \tau, 1)^3},$$

the two expressions in τ being equal to each other in virtue of the foregoing equation in τ.

I verify this result in the case of the special quintic $(a, 0, 0, 0, e, f \between x, y)^5$. Writing with Mc Clintock $\phi^2 = w$, the equation in ϕ^2, or w, here is

$$(w^3 - 100\, aew^2 + 6000\, a^2e^2w + 40000\, a^3e^3)^2 - 320000\, (a^6f^4 + 256\, a^5e^5)\, w = 0;$$

and it is to be shown that, assuming as a definition of w the foregoing expressions in terms of τ, viz. for the form in question the expressions

$$\frac{w}{500} = \frac{ae^2\tau}{a\tau^5 + 5e\tau + f}, = \frac{ae\tau - af}{25\tau},$$

(implying of course $(a\tau^5 + 5e\tau + f)(e\tau - f) - 25\, e^2\tau^2 = 0$ for the sextic equation in τ) the elimination of τ from these equations leads to the just mentioned sextic equation in w.

The two equations are

$$aw\tau^5 + 5e(w - 100\,ae)\,\tau + wf = 0, \quad \tau(w - 20\,ae) + 20\,af = 0,$$

or, if for greater convenience we write $20\,ae = \theta$, then

$$aw\tau^5 + 5e(w - \theta)\,\tau \quad + wf = 0, \quad \tau(w - \theta) \quad + 20\,af = 0,$$

we have

$$\tau = -\frac{20\,af}{w - \theta},$$

and thence

$$aw \,.\, \frac{-3200000\,a^5 f^5}{(w-\theta)^5} - \frac{100\,aef(w - 5\theta)}{w - \theta} + wf = 0,$$

that is

$$w(w-\theta)^5 - 5\theta\,(w - 5\theta)(w-\theta)^4 - 3200000\,a^6 f^4 w = 0,$$

which should be identical with the before mentioned equation in w, that is with

$$(w^3 - 5\theta w^2 + 15\theta^2 w + 5\theta^3)^2 \qquad - (3200000 + 256\,\theta^5)\,w = 0,$$

and it is in fact at once seen that each of these equations is

$$\begin{aligned} 0 = \;& w^6 \\ & + w^5 \,. - 10\,\theta \\ & + w^4 \,. \quad 55\,\theta^2 \\ & + w^3 \,. - 140\,\theta^3 \\ & + w^2 \,. \quad 175\,\theta^4 \\ & + w \;. - 106\,\theta^5 - 3200000\,a^6 f^4 \\ & + w^0 \quad 25\,\theta^6\,; \end{aligned}$$

which completes the proof. The proof for the general form $(a, b, c, d, e, f \between x, y)^5$ is similar in principle, viz. treating for the moment ϕ^2 or w as a constant, we have in τ a quintic equation and a cubic equation, in each of which the coefficients contain w linearly; and the elimination of τ leads to the required sextic equation in w, but there would probably be considerable difficulty in effecting the calculations.

It thus appears that assuming the solution of the central resolvent equation for τ, $=\dfrac{\chi}{\phi}$, we also know ϕ: I recall that for the quintic equation whose roots are x_1, x_2, x_3, x_4, x_5, the significations of these quantities are

$$\tau = \frac{\chi}{\phi} = \frac{(12345) - (24135)}{12345 - 24135}, \quad \phi = 12345 - 24135,$$

where

$12345 \;=\; 12 + \;\, 23 + \;\, 34 + \;\, 45 + \;\, 51$, meaning thereby $x_1x_2 \;\; + x_2x_3 \;\; + x_3x_4 \;\; + x_4x_5 \;\; + x_5x_1$,

$(12345) = 123 + 234 + 345 + 451 + 512$ „ „ $x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$, viz. from the values just written down, we obtain by permutation of the roots the six values of each of the functions τ and ϕ.

