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ON THE THEORY OF INVOLUTION.

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THREE or more quantics which satisfy identically a linear equation such as

$$\lambda U + \lambda' U' + \lambda'' U'' + \ldots = 0,$$

where λ, λ', λ'',... are constants, are said to be in Involution. In particular any quantic $U + kV$, where k is a constant, is in involution with the quantics U, V; and the entire system of such quantics, k having any value whatever, is a system in involution with the quantics U, V. And in like manner the equation $U + kV = 0$, or the locus or system of loci thereby represented is said to be in involution, or to form a system in involution with the equations or loci $U = 0$, $V = 0$. If U, V are binary quantics then the equation $U + kV = 0$ may be considered as representing a range of points in involution with the ranges $U = 0$, $V = 0$. And similarly, if U, V are ternary quantics, then the equation $U + kV = 0$ may be considered as representing a curve in involution with the curves $U = 0$, $V = 0$.

In the case of a range $U + kV = 0$, the constant k may be determined so that the range shall have a twofold[1] point. The condition for this may be written

$$\text{Disc}^{\text{t}}.\ (U + kV) = 0,$$

[1] The series of epithets *onefold*, *twofold*, &c., seems preferable to the series *single*, *double*, &c., as avoiding ambiguities which would sometimes be occasioned by the use of these last. The double point of a curve I call a *node*, viz. a *crunode* when it is a double point with two real branches, and an *acnode* when it is a conjugate or isolated point. The subject-matter and context will in general show whether the term node is to be considered as including or as not including a cusp.

it being understood that the discriminant of the function $U+kV$ is taken in regard to the coordinates. And this being so, we may write $\text{Disc}^t.\,\text{Disc}^t.(U+kV)$ to denote the discriminant in regard to k of the function $\text{Disc}^t.(U+kV)$. The quantity in question (viz. $\text{Disc}^t.\,\text{Disc}^t.(U+kV)$, or say for shortness $\square$) is a function of the coefficients of U, V, homogeneous as regards each set of coefficients separately, and it breaks up into factors in the form

$$\square = RQ^3P^2,$$

where $R=0$ is the condition in order that the ranges $U=0$, $V=0$ may have a point in common; or what is the same thing, in order that there shall be a range $U+kV=0$ having a twofold point at a common point of the ranges $U=0$, $V=0$, (R is in fact the resultant of the quantics U, V): $Q=0$ is the condition in order that there may be a range $U+kV=0$ having a threefold point: and $P=0$ is the condition in order that there may be a range $U+kV=0$ having a pair of twofold points.

And similarly, when $U=0$, $V=0$ are curves, then we have the like equation

$$\square = RQ^3P^2,$$

where $R=0$ is the condition in order that the curves $U=0$, $V=0$ may have a point of twofold intersection, that is, that the two curves may touch each other, (R is the Tactinvariant of the quantics U, V); or what is the same thing, it is the condition in order that there may be a curve $U+kV=0$ having a node at a point of twofold intersection of the curves $U=0$, $V=0$; moreover $Q=0$ is the condition in order that there may be a curve $U+kV=0$ having a cusp: and $P=0$ is the condition in order that there may be a curve $U+kV=0$ having a pair of nodes.

The establishment and illustration of the foregoing theorems form the chief object of the present memoir.

Article Nos. 1 to 16, *relating to two Binary Quantics.*

1. Let $U=(a, \ldots \between x, y)^2$, $V=(a', \ldots \between x, y)^2$, be two binary quantics of the same order n; and write $W=U+kV=(a+ka', \ldots \between x, y)^2$, so that $W=U+kV=0$ is the equation of a range in involution with the ranges $U=0$, $V=0$. But for greater distinctness it is in general convenient to retain $U+kV$ instead of replacing it by the single letter W.

2. In order that the range $U+kV=0$ may have a twofold point we must have simultaneously

$$\delta_x(U+kV)=0,$$

$$\delta_y(U+kV)=0,$$

and eliminating (x, y) from these equations we find

$$\text{Disc}^t.\,(U+kV)=0,$$

which is an equation of the order $2(n-1)$ as regards k, and of the same order as regards the coefficients of U and V conjointly. And to each of the $2(n-1)$ values of k there corresponds a point (x, y) satisfying the required conditions; that is, a point which is a twofold point of the range $U+kV=0$. The points in question may be termed the 'critic centres' of the involution.

3. The elimination of k from the before-mentioned two equations gives

$$\begin{vmatrix} \delta_x U, & \delta_y U \\ \delta_x V, & \delta_y V \end{vmatrix} = 0,$$

where the determinant, which for shortness I call J, is the Jacobian of the two functions U, V. The equation $J=0$ gives a range of $2(n-1)$ points which are in fact the critic centres; and for each of these points we have

$$\delta_x U : \delta_x V = \delta_y U : \delta_y V = -k : 1,$$

which gives the value of k corresponding to the point in question.

4. The condition in order that the equation in k may have a twofold root is

$$\text{Disc}^{\text{t}}.\ \text{Disc}^{\text{t}}.\ (U+kV)=0,$$

or say

$$\square = 0,$$

where $\text{Disc}^{\text{t}}.\ \text{Disc}^{\text{t}}.\ (U+kV)$, $=\square$, is a function of the degree $2(2n-2)(2n-3)$ in regard to the coefficients of U, V conjointly; but it is separately homogeneous, and therefore of the degree $(2n-2)(2n-3)$ in regard to each of the two sets of coefficients.

