350.

ON THE CLASSIFICATION OF CUBIC CURVES.

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THE notion of a curve of a given order may be considered as arising from Descartes' invention of his method of coordinates; and one of the earliest applications of the method was made by Sir Isaac Newton in the *Enumeratio linearum tertii Ordinis* (1706), a work worthy of its author, and which opened a new field of geometrical science. The classification is according to the nature of the infinite branches; there are fourteen genera containing together seventy-two species, but four species were added by Stirling in his *Lineæ tertii Ordinis Newtonianæ; sive Illustratio &c.* (1717), and two more by Murdoch or Cramer[1], making in all seventy-eight species. A new classification was made by Plücker in his *System der Analytischen Geometrie,* 1835; this is likewise according to the nature of the infinite branches, but after his six head divisions, and some subordinate divisions thereof, Plücker establishes the divisions called Groups, which have nothing analogous to them in the Newtonian theory; there are sixty-one groups, and the total number of species is 219.

The present Memoir contains an exposition of the foregoing classifications, and of the principles on which they are founded, in so far as relates to the superior divisions of the two classifications: and in particular I develope more completely than was done by Plücker the theory of the division into groups. I do not however consider otherwise than very slightly the ultimate division into species.

The above-mentioned work of Newton contains, under the heading "Genesis Curvarum per Umbras," the remarkable theorem that the curves of the third order may all of them be considered as the shadows of the five Divergent Parabolas; I reserve for a separate Memoir the whole series of considerations to which this theorem gives rise.

[1] The two additional species are, I believe, first mentioned in Murdoch's *Genesis Curvarum per Umbras* (1746), but one of them is there ascribed to Cramer.

I commence by establishing the theory of the classification of cubic curves according to the nature of their infinite branches, in what appears to me the scientifically correct manner as follows:

The Seven Head Divisions, Article Nos. 1 to 4.

1. A line in general, and therefore the line Infinity, meets a cubic curve in three points, and these may be

Three onefold points,

A twofold point and a onefold (or, as it may also be termed, a one-with-twofold) point,

A threefold point.

2. But in the second case the line Infinity

may be a proper tangent to the curve,

may pass through a node,

may pass through a cusp;

and in the third case the line Infinity

may touch the curve at an inflexion,

may at a node touch one of the two branches,

may touch the curve at a cusp.

3. The first case, the three divisions of the second case, and the three divisions of the third case, give in all seven divisions, which, as will appear in the sequel, fall in with Newton's classification, and can be named in his language, viz.

Three onefold points,		The Hyperbolas.
A onefold and a twofold point;		
	Infinity a proper tangent,	The Parabolic Hyperbolas.
	Do. through a node,	The Central Hyperbolisms.
	Do. through a cusp,	The Parabolic Hyperbolisms.
A threefold point;		
	Infinity a tangent at an inflexion,	The Divergent Parabolas.
	Do. Do. at a node, to one branch,	The Trident Curve.
	Do. Do. at a cusp,	The Cubical Parabola.

4. As regards the signification of these terms, it may be remarked that the Hyperbolas have hyperbolic branches, the Parabolic Hyperbolas, hyperbolic and parabolic branches; where by a hyperbolic branch is meant one having an asymptote, and by

a parabolic branch one not having an asymptote. The hyperbolism of any curve is the curve derived from it by altering the ordinate in the ratio of the abscissa to any given line

$$\left(y' = \frac{m}{x} y, \text{ or say } y' = \frac{y}{x}\right);$$

the expression Central Hyperbolism is used to include Newton's hyperbolisms of the hyperbola and ellipse; and the expression Parabolic Hyperbolism to denote his hyperbolism of the parabola. The Divergent Parabolas are curves the branches of which ultimately diverge from each other as in the semicubical parabola $y^2 = x^3$, which is in fact one of these curves. The names Trident Curve and Cubical Parabola are not generic but specific; it so happens that the genera to which they respectively belong contain each only a single species. The names for the several kinds of curves are not scientifically-devised ones, but it is convenient to have them such as they are.

The foregoing seven divisions, uniting in one the Central Hyperbolisms and the Parabolic Hyperbolisms, are the six head divisions of Plücker.

Asymptotes, &c. Equations for the Seven Head Divisions. Article Nos. 5 to 22.

5. For a Hyperbola there is at each of the points at infinity a tangent, which is an asymptote; and the hyperbola has thus three asymptotes.

6. For a Parabolic Hyperbola there is at the onefold point at infinity a tangent, which is an asymptote. There may be described a conic having with the curve at the twofold point at infinity a five-pointic intersection(1). Such conic, as having the line infinity for a tangent, is a parabola, and it may be termed the asymptotic parabola: the Parabolic Hyperbola has thus an asymptote and an asymptotic parabola.