By means of these data, or say of $t, = \tau + a^{-1}b$, and $v, = \dfrac{\phi}{10\sqrt{5}}$ (I write v instead of his $\sqrt{v}$) Mc Clintock completes in a very elegant manner the determination of the roots x_1, x_2, x_3, x_4, x_5 of the quintic equation: the solution contains also the coefficients $\gamma, \delta, \epsilon, \zeta$ which belong to the equation deprived of its second term, viz. for the definition of these, we have

$$(a, b, c, d, e, f \between x, 1)^5 = a(1, 0, \gamma, \delta, \epsilon, \zeta \between x + a^{-1}b, 1)^5 = 0.$$

I reproduce this solution: writing as usual

$$\begin{aligned} 5u_1 &= x_1 + \omega\, x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5, \\ 5u_2 &= x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega\, x_4 + \omega^3 x_5, \\ 5u_3 &= x_1 + \omega^3 x_2 + \omega\, x_3 + \omega^4 x_4 + \omega^2 x_5, \\ 5u_4 &= x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega\, x_5, \end{aligned}$$

(ω an imaginary fifth root of unity), we find the four "Eulerian" equations

$$\begin{aligned} -2\gamma &= u_1u_4 + u_2u_3 \\ -2\delta &= u_1^2u_3 + u_4^2u_2 + u_2^2u_1 + u_3^2u_4 \\ -\epsilon + 4\gamma^2 - 3u_1u_2u_3u_4 &= u_1^3u_2 + u_4^3u_3 + u_2^3u_4 + u_3^3u_1 \\ -\zeta - 20\gamma\delta &= u_1^5 + u_2^5 + u_3^5 + u_4^5 - 10\,(u_1^3u_3u_4 + u_4^3u_2u_1 + u_2^3u_1u_3 + u_3^3u_4u_2), \end{aligned}$$

from which u_1, u_2, u_3, u_4 are to be obtained.

It is found that the first and second equations may be replaced by the two pairs of equations

$$u_1u_4 = -\gamma - v, \qquad u_2u_3 = -\gamma + v$$

$$u_1^2u_3 + u_4^2u_2 = -\delta - tv, \qquad u_2^2u_1 + u_3^2u_4 = -\delta + tv.$$

As to the first of these pairs, we find

$$\begin{aligned} 25u_1u_4 = {} & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ & + (\omega + \omega^4)(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) \\ & + (\omega^2 + \omega^3)(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2), \end{aligned}$$

say this is

$$25u_1u_4 = \Sigma x^2 + (\omega + \omega^4)\, 12345 + (\omega^2 + \omega^3)\, 13524;$$

and then similarly

$$25u_2u_3 = \Sigma x^2 + (\omega^2 + \omega^3)\, 12345 + (\omega + \omega^4)\, 13524;$$

whence

$$25\,(u_1u_4 - u_2u_3) = (\omega + \omega^4 - \omega^2 - \omega^3)(12345 - 13524), = \sqrt{5}\,\phi, = 50\,v,$$

if as above $v = \frac{\phi}{10\sqrt{5}}$; that is

$$u_1u_4 - u_2u_3 = \quad 2v,$$

and combining herewith the equation

$$u_1u_4 + u_2u_3 = -2\gamma,$$

we have the first pair of equations. The second pair of equations is obtained by a similar process, but the work is longer. We have

$$125\, u_1^2u_3 = F + A\omega + B\omega^2 + C\omega^3 + D\omega^4$$

$$125\, u_2^2u_1 = F + A\omega^2 + B\omega^4 + C\omega + D\omega^3$$

$$125\, u_3^2u_4 = F + A\omega^3 + B\omega + C\omega^4 + D\omega^2$$

$$125\, u_4^2u_2 = F + A\omega^4 + B\omega^3 + C\omega^2 + D\omega$$

where

$$F = \Sigma x_1^3 + 2\Sigma x_1x_2x_3$$

$$A = \{24135\} + 2\,\{12345\} + 2\,(24135)$$

$$B = \{54321\} + 2\,\{24135\} + 2\,(54321)$$

$$C = \{12345\} + 2\,\{53142\} + 2\,(12345)$$

$$D = \{53142\} + 2\,\{54321\} + 2\,(53142)$$

where

$$\{24135\} = x_2^2x_4 + x_4^2x_1 + x_1^2x_3 + x_3^2x_5 + x_5^2x_2, \text{ \&c.}:$$

and as before

$$(24135) = x_2x_4x_1 + x_4x_1x_3 + x_1x_3x_5 + x_3x_5x_2 + x_5x_2x_4, \text{ \&c.}$$