5. To each point of the range $J=0$, there corresponds a value of k; hence if the range J have a twofold point, then the equation in k will have a twofold root. Now *first* if the ranges $U=0$, $V=0$ have a common point, then this is a twofold point of the range $V=0$. But *secondly*, without a common point in the ranges $U=0$, $V=0$, the range $J=0$ may have a twofold point; and in this case also we have a twofold root of the equation in k. And *thirdly*, without a twofold point in the range $V=0$, there may be in this range two onefold points giving each of them the same value of k, and so giving a twofold root of the equation in k. And the three suppositions correspond respectively to the cases of there being a range $U+kV=0$ having a twofold point at a common point of the ranges $U=0$, $V=0$, having a threefold point, and having a pair of twofold points.

6. *First*, if the ranges $U=0$, $V=0$ have a common point we may write

$$U=(x-\alpha y)\,U',\ V=(x-\alpha y)\,V',$$

and these give

$$U+kV=(x-\alpha y)(U'+kV').$$

Now in general

$$\text{Disc}^t.\ PQ = \text{Disc}^t.\ P\,.\,\text{Disc}^t.\ Q\,.\,[\text{Result.}\ (P,\ Q)]^2,$$

and hence in the present case

$$\text{Disc}^t.\ (U+kV) = \text{Disc}^t.\ (U'+kV')\,.\,(U_0'+kV_0')^2,$$

where $U_0'+kV_0'$ is what $U'+kV'$ becomes on writing therein $x=\alpha y$ and neglecting the factor y^{n-1} which then presents itself. We see therefore that in this case the equation k has a twofold root $k=-U_0'\div V_0'$; a result which might also have been obtained from the consideration of the Jacobian.

7. The condition in order that the ranges $U=0$, $V=0$ may have a common point is

$$\text{Result.}\ (U,\ V)=0,$$

say $P=0$. P is of the degree n in regard to the coefficients of U, V respectively.

8. *Secondly*, suppose that the functions U, V are such that there exists a range $U+kV=0$ having a threefold point. If k_1 be the proper value of k, then we have $U+k_1V=(x-\alpha y)^3\,\Theta$, and therefore $U=-k_1V+(x-\alpha y)^3\,\Theta$. Hence forming the Jacobian of U, V, the equation for the determination of the critic centres will be

$$\begin{vmatrix} \delta_x V, & \delta_x\,.\,(x-\alpha y)^3\,\Theta \\ \delta_y V, & \delta_y\,.\,(x-\alpha y)^3\,\Theta \end{vmatrix} = 0,$$

which is of the form

$$(x-\alpha y)^2\,\Omega=0\,;$$

or we have $(x-\alpha y)^2=0$, a twofold critic centre. The corresponding value of k given by the equation $-k:1=\delta_x U:\delta_x V$ is $k=k_1$, and we have thus $k=k_1$ as a twofold root of the equation in k.

9. But if the range $W=U+kV=0$ has a threefold point, or what is the same thing, if the equation $W=0$ has a threefold root; then we must have between the coefficients of W a plexus of equations equivalent to two relations. Such plexus is known to be of the order $3(n-2)$. This comes to saying that if the coefficients of W are assumed to be of the form $a+ka'+la''$, ... and if between the several equations of the plexus we eliminate k, we obtain for l an equation $Q=0$ of the degree $3(n-2)$. The equation in question would be of the form Funct. $(a+la'',\ a',\ ..)=0$. Hence Q is of the degree $3(n-2)$ in the coefficients $(a,\ldots)$ of U. And in a similar manner Q is of the degree $3(n-2)$ in the coefficients $(a',\ldots)$ of V. And omitting altogether the terms in l, or taking the coefficients of W to be $a+ka'$, ... if from the equations of the plexus we eliminate k, we find an equation $Q=0$, where Q is a function of the degree $3(n-2)$ as regards the coefficients of U, and of the same degree as regards the coefficients of V. We have thus found the form of the condition $Q=0$ which expresses that there may be a range $U+kV=0$ having a threefold point.

10. It may be proper to remark conversely that given the equation $Q=0$, if in this equation we write $a=(a+ka')-ka', \ldots$ so that Q becomes a function of $a+ka', \ldots a', \ldots k$, then the equation $Q=0$ will be satisfied irrespectively of the values of $a', \ldots k$ by a plexus of equations involving only the coefficients $a+ka', \ldots$ and which is in fact the very plexus (equivalent therefore to two relations) which gives the conditions in order that the equation $W=0$ may have a threefold root.

11. *Thirdly*, suppose that the functions U, V are such that there exists a range $U+kV=0$ having a pair of twofold points. If k_1 be the proper value of k, then we have $U+k_1V=(x-\alpha y)^2(x-\beta y)^2\Theta$, and therefore $U=-k_1V+(x-\alpha y)^2(x-\beta y)^2\Theta$. Hence forming the Jacobian of U, V, we have for the determination of the critic centres the equation

$$\begin{vmatrix} \delta_x V, & \delta_x . (x-\alpha y)^2 (x-\beta y)^2 \Theta \\ \delta_y V, & \delta_y . (x-\alpha y)^2 (x-\beta y)^2 \Theta \end{vmatrix} = 0,$$

which is of the form

$$(x-\alpha y)(x-\beta y)\Omega=0;$$

or, we have $x-\alpha y=0$, or $x-\beta y=0$, a pair of critic centres; and for each of these the corresponding value of k_1 given by the equation $-k:1=\delta_x U:\delta_x V$ is $k=k_1$, so that $k=k_1$ is a twofold root of the equation in k.