7. For a Central Hyperbolism there is at the onefold point at infinity a tangent which is an asymptote, and which for distinction may be called the onefold asymptote; and at the node or twofold point at infinity there is a pair of tangents which are the parallel asymptotes.

8. For a Parabolic Hyperbolism there is at the onefold point at infinity a tangent which is an asymptote, and which may be called the onefold asymptote; and at the cusp or twofold point at infinity a twofold tangent which is an asymptote, and which may be called the twofold asymptote.

9. For a Divergent Parabola there is not any asymptote or asymptotic conic; but we may consider an asymptotic cubic, viz. this will be a semicubical parabola ($y^2 = x^3$), which is in fact one of the divergent parabolas, the cuspidal divergent parabola, and which may be in general so determined as to have at the inflexion or threefold point

[1] I have elsewhere spoken of the conic of five-pointic contact: the expression five-pointic *intersection* is more accurate.

at infinity a seven-pointic intersection. For the asymptotic cubic, in order that it may have a cusp, must satisfy two conditions; it may therefore be made to satisfy seven more conditions, or to have a seven-pointic intersection; and then the original curve having an inflexion or threefold point at infinity, the asymptotic cubic will *ipso facto* have the same point as an inflexion, or threefold point at infinity, and be thus a cuspidal divergent parabola.

10. For the Trident Curve, we have at the node or threefold point at infinity, viz. to the branch which is not touched by the line infinity, a tangent which is an asymptote: this cuts at the node the other branch of the curve, and it is therefore an asymptote of three-pointic intersection. We may describe a conic having at the node a five-pointic intersection with the other branch of the curve; such conic as touching the line infinity is a parabola, and it may be called the asymptotic parabola; since the parabola cuts at the node the first-mentioned branch of the curve, viz. the branch not touched by the line infinity, the parabola is in fact a parabola of six-pointic intersection. The Trident Curve has thus an asymptote and an asymptotic parabola of six-pointic intersection.

11. For the Cubical Parabola there is not any asymptote or asymptotic conic: the curve *quâ* curve having a cusp (viz. the cusp or threefold point at infinity) has a single inflexion; and the line joining the cusp with the inflexion, regarded as a threefold line, has with the curve a six-pointic intersection at infinity, and may be considered as an asymptotic cubic.

12. We have in every case a cubic curve $V=0$ having with the original curve an intersection at infinity which is at least six-pointic, and which I call the asymptotic aggregate: viz. the asymptotic aggregate is

For the Hyperbolas; the three asymptotes, intersection six-pointic.

For the Parabolic Hyperbolas; the asymptote and the asymptotic parabola, intersection seven-pointic.

For the Central Hyperbolisms; the onefold asymptote and the parallel asymptotes, intersection eight-pointic.

For the Parabolic Hyperbolisms; the onefold asymptote and the twofold asymptote regarded as a twofold line; intersection eight-pointic.

For the Divergent Parabolas; the asymptotic semicubical parabola, intersection seven-pointic.

For the Trident Curve; the asymptote and the asymptotic parabola, intersection nine-pointic.

For the Cubical Parabola; the line joining the cusp at infinity with the inflexion, regarded as a threefold line, intersection six-pointic.

13. I have said that the intersection at infinity is at least six-pointic; but more than this, the intersection at any onefold point at infinity is at least two-pointic; at

a twofold point at infinity it is at least four-pointic; and at a threefold point at infinity it is at least six-pointic.

It follows that the intersections at infinity of the cubic and the asymptotic aggregate include the six intersections of the cubic by the line infinity considered as a twofold line; and hence the remaining three intersections of the cubic and the asymptotic aggregate must lie in a line $s=0$ (Plücker's line S), which I call the satellite line. And writing $z=0$ for the equation of the line infinity, the equation of the cubic is of the form $U=V+\mu z^2 s=0$. It is to be observed moreover, that when, as for the Hyperbolas and the Cubical Parabola, the intersection at infinity is six-pointic, the line $s=0$ is an arbitrary line; when as for the Parabolic Hyperbolas, and the Divergent Parabolas, the intersection is seven-pointic, the line $s=0$ meets the cubic in a given point at infinity, viz. the twofold or the threefold point at infinity; and when as for the Central Hyperbolisms and the Parabolic Hyperbolisms the intersection is eight-pointic, the line $s=0$ has with the cubic a given twofold intersection at infinity; this however merely implies that the line $s=0$ passes through the node or cusp at infinity, and so imposes only one condition on the line $s=0$. Finally, when as in the Trident Curve the intersection is nine-pointic, the line $s=0$ has with the curve a given threefold intersection at infinity; that is, it coincides with the line infinity, $z=0$.