Hence

$$125\,(u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1) = (\omega + \omega^4 - \omega^2 - \omega^3)(A + D - B - C) = \sqrt{5}\,(A + D - B - C).$$

Here

$$A + D - B - C = \quad \{12345\} - \{24135\} - \{53142\} + \{54321\}$$

$$- 2\,[(12345) - (24135) - (53142) + (54123)],$$

where, substituting the values, the first line is

$x_1^2(x_2 - x_3 - x_4 + x_5) + x_2^2(x_3 - x_4 - x_5 + x_1) + x_3^2(x_4 - x_5 - x_1 + x_2) + x_4^2(x_5 - x_1 - x_2 + x_3) + x_5^2(x_1 - x_2 - x_3 + x_4)$, $= \Sigma'$ suppose: and the second line observing that (54123) and (53142) are equal to (12345) and (24135) respectively, is $= -4\,[(12345) - (24135)]$, $= -4\chi$: hence

$$A + D - B - C = \Sigma' - 4\chi.$$

But from the equations

$$a^{-1}\phi = 12345 - 24135, \; = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 - x_2x_4 - x_4x_1 - x_1x_3 - x_3x_5 - x_5x_2,$$

and

$$-5b \qquad = x_1 + x_2 + x_3 + x_4 + x_5,$$

we easily obtain

$$-5a^{-1}b\phi = \Sigma' + \chi,$$

and the last result thus is

$$A + D - B - C = -5a^{-1}b\phi - 5\chi, \ = -5a^{-1}b\phi - 5\phi\tau, \ = -5\phi(a^{-1}b + \tau),$$

or substituting for ϕ and $a^{-1}b + \tau$ their values $10\sqrt{5}\,v$, and t, this is $= -50\sqrt{5}\,tv$. Hence the value of $125(u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1)$ is $= -250\,tv$, that is we have

$$u_1^2u_3 + u_4^2u_2 - u_3^2u_4 - u_2^2u_1 = -2tv,$$

and combining herewith

$$u_1^2u_3 + u_4^2u_2 + u_3^2u_4 + u_2^2u_1 = -2\delta,$$

we have the second pair of equations.

We next write

$$2u_1^2u_3 = -\delta - tv + 2n_1,$$
$$2u_4^2u_2 = -\delta - tv - 2n_1,$$
$$2u_2^2u_1 = -\delta + tv + 2n_2,$$
$$2u_3^2u_4 = -\delta + tv - 2n_2,$$

whence

$$4n_1^2 = (\delta + tv)^2 + 4(\gamma^2 - v^2)(\gamma + v),$$
$$4n_2^2 = (\delta - tv)^2 + 4(\gamma^2 - v^2)(\gamma - v),$$

which equations determine n_1, n_2, so that $u_1^2u_3$, $u_4^2u_2$, $u_2^2u_1$ and $u_3^2u_4$ are now known; and we then have

$$u_1^5 = (u_1^2u_3)^2(u_2^2u_1) \div (u_2u_3)^2,$$
$$u_2^5 = (u_2^2u_1)^2(u_4^2u_2) \div (u_1u_4)^2,$$
$$u_4^5 = (u_4^2u_2)^2(u_3^2u_4) \div (u_2u_3)^2,$$
$$u_3^2 = (u_3^2u_4)^2(u_1^2u_3)^2 \div (u_1u_4)^4,$$

which determine u_1, u_2, u_3, u_4. (Compare herewith the formulæ, $A = \dfrac{A'A''}{\beta\gamma}$, &c. p. 54 of my paper "On a solvible quintic Equation," *Amer. Math. Jour.* t. XIII. (1890), pp. 53—58.)