12. By the like considerations as for the threefold root (observing that if the equation $W=0$ has a pair of twofold roots we must have between the coefficients of W a plexus equivalent to two relations, and of the order $2(n-2)(n-3)$), we see that the condition for the existence of a range $U+kV=0$ having a pair of twofold points is of the form $P=0$, where P is a function of the degree $2(n-2)(n-3)$ as regards the coefficients of U, and of the same degree as regards the coefficients of V; and conversely that, given the equation $P=0$, we may find the plexus.

13. The equation $\square=0$ will be satisfied if $R=0$, or if $Q=0$, or if $P=0$; *and in no other cases.* To prove this, suppose that $x-\alpha y=0$ is the critic centre corresponding to a twofold root k_1 of the equation in k. We have $U=-k_1V+(x-\alpha y)^2\Theta$, and thence the equation for the critic centres is

$$\begin{vmatrix} \delta_x V, & \delta_x (x-\alpha y)^2 \Theta \\ \delta_y V, & \delta_y (x-\alpha y)^2 \Theta \end{vmatrix} = 0,$$

which is an equation of the form $(x-\alpha y)\Omega=0$; and where, corresponding to the root $x-\alpha y=0$, the equation $-k:1=\delta_x U:\delta_x V$ gives $k=k_1$. Since k_1 is a twofold root, there must be another critic centre also giving the value k_1 of k. This new critic centre may be either $x-\alpha y=0$ (the same as the first mentioned critic centre) or it may be a distinct critic centre $x-\beta y=0$. In the former case

$$\begin{vmatrix} \delta_x V, & \delta_x (x-\alpha y)^2 \Theta \\ \delta_y V, & \delta_y (x-\alpha y)^2 \Theta \end{vmatrix}$$

must contain, instead of the factor $(x-\alpha y)$, the factor $(x-\alpha y)^2$. In order that this may be so, we must have

$$\left|\begin{array}{ll}\delta_x V, & \Theta \\ \delta_y V, & -\alpha\Theta\end{array}\right|,$$

that is, $(\alpha\delta_x V+\delta_y V)\Theta$ divisible by $(x-\alpha y)$, that is, either $\alpha\delta_x V+\delta_y V$, or else Θ, divisible by $x-\alpha y$; or, what is the same thing, either V, or else Θ, divisible by $x-\alpha y$. But if V be divisible by $x-\alpha y$, then U, V have the common factor $x-\alpha y$, and we have the case *first* above considered. And again if Θ contain the factor $x-\alpha y$, then we have

$$U=-k_1 V+(x-\alpha y)^3\Theta',$$

and we have the case *secondly* above considered. Finally if the new critic centre be the distinct centre $x-\beta y=0$, then for $x-\beta y=0$ the equation

$$-k\;:\;1=\delta_x U\;:\;\delta_x V=\delta_y U\;:\;\delta_y V$$

should give $k=k_1$; but this will only happen if $\delta_x.(x-\alpha y)^2\Theta$, $\delta_y.(x-\alpha y)^2\Theta$ vanish for $x-\beta y=0$, that is, if Θ contains the factor $(x-\beta y)^2$; and when this is so,

$$U=-k_1 V+(x-\alpha y)^2(x-\beta y)^2\Theta',$$

or we have the case *thirdly* above considered.

14. Hence the equation $\square=0$ being satisfied if $R=0$, or else if $Q=0$, or else if $P=0$, and in no other cases, the function $\square$ must be made up of the factors R, Q, P, each taken the proper number of times, and knowing the degrees of the several functions, it follows that we must have

$$\square=RQ^3P^2,$$

in fact, considering the coefficients of either U or V, the comparison of the degrees gives

$$2(n-1)(2n-3)=n+9(n-2)+4(n-2)(n-3),$$

where the function on the right-hand side is

$$\begin{aligned} = &\quad n \\ &+\;9n-18 \\ &+4n^2-20n+24 \\ = &\;\;4n^2-10n+\;6, \end{aligned}$$

which is the value of the function on the left-hand side.

15. In the very particular case $n=2$, Q and P are each of them of the degree $=0$; and we have simply $\square=R$, that is, the resultant of the two quadric functions

$$U=(a,\,b,\,c\between x,\,y)^2,\;V=(a',\,b',\,c'\between x,\,y)^2$$

is

$$\begin{aligned} &=\text{Disc}^t.\;(U+kV) \\ &=\text{Disc}^t.\;(ac-b^2,\;ac'+a'c-2bb',\;a'c'-b'^2\between 1,\,k)^2, \end{aligned}$$

which is Prof. Boole's ancient theorem referred to in my Fifth Memoir on Quantics[1], but which is now first exhibited in connexion with the general theory to which it belongs.

16. It may be noticed that the condition for a twofold critic centre, or (what is the same thing) a twofold factor of the Jacobian {which condition is of the degree $2(2n-3)$ in regard to the coefficients of U or V} is $RQ=0$; and that we in fact have

$$2(2n-3)=n+3(n-2).$$

This remark is due to Dr Salmon.

Article, Nos. 17 to 42, *relating to two Ternary Quantics.*

17. Suppose now that $U=(a,\ldots \between x, y, z)^n$, $V=(a',\ldots \between x, y, z)^n$ are two ternary quantics of the same order n, and write $W=U+kV=(a+ka',\ldots \between x, y, z)^n$, so that

$$W=U+kV=0$$

is the equation of a curve in involution with the curves $U=0$, $V=0$. But for greater distinctness it is in general proper to retain $U+kV$ in place of W.

18. In order that the curve $U+kV=0$ may have a node, we must have simultaneously

$$\delta_x(U+kV)=0,$$
$$\delta_y(U+kV)=0,$$
$$\delta_z(U+kV)=0,$$

and eliminating (x, y, z) from these equations we have

$$\text{Disc}^t.\,(U+kV)=0,$$

which is an equation of the degree $3(n-1)^2$ as regards k, and of the same order as regards the coefficients of U, V conjointly.