But, as Mc Clintock remarks, it is possible to obtain better formulæ: viz. these are

$$u_1^5 = \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 + \sqrt{s_1 + s_2},$$
$$u_2^5 = \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 + \sqrt{s_1 - s_2},$$
$$u_3^5 = \tfrac{1}{4}r_1 - \tfrac{1}{4}r_2 - \sqrt{s_1 - s_2},$$
$$u_4^5 = \tfrac{1}{4}r_1 + \tfrac{1}{4}r_2 - \sqrt{s_1 + s_2},$$

where

$$r_1 = -\zeta + 20\,tv^2,$$

$$r_2 = (\gamma^2 - v^2)^{-1}\, v^{-1} \left\{ \begin{array}{l} (-\delta + 2\gamma t - t^3)\, v^4 \\ + \{\delta\epsilon + 2\gamma^2\delta + (-2\gamma\epsilon + \delta^2 + 4\gamma^3)\, t + \gamma\delta t^2\}\, v^2 \\ + \{\gamma^2\delta\epsilon - \gamma\delta^3\} \end{array} \right\},$$

$$s_1 = \tfrac{1}{16} r_1^2 + \tfrac{1}{16} r_2^2 + \gamma^5 + 10\gamma^3 v^2 + 5\gamma v^4,$$

$$s_2 = \tfrac{1}{8} r_1 r_2 \quad - 5\gamma^4 v - 10\gamma^2 v^3 - v^5.$$

To prove these results write for shortness

$2m_1 = -\delta - tv$, $2m_2 = -\delta + tv$, and (as above) $2n_1 = u_1^2 u_3 - u_4^2 u_2$, $2n_2 = u_2^2 u_1 - u_3^2 u_4$, then

$$u_1^2 u_3 = m_1 + n_1, \quad u_4^2 u_2 = m_1 - n_1, \quad u_2^2 u_1 = m_2 + n_2, \quad u_3^2 u_4 = m_2 - n_2;$$

we have $u_1 u_2 u_3 u_4 = u_1 u_4 \,.\, u_2 u_3 = (-\gamma + v)(-\gamma - v), = \gamma^2 - v^2$, and the third Eulerian equation thus becomes

$$\epsilon = \gamma^2 + 3v^2 + (v^2 - \gamma^2)^{-1} \{u_1^4 u_2^2\, u_3 u_4 + u_4^4 u_3^2\, u_1 u_2 + u_2^4 u_3^2\, u_1 u_2 + u_3^4 u_1^2\, u_2 u_4\}.$$

The terms within the { } are equal to

$$(m_1 + n_1)(m_2 + n_2)(-\gamma + v), \quad (m_1 - n_1)(m_2 - n_2)(-\gamma + v), \quad (m_1 - n_1)(m_2 + n_2)(-\gamma - v),$$
$$(m_1 + n_1)(m_2 - n_2)(-\gamma - v),$$

respectively, and their sum is $= -4\gamma m_1 m_2 + 4v n_1 n_2$.

Hence putting for shortness $p = 4v n_1 n_2$, $= 4v\,(u_1^2 u_3 - u_4^2 u_2)(u_2^2 u_1 - u_3^2 u_4)$, the third equation becomes

$$\epsilon = \gamma^2 + 3v + (v^2 - \gamma^2)^{-1} (\gamma t^2 v^2 - \gamma\delta^2 + p),$$

or, what is the same thing,

$$p = \gamma\delta^2 - \gamma t^2 v^2 + (v^2 - \gamma^2)(\epsilon - \gamma^2 - 3v).$$

Again, writing the fourth Eulerian equation in the form

$$\Sigma u^5 = -\zeta - 20\gamma\delta + 10\,\{(m_1 + n_1)\, u_1 u_4 + (m_1 - n_1)\, u_1 u_4 + (m_2 + n_2)\, u_2 u_3 + (m_2 - n_2)\, u_2 u_3\},$$

the term in { } is $2m_1 u_1 u_4 + 2m_2 u_2 u_3$, $= (-\delta - tv)(-\gamma + v) + (-\delta + tv)(-\gamma - v)$, $= 2\gamma\delta - 2tv^2$, so that writing $\Sigma u^5 = r_1$, the equation becomes

$$r_1 = -\zeta - 20tv^2,$$

viz. this is the above-mentioned value of r_1.