19. To each of the $3(n-1)^2$ values of k there corresponds a point satisfying the conditions in question, and which is therefore a node of the corresponding nodal curve

$$U+kV=0;$$

the points in question are the critic centres of the involution.

20. The critic centres may be differently obtained as follows; viz. if from the three equations we eliminate k, we find

$$\left\|\begin{matrix}\delta_x U, & \delta_y U, & \delta_z U\\ \delta_x V, & \delta_y V, & \delta_z V\end{matrix}\right\|=0,$$

[1] *Phil. Trans.* vol. CXLVIII. (1858), pp. 415—427, [156].

a plexus of three curves, each of them of the order $2(n-1)$; any two of the three curves intersect in $4(n-1)^2$ points; but $(n-1)^2$ of these do not lie on the third curve; the remaining $3(n-1)^2$ of them lie on all three of the curves, and they are the critic centres of the involution.

21. More generally the critic centres lie on any curve whatever of the form

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \delta_x U, & \delta_y U, & \delta_z U \\ \delta_x V, & \delta_y V, & \delta_z V \end{vmatrix} = 0,$$

and any such curve, viz. any curve of the order $2(n-1)$ passing through the $3(n-1)^2$ critic centres, may be termed a diacritic curve.

22. For any one of the critic centres we have

$$\delta_x U : \delta_y U : \delta_z U = \delta_x V : \delta_y V : \delta_z V = k : -1,$$

which gives the value of k corresponding to the point in question.

23. The condition in order that the equation in k may have a twofold root is

$$\text{Disc}^t. \ \text{Disc}^t. \ (U+kV) = 0,$$

or say

$$\square = 0,$$

where $\text{Disc}^t. \ \text{Disc}^t. \ (U+kV), = \square$, is a function of the degree $2 \,.\, 3(n-1)^2\{3(n-1)^2-1\}$ in regard to the coefficients of U, V conjointly; but it is separately homogeneous, and therefore of the degree $3(n-1)^2\{3(n-1)^2-1\}$ in regard to each set of coefficients.

24. To each of the critic centres there corresponds a value of k. Hence if two of the critic centres coincide, or say if there is a twofold critic centre, the equation in k will have a twofold root. Now *first* if the curves $U=0$, $V=0$ touch each other (have a point of contact or twofold intersection) then the diacritic curves will all touch (have a point of twofold intersection) at the point in question, which is therefore a twofold critic centre. It may be remarked in passing that the diacritic curves do not at the twofold critic centre touch the curves $U=0$, $V=0$. But *secondly* the diacritic curves may touch at a point which is not a point of contact of the curves $U=0$, $V=0$. Such a point is a twofold critic centre. In each of these two cases the equation in k has a twofold root. Moreover, in the first case the curve $U+kV=0$ corresponding to the twofold root has a node at the point of contact of the two curves $U=0$, $V=0$; in the second case the curve $U+kV=0$ corresponding to the twofold root has the twofold centre (not a mere node but) a cusp. And *thirdly*, without any twofold critic centre, two distinct critic centres may give by the equations

$$\delta_x U : \delta_y U : \delta_z U = \delta_x V : \delta_y V : \delta_z V = k : -1$$

the same value of k, and then the curve $U+kV=0$ corresponding to such value of k is a curve having a node at each of the critic centres in question, that is, it has two nodes.

25. *First*, if the curves $U=0$, $V=0$ touch each other, then, (x, y, z) being the coordinates of the point of contact, we have $U=0$, $V=0$,

$$\delta_x(U+k_1V)=0,$$
$$\delta_y(U+k_1V)=0,$$
$$\delta_z(U+k_1V)=0,$$

where k_1 denotes the value given by the equations

$$\delta_xU : \delta_yU : \delta_zU = \delta_xV : \delta_yV : \delta_zV = k_1 : -1$$

belonging to the point of contact. It at once follows that every diacritic curve passes through the point in question. But it is somewhat more difficult to show that the diacritic curves touch at this point.

26. I represent for shortness the first and second differential coefficients of U by (L, M, N), (a, b, c, f, g, h), and similarly those of V by (L', M', N'), (a', b', c', f', g', h'), these values all belonging to the point of contact: we have therefore

$$L+k_1L'=0,\ M+k_1M'=0,\ N+k_1N'=0.$$

The equation of the diacritic curve is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ L', & M', & N' \end{vmatrix}=0;$$

to find the tangent we must operate on the left-hand side with $X\delta_x+Y\delta_y+Z\delta_z$, where X, Y, Z are current coordinates. Calling the foregoing symbol D, this gives

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ DL', & DM', & DN' \end{vmatrix} + \begin{vmatrix} \alpha, & \beta, & \gamma \\ DL, & DM, & DN \\ L', & M', & N' \end{vmatrix}=0;$$

or, what is the same thing,

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ L, & M, & N \\ DL', & DM', & DN' \end{vmatrix} - \begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ DL, & DM, & DN \end{vmatrix}=0;$$

or, substituting in the first determinant for L, M, N their values $-kL', -kM', -kN'$, and transferring the factor k_1 from the second to the third line, we obtain

$$-\begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ k_1DL', & k_1DM', & k_1DN' \end{vmatrix} - \begin{vmatrix} \alpha, & \beta, & \gamma \\ L', & M', & N' \\ DL, & DM, & DN \end{vmatrix}=0,$$

which may also be written

$$\begin{vmatrix} \alpha\ , & \beta\ , & \gamma \\ L'\ , & M'\ , & N' \\ DL + k_1 DL'\ , & DM + k_1 DM'\ , & DN + k_1 DN \end{vmatrix} = 0$$

or, more symmetrically, θ being any quantity whatever, it is

$$\begin{vmatrix} \alpha\ , & \beta\ , & \gamma \\ L + \theta L'\ , & M + \theta M'\ , & N + \theta N' \\ DL + k_1 DL'\ , & DM + k_1 DM'\ , & DN + k_1 DN' \end{vmatrix} = 0;$$

or, substituting for D its value, the equation of the tangent is

$$\begin{vmatrix} \alpha\ , & \beta\ , & \gamma \\ L + \theta L'\ , & M + \theta M'\ , & N + \theta N' \\ (a + ka', .\,)(X,\ Y,\ Z),\ & (h + kh', .\,)(X,\ Y,\ Z),\ & (g + kg', .\,)(X,\ Y,\ Z) \end{vmatrix} = 0.$$

27. Now if the diacritics touch, this equation should be independent of α, β, γ. Putting for shortness $a + k_1 a' = a$, &c., and also taking as we may do $\theta = 0$, the parts multiplied by α, β, γ respectively are

$$M(gX + fY + cZ) - N(hX + bY + fZ),$$
$$N(aX + hY + gZ) - L(gX + fY + cZ),$$
$$L(hX + bY + fZ) - M(aX + hY + gZ),$$

and we ought therefore to have

$$\begin{aligned} & Mg - Nh \ : \ Mf - Nb \ : \ Mc - Nf \\ = {} & Na - Lg \ : \ Nh - Lf \ : \ Ng - Lc \\ = {} & Lh - Ma \ : \ Lb - Mh \ : \ Lf - Mg, \end{aligned}$$

equations which are in fact satisfied; for take any one of them, for example, the equation

$$\frac{Mg - Nh}{Na - Lg} = \frac{Mf - Nh}{Nh - Lf},$$

this is

$$MN(gh - af) - N^2(h^2 - ab) - LM(fg - fg) + LN(fh - bg) = 0,$$

or, omitting the term in LM, and throwing out the factor N,

$$L(hf - bg) + M(gh - af) + N(ab - h^2) = 0.$$

But the equations $L + kL' = 0$, &c., give

$$ax + hy + gz = 0,$$
$$hx + by + fz = 0,$$
$$gx + fy + cz = 0,$$

that is

$$\begin{aligned} x : y : z &= bc - f^2 : fg - ch : hf - bg \\ &= fg - ch : ca - g^2 : gh - af \\ &= hf - bg : gh - af : ah - h^2, \end{aligned}$$

so that the above written equation is $Lx + My + Nz = 0$, which is true in virtue of the equation $U = 0$; and similarly for all the other equations which were to be verified.

28. It is to be noticed that the determination of the tangent of the diacritics depends only on the second differential coefficients (a, b, c, f, g, h), (a', b', c', f', g', h'), of U, V. The tangent in question will be the same if instead of the curves, $U = 0$, $V = 0$ we have the conics $(a, b, c, f, g, h \between x, y, z)^2 = 0$, $(a', b', c', f', g', h' \between x, y, z)^2 = 0$: these conics pass through the point of contact of the two curves, and their tangents are coincident with those of the two curves $U = 0$, $V = 0$ respectively; that is, the conics touch at the point in question. They consequently intersect in two more points; the chord of intersection or line joining the last-mentioned two points, meets the common tangent in a point; the polars of this point in regard to the two conics respectively, pass through the point of contact, and moreover they are one and the same line; this line is the required tangent of the diacritics. The proof will be given, *post*, No. 41.

29. Let $R = 0$ be the condition in order that the two curves $U = 0$, $V = 0$ may touch each other, or say let R be the Tactinvariant of U, V. When the curves U, V are of the degrees m, n respectively, then R is of the degrees $n(2m + n - 3)$, $m(m + 2n - 3)$ in regard to the coefficients of U, V respectively. Hence in the present case where U, V are each of the degree n, R is of the degree $3n(n - 1)$ in regard to each set of coefficients.

30. *Secondly*, if the functions U, V are such that there exists a curve $U + kV = 0$ (say the curve $U + k_1V = 0$) which has a cusp, then it is to be shown that $k = k_1$ is a twofold root of the equation in k; and to do this it has to be shown that the cusp is a twofold critic centre; or that the diacritic curves touch at the cusp: it may be added that the cuspidal tangent is the common tangent of the diacritic curves. Now the cuspidal curve being $U + k_1V = 0$, then at the cusp the first derived functions $L + k_1L'$, $M + k_1M'$, $N + k_1N'$ vanish identically; and moreover the second derived functions $a + ka', \ldots$ are such that (X, Y, Z) being any magnitudes whatever, $(a + k_1a', \ldots \between X, Y, Z)^2$ is a perfect square, $= (\lambda X + \mu Y + \nu Z)^2$ suppose. Now X, Y, Z being current coordinates, and D denoting the operation $D = X\delta_x + Y\delta_y + Z\delta_z$, the equation of the tangent to the diacritic curve (by an investigation similar to that for this same tangent in the case first above considered) is found to be

$$\left| \begin{array}{ccc} \alpha\ , & \beta\ , & \gamma \\ L + \theta L'\ , & M + \theta M'\ , & N + \theta N' \\ (a + k_1a', \ldots \between X, Y, Z), & (h + k_1h', \ldots \between X, Y, Z), & (g + k_1g', \ldots \between X, Y, Z) \end{array} \right| = 0,$$

and we have

$$(a+k_1a',..\between X,\ Y,\ Z)=\tfrac{1}{2}\delta_X(a+k_1a',..\between X,\ Y,\ Z)^2=\tfrac{1}{2}\delta_X(\lambda X+\mu Y+\nu Z)^2=\lambda(\lambda X+\mu Y+\nu Z):$$

and similarly the values of $(h+k_1h',..\between X, Y, Z)$ and $(g+k_1g',..\between X,\ Y,\ Z)$ are

$$=\mu(\lambda X+\mu Y+\nu Z) \text{ and } =\nu(\lambda X+\mu Y+\nu Z) \text{ respectively.}$$