We then have $r_2 = u_1^5 + u_4^5 - u_2^5 - u_3^5$, and for calculating this, we have

$$(\gamma^2 - v^2)^2\, u_1^5 = (m_1 + n_1)^2 (m_2 + n_2)(\gamma - v)^2,$$

$$(\gamma^2 - v^2)^2\, u_4^5 = (m_1 - n_1)^2 (m_2 - n_2)(\gamma - v)^2,$$

$$(\gamma^2 - v^2)^2\, u_2^5 = (m_2 + n_2)^2 (m_1 - n_1)(\gamma + v)^2,$$

$$(\gamma^2 - v^2)\;\; u_3^5 = (m_2 - n_2)^2 (m_1 + n_1)(\gamma + v)^2,$$

and thence, instead of $4vn_1n_2$ which occurs on the right-hand side writing its value $=p$, we find

$$\begin{aligned}(\gamma^2-v^2)^2 r_2 = &\quad (m_1+m_2)\, v^{-1}\{p\,(\gamma^2+v^2)-4m_1m_2\gamma\}\\ &+2\,(m_1-m_2)\quad \{m_1m_2\,(\gamma^2+v^2)-\ \ p\gamma\}\\ &+2\,(n_1{}^2m_2-n_2{}^2m_1)\quad (\gamma^2+v^2)\\ &-4\,(n_1{}^2m_1+n_2{}^2m_1)\quad \gamma v.\end{aligned}$$

We have

$$4n_1{}^2=(\delta+tv)^2+4\,(\gamma^2-v^2)\,(\gamma-v),\quad 4n_2{}^2=(\delta-tv)^2+4\,(\gamma^2-v^2)\,(\gamma+v),$$

p is given as above, and moreover $m_1+m_2=-\delta,\ \ m_1-m_2=-tv,\ \ m_1m_2=\tfrac{1}{4}\,(\delta^2-t^2v^2)$. Substituting these values and dividing out by $(\gamma^2-v^2)^2$, we find after some reductions the value given above for r_2.

Finally we have

$$4\,(\gamma-v)^5=-4\,(u_1u_4)^5=(u_1{}^5+u_4{}^5)^2-4\,(u_1u_4)^5-\tfrac{1}{2}\,(r_1+r_2)\,(u_1{}^5+u_4{}^5)=(u_1{}^5-u_4{}^5)^2-\tfrac{1}{2}\,(r_1+r_2)\,(u_1{}^5+u_4{}^5),$$

that is

$$4\,(\gamma-v)^5=4\,(s_1+s_2)-\tfrac{1}{4}\,(r_1+r_2)^2,$$

or say

$$s_1+s_2=(\tfrac{1}{4}r_1+\tfrac{1}{4}r_2)^2+(\gamma-v)^5,$$

and similarly

$$s_1-s_2=(\tfrac{1}{4}r_1-\tfrac{1}{4}r_2)^2+(\gamma+v)^5$$

which equations give the above-mentioned values of s_1 and s_2.

As to Jacobi's Memoir spoken of in the Addition, I refer to the paper Kronecker, "Ueber eine stelle in Jacobi's Aufsatz, Observatiunculæ ad theoriam æquationum pertinentes," *Crelle*, t. CVII. (1891), pp. 349—352, which incorporates some remarks of mine in regard thereto.

284, 294. The fundamental idea of these two papers is *not* that of "the six coordinates of a line," but (as indeed appears from the title) a somewhat different one, viz. I say that a curve in space may be represented by a homogeneous equation $V=0$ between six coordinates $(p,\ q,\ r,\ s,\ t,\ u)$ such that $ps+qt+ru=0$; this equation being the equation of a cone of arbitrary vertex passing through the curve in question: taking $x,\ y,\ z,\ w$ to be current point-coordinates and $\alpha,\ \beta,\ \gamma,\ \delta$ to be the point-coordinates of the arbitrary vertex, then $p,\ q,\ r,\ s,\ t,\ u$ are the six determinants of the matrix

$$\left|\begin{matrix} x, & y, & z, & w\\ \alpha, & \beta, & \gamma, & \delta\end{matrix}\right|,$$