Hence the equation of the tangent to the diacritic curve is

$$\lambda X+\mu Y+\nu Z=0,$$

that is, the tangent being independent of the values of $(\alpha,\ \beta,\ \gamma)$ is the same for all the diacritic curves, and is the tangent at the cusp of the cuspidal curve $U+k_1V=0$.

31. The conditions in order that the curve $W=U+kV=0$ may have a cusp are given by a plexus equivalent to three relations between the coefficients $a+ka',..$ of W, and using for a moment β to denote the degree of the plexus or system, then eliminating k between the equations of the plexus we find between the coefficients $a,...$ of U and the coefficients $a',...$ of V an equation $Q=0$ of the degree β in regard to the two sets of coefficients respectively. Conversely, given the equation $Q=0$, we may find the plexus between the coefficients $a+ka',...$ of W. The value of β, as will be shewn *post*, Annex, is

$$=12(n-1)(n-2).$$

32. *Thirdly*, when the functions U, V are such that there exists a curve $U+kV=0$ (suppose the curve $U+k_1V=0$) which has a pair of nodes; each of these nodes is a critic centre, and (by means of the equation $-k:1=\delta_xU:\delta_xV$) gives the value k_1 of k, that is, k_1 is a twofold root of the equation in k.

33. The conditions in order that the curve $W=U+kV=0$ may have a pair of nodes are given by a plexus of the degree α; then the coefficients being $a+ka',...$ if we eliminate k between the equations of the plexus, we find between the coefficients $a,...$ of U and $a',...$ of V an equation $P=0$ of the degree α in the two sets of coefficients respectively. And conversely, given the equation $P=0$, we may find the plexus between the coefficients $a+ka',...$ of W. I have not succeeded in finding directly the value of α, but only derive it from the equation $\square=RQ^3P^2$, which if α had been found independently, would have been verified by means of such value of α; the value is $\alpha=\tfrac{1}{2}.3(n-1)(n-2)(3n^2-3n-11)$.

34. The equation $\square=0$ is satisfied if $R=0$, or if $Q=0$, or if $P=0$, and it may be seen that it is not satisfied in any other case. Hence $\square$ is made up of the factors P, Q, R, and I assume that its form is the same as in the case of a binary quantic, that is, that we have

$$\square=RQ^3P^2.$$

35. Comparing the degrees of the two sides we have

$$\begin{aligned}3(n-1)^2(3n^2-6n+2) = {} & 3n(n-1) \\ & + 36(n-1)(n-2) \\ & + 3(n-1)(n-2)(3n^2-3n-11);\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}(n-1)(3n^2-6n+2) = {} & n \\ & + 12(n-2) \\ & + (n-2)(3n^2-3n-11),\end{aligned}$$

which is true, but, as just remarked, this equation itself was used to find the value

$$\alpha = \tfrac{1}{2}\,.\,3(n-1)(n-2)(3n^2-3n-11).$$

36. Recapitulating, the equation in k will have a twofold root

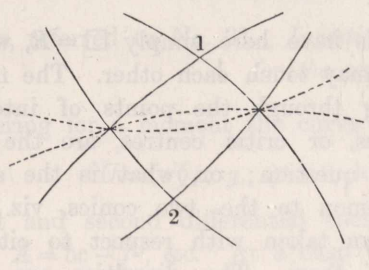
1°, if $R=0$, that is, if the curves $U=0$, $V=0$ touch each other, and in this case there is a twofold critic centre at the point of contact:

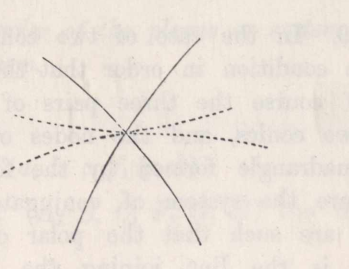
2°, if $Q=0$, that is, if there be a curve $U+kV=0$ having a cusp, and in this case the cusp is a twofold critic centre:

3°, if $P=0$, that is, if there is a curve $U+kV=0$ having a pair of nodes.

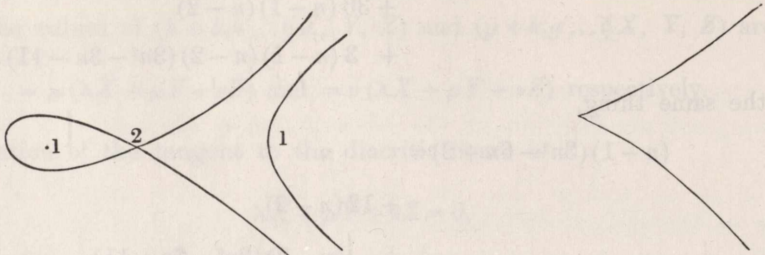
37. The three cases may be geometrically illustrated by supposing that the curves $U=0$, $V=0$ are in the first instance nearly, but not exactly, in the several relations in question.

First, if the curves $U=0$, $V=0$ are about to touch each other, that is, if there are two points of intersection about to coincide with each other. There are here two critic centres in the neighbourhood of the two points of intersection, and which, when the two points of intersection become a point of contact, coincide each with the point of contact.

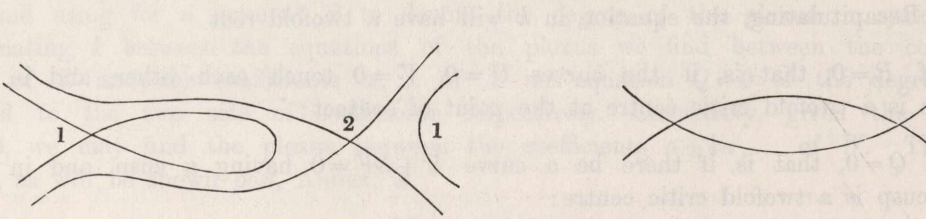




Secondly, when the curves are such that there are two critic centres which become ultimately a twofold centre.



And, *thirdly*, when the curves are such that there are two critic centres which remaining distinct from each other belong ultimately to the same critic curve.



38. The curves 1 and 2 in the left-hand figures respectively represent the nodal curves corresponding to slightly different values of k, which in the right-hand figures respectively give the curve corresponding to a twofold value of k. In the first pair of figures the curves $U=0$ and $V=0$, about to touch in the left-hand figure, touching in the right-hand figure, are shown by dotted lines. It will be observed that in the second case in the left-hand figure the two nodes which give rise to a cusp are the one of them an acnode and the other a crunode; this is in fact the only mode of drawing the figure so that a cusp shall present itself. The transition of form is one of ordinary occurrence in cubic curves and in curves of a higher order; thus if $y^2=(x-a)(x-b)(x-c)$, where $a<b<c$, then if $a=b$, we have an acnodal curve, if $b=c$ a crunodal curve, and if $a=b=c$ a cuspidal curve.

39. In the case of two conics, $n=2$. We have here simply $\square = R$, where $R=0$ is the condition in order that the two conics may touch each other. The nodal curves are of course the three pairs of lines passing through the points of intersection of the two conics, and the nodes of these curves, or critic centres, are the centres of the quadrangle formed by the four points in question; or, what is the same thing, they are the system of conjugate points common to the two conics, viz. the points which are such that the polar of one of them taken with respect to either of the conics is the line joining the other two of them. The diacritics are any conics passing through the three points.

40. If the two conics are

$$(a\,,\ b\,,\ c\,,\ f\,,\ g\,,\ h\between x,\ y,\ z)^2=0,$$
$$(a',\ b',\ c',\ f',\ g',\ h'\between x,\ y,\ z)^2=0,$$

and if the determinant formed with $a+ka'$, &c., is denoted by

$$(K,\ \Theta,\ \Theta',\ K'\between 1,\ k)^3,$$

so that

$$\begin{aligned} K &= abc \quad - af^2 - bg^2 - ch^2 + 2fgh,\\ \Theta &= a'(bc - \ f^2) + \&c.,\\ \Theta' &= a(b'c' - \ f'^2) + \&c.,\\ K' &= a'b'c' - a'f'^2 - b'g'^2 - c'h'^2 + 2f'g'h', \end{aligned}$$

then the before-mentioned equation $\square = R = 0$ which gives the condition that the conics may touch is

$$\text{Disc}^t.\ (K,\ \Theta,\ \Theta',\ K'\between 1,\ k)^2=0,$$

where the left-hand side is of the order 6 in the coefficients of the two conics respectively: this is a known formula.

41. If the equation in k have a twofold root the two conics will touch: two of the critic centres will then coincide at the point of contact, or this point is a twofold critic centre: the remaining or onefold critic centre is the intersection of the common tangent and of the line joining the two points of intersection of the conics. In virtue of the general property, the first-mentioned two centres must be considered as lying on the line which is the polar of the onefold critic centre in regard to either of the conics. The diacritics pass through the critic centres, that is, they pass through the onefold centre, and touch the polar in question at the point of contact of the two conics, or twofold critic centre; this is in fact the property mentioned *ante*, No. 28.

42. The equation

$$\text{Disc}^t.\ \text{Disc}^t.\ (U+kV)=0,$$

as applied to a conic and a circle leads at once to the equation of the curve parallel to a given conic; such parallel curve is in fact the envelope of the circles of a given constant radius which touch the given conic. This method is in effect due to Dr Salmon, who applied the corresponding theorem *in solido* to the determination of the surface parallel to an ellipsoid.

Annex, referred to No. 31. *Investigation of the order of the plexus or system for the existence of a Cusp.*

Considering for a moment the curve

$$U=(*\between x,\ y,\ z)^n=0, \text{ let } (L,\ M,\ N),\ (a,\ b,\ c,\ f,\ g,\ h)$$

be the first and second differential coefficients of U; $(A,\ B,\ C,\ F,\ G,\ H)$ the inverse system, viz. $A=bc-f^2$, &c. At a cusp we have

$$L=0,\ M=0,\ N=0,\ A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0,$$

a system of equations which is contained in the system, $L=0$, $M=0$, $N=0$, $A=0$. But this system contains besides the cusp system the irrelevant system $L=0$, $M=0$, $N=0$, $x^2=0$. In fact the equations $L=0$, $M=0$, $N=0$ give