or, what is the same thing, we have

$$\begin{aligned} p&=\gamma y-\beta z, & s&=\delta x-\alpha w,\\ q&=\alpha z-\gamma x, & t&=\delta y-\beta w,\\ r&=\beta x-\alpha y, & u&=\delta z-\gamma w,\end{aligned}$$

values which satisfy $ps+qt+ru=0$. And I accordingly say that the equation of a line in space is

$$Ap+Bq+Cr+Fs+Gt+Hu=0,$$

viz. this is the equation of the cone of arbitrary vertex $(\alpha, \beta, \gamma, \delta)$ (that is, of the plane through the point $(\alpha, \beta, \gamma, \delta)$) which passes through the line in question. But I go on to say that if $(\alpha', \beta', \gamma', \delta'')$, $(\alpha'', \beta'', \gamma'', \delta'')$ are the coordinates of any two points on the given line, or if the line be given as the intersection of the two planes $ax+by+cz+dw=0$, $a'x+b'y+c'z+d'w=0$, then in the first case

$$A=\alpha'\delta''-\alpha''\delta', \quad F=\beta'\gamma''-\beta''\gamma',$$
$$B=\beta'\delta''-\beta''\delta', \quad G=\gamma'\alpha''-\gamma''\alpha',$$
$$C=\gamma'\delta''-\gamma''\delta', \quad H=\alpha'\beta''-\alpha''\beta',$$

and in the second case

$$A=bc'-b'c, \quad F=ad'-a'd,$$
$$B=ca'-c'a, \quad G=bd'-b'd,$$
$$C=ab'-a'b, \quad H=cd'-c'd,$$

so that in each case $AF+BG+CH=0$. I thus in effect, although not quite explicitly, define (A, B, C, F, G, H) as the "six coordinates of a line"; and after giving in terms of these quantities for any two lines the condition of the intersection of the two lines I say that any other problems in relation to the line, for instance...&c., may also be solved "by means of the new coordinates."

Plücker's Memoir "On a New Geometry of Space" is published *Phil. Trans.* t. CLV. (for 1865), pp. 725—791, the paper being received Dec. 22, 1864, and the Additional Note appended thereto, Dec. 11, 1865. My two papers are referred to in the foot-note p. 784, belonging to this additional note as follows: "In two remarkable papers 'On a New Analytical Representation of Curves in Space' published in the third and fifth volumes of the *Quarterly Journal of Mathematics*, Professor Cayley employed before me in order to represent cones the six coordinates of a right line depending upon any two of its points. Having lately only seen the papers I hasten to mention it now. But besides the coincidence referred to the leading views of Professor Cayley's paper and mine have nothing in common. On this occasion I may state that the principles upon which my paper is based were advanced by me nearly twenty years ago (Geometry of Space, No. 258), but this had entirely escaped my memory when I recurred to Geometry some time since."

In the work referred to, "System der Geometrie des Raumes u. s. w." (4to. Düsseldorf, 1846), No. 258, Plücker remarks that a straight line depends upon four constants, viz. its equations in point-coordinates being $x=\kappa z+\lambda$, $y=\mu z+\nu$, or in line-coordinates being $t=\kappa v+\lambda w$, $u=\mu v+\nu w$, then in either case the constants are κ, λ, μ, ν; and he defines these four quantities as the "four coordinates of a line."

The leading idea of Plücker's memoir appears in the first words thereof, "I. On linear Complexes of Right lines." He works at first with the *four* coordinates of a line; as long as these are arbitrary the line is any line whatever; but considering them as connected by a single equation, then there is he says "a Complex," considering them as connected by two equations "a Congruency", and considering them as connected by three equations "a Configuration" or ruled surface. The establishment of these notions of a Complex and a Congruency, and the general idea of regarding the line as an element in the Geometry of Space are absolutely Plücker's, there is no anticipation of them in my two papers. Later on five coordinates $r, s, \sigma, \rho, s\rho - r\sigma$ are introduced, but the *six* coordinates are first used in the Additional Note, viz. here instead of a non-homogeneous equation or equations between four coordinates we have a homogeneous equation or equations between six quantities connected by a homogeneous quadric equation. It is hardly necessary to refer to the posthumous work *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Leipzig, 1868 and 1869, or to later developments of the theory.

END OF VOL. IV.