$$ax+hy+gz=0,$$
$$hx+by+fz=0,$$
$$gx+fy+cz=0,$$

and thence

$$x^2 : y^2 : z^2 : yz : zx : xy = A : B : C : F : G : H.$$

Hence the equation $A=0$, if $x^2=0$, implies the entire system

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0.$$

But if $L=0$, $M=0$, $N=0$, $x^2=0$, then these equations give $A=0$ (and also $H=0$, $G=0$), but they do not give the remaining equations $B=0$, $C=0$, $F=0$. Or the same thing may be shown in a less symmetrical form, but more clearly thus; we have identically

$$-cx(ax+hy+gz)+(hx-fz)(hx+by+fz)+bz(gx+fy+cz)-(bc-f^2)z^2+(ab-h^2)x^2=0,$$

whence the equations $L=0$, $M=0$, $N=0$, $A=0$ give $Cx^2=0$, that is, $C=0$, or $x^2=0$. But the equations $L=0$, $M=0$, $N=0$, $A=0$, $C=0$ give (as it is easy to show) the entire system $A=0$, $B=0$, $C=0$, $F=0$, $G=0$, $H=0$. That is, the system

$$L=0,\ M=0,\ N=0,\ A=0$$

is made up of the cusp system, and of the system ($L=0$, $M=0$, $N=0$, $A=0$, $x^2=0$); or since $A=0$ is a consequence of the other equations, the second system is

$$(L=0,\ M=0,\ N=0,\ x^2=0).$$

Consider now the curve $\lambda U+\mu U'+\nu U''=0$, which will have a cusp if the ratios $\lambda : \mu : \nu$ are properly determined. And to each set of values of $\lambda : \mu : \nu$ there corresponds a set of values (x, y, z), the coordinates of a cusp of the curve; so that the number of such sets, that is, the number of points each whereof is the cusp of a corresponding curve $\lambda U+\mu V+\nu W=0$ is precisely equal to the number of sets of values of $\lambda : \mu : \nu$: or it is equal to the order of the system of conditions for the existence of a cusp.

Denoting as before the first and second differential coefficients of U by

$$L,\ M,\ N,\ a,\ b,\ c,\ f,\ g,\ h,$$

and those of U', U'' in a corresponding manner, and taking for the cusp the system before represented by $L=0$, $M=0$, $N=0$, $A=0$, we have

$$\lambda L+\mu L'+\nu L''=0,$$
$$\lambda M+\mu M'+\nu M''=0,$$
$$\lambda N+\mu N'+\nu N''=0,$$
$$(\lambda b+\mu b'+\nu b'')(\lambda c+\mu c'+\nu c'')-(\lambda f+\mu f'+\nu f'')^2=0,$$

which last equation, to denote that it is of the second order in regard to the differential coefficients a, b, &c., a', &c. I represent by

$$((a, b, \dots)^2 \between \lambda, \mu, \nu)^2 = 0.$$

But this system of four equations contains not only the cusp system, but the system made of the three linear equations and the equation $x^2 = 0$. Eliminating λ, μ, ν, the last-mentioned system is

$$\begin{vmatrix} L, & L', & L'' \\ M, & M', & M'' \\ N & N' & N'' \end{vmatrix} = 0, \quad x^2 = 0,$$

where the first equation is that of a curve of the order $3(n-1)$. And the two equations give together $6(n-1)$ points, viz. the points of intersection of the curve by the line $x = 0$, each reckoned as a twofold point.

Returning to the first-mentioned system, this may be replaced by

$$\begin{vmatrix} L, & L', & L'' \\ M & M' & M'' \\ N & N' & N'' \end{vmatrix} = 0, \quad ((a, b, \dots)^2 \between L'M'' - L''M', \; L''M - LM'', \; LM' - L'M)^2 = 0,$$

which are curves of the orders $3(n-1)$ and $6n-8$ respectively. But each of these curves passes through the $3(n-1)^2$ points given by the equations

$$\begin{Vmatrix} L, & L', & L'' \\ M, & M', & M'' \end{Vmatrix} = 0,$$

and these points are moreover nodes on the curve of the order $6n-8$; hence the points in question reckon as $6(n-1)^2$ intersections of the two curves. The number of the remaining intersections is

$$3(n-1)(6n-8) - 6(n-1)^2 = 6(n-1)\big(3n-4-(n-1)\big) = 6(n-1)(2n-3),$$

but among these are included the $6(n-1)$ intersections of the curve of the order $3(n-1)$ by the twofold line $x^2 = 0$; or, subtracting these, the number of the remaining points is

$$6(n-1)(2n-3-1) = 12(n-1)(n-2);$$

which number is consequently the order of the cusp system.

It may be remarked that considering the entire series of equations at first denoted by $(L=0, M=0, N=0)$, $(A=0, B=0, C=0, F=0, G=0, H=0)$, the elimination of λ, μ, ν from the three linear equations gives as before

$$\begin{vmatrix} L, & L', & L'' \\ M, & M' & M'' \\ N & N' & N'' \end{vmatrix} = 0,$$

which is a curve of the order $3(n-1)$: and the eliminating of λ^2, μ^2, ν^2, $\mu\nu$, $\nu\lambda$, $\lambda\mu$ from the six quadric equations gives

$$\left|\begin{array}{l} bc-f^2,\ b'c'-f'^2,\ b''c''-f''^2,\ b'c''+b''c'-2f'f'',\ b''c+bc''-2ff'',\ bc'+b'c-2ff' \\ ca-g^2,\ \&c., \\ \vdots \end{array}\right|=0,$$

which is a curve of the order $12(n-2)$; the two curves would intersect in $36(n-1)(n-2)$ points, but as this is precisely three times the number $12(n-1)(n-2)$, I infer that these are in fact the $12(n-1)(n-2)$ points three times repeated, that is, that each of these is a point of threefold intersection of the two curves.

Cambridge, 7th November, 1863.