14. The preceding considerations in regard to the asymptotic aggregate $V=0$, lead very directly to the best analytical form of the function V, and therefore to that of the equation $U=V+\mu z^2 s=0$, of the cubic.

15. For the Hyperbolas; the equations of the asymptotes being $p=0$, $q=0$, $r=0$, then we have $V=pqr=0$ for the asymptotic aggregate; the satellite line is arbitrary, and hence

Equation of the Hyperbolas is

$$pqr+\mu z^2 s=0.$$

16. For the Parabolic Hyperbolas. Imagine parallel to the asymptote a line $p=0$ touching the asymptotic parabola; and let the line joining the point of contact with the twofold point at infinity have for its equation $q=0$; the equation of the asymptote is

$$p+\kappa z=0,$$

that of the asymptotic parabola is $q^2+\lambda pz=0$, and hence the equation of the asymptotic aggregate is $(p+\kappa z)(q^2+\lambda pz)=0$; the satellite line passes through the twofold point at infinity, or its equation is $q+\sigma z=0$; hence

Equation of the Parabolic Hyperbolas is

$$(p+\kappa z)(q^2+\lambda pz)+\mu z^2(q+\sigma z)=0.$$

17. For the Central Hyperbolisms; the equation of the onefold asymptote is taken to be $p=0$, and that of the parallel asymptotes to be $q^2+\kappa z^2=0$; hence the equation

of the asymptotic aggregate is $p(q^2+\kappa z^2)=0$, the satellite line passes through the twofold point at infinity, its equation is $q+\sigma z=0$; hence

Equation of the Central Hyperbolisms is

$$p(q^2+\kappa z^2)+\mu z^2(q+\sigma z)=0.$$

18. For the Parabolic Hyperbolisms: the only difference is that instead of the parallel asymptotes $q^2+\kappa z^2=0$ we have the twofold asymptote $q^2=0$; hence

Equation of the Parabolic Hyperbolisms is

$$pq^2+\mu z^2(q+\sigma z)=0.$$

19. For the Divergent Parabolas: the asymptotic aggregate is a semicubical parabola; let $q=0$ be the equation of the cuspidal tangent, $p=0$ the equation of the line joining the cusp with the inflexion at infinity, then the equation is $p^3+\lambda q^2 z=0$. The satellite line passes through the threefold point at infinity, its equation is $p+\sigma z=0$, hence

Equation of the Divergent Parabolas is

$$p^3+\lambda q^2 z+\mu z^2(p+\sigma z)=0.$$

20. For the Trident Curve: let $p=0$ be the equation of the asymptote, $q=0$ that of the tangent to the asymptotic parabola at the point not at infinity where it is met by the asymptote, then the equation of the parabola is $p^2+\lambda qz=0$, and that of the asymptotic aggregate is $p(p^2+\lambda qz)=0$; the satellite line is the line infinity, $z=0$; hence

Equation of the Trident Curve is

$$p(p^2+\lambda qz)+\mu z^3=0.$$

21. For the Cubical Parabola: let $p=0$ be the equation of the line joining the inflexion with the cusp at infinity, then the asymptotic aggregate is this line taken as a threefold line, or the equation is $p^3=0$; the satellite line is arbitrary; hence

Equation of the Cubical Parabola is

$$p^3+\mu z^2 s=0.$$

22. It is convenient to notice here that for the Hyperbolas the line $s=0$ is determined as follows, viz. the line infinity meets the curve in three points, and the tangents at these points (the asymptotes) again meet the curve in three points lying in a line which is the line in question; in other words, the line $s=0$ is (in the sense in which I have elsewhere used the term) the satellite line of infinity. For the other kinds of cubic curves, the line $s=0$ is *not*, in the sense just referred to, the satellite line of infinity: but in the present Memoir I shall in every case call the line, $s=0$, the satellite line.

The Thirteen Divisions. Article Nos. 23 to 33.

23. The characters of the foregoing seven divisions are irrespective of *reality;* and before going further it may be remarked, that as to the Hyperbolas and the Parabolic Hyperbolas a subdivision also irrespective of reality may be made as follows.

24. For a Hyperbola, the three asymptotes may not meet in a point, or they may meet in a point. For shortness I say that in the former case we have a Hyperbola Δ, in the latter case a Hyperbola $\odot$. I consider more particularly (*post,* No. 41) the special case of a Hyperbola $\odot$.

25. For a Parabolic Hyperbola, the asymptote may meet the asymptotic parabola in two onefold points; or in a twofold point.

26. I come now to the divisions which depend on *reality:* it is assumed that the curve is real.

27. For the Hyperbola the three points at infinity may be all real or else one real, two imaginary. In the former case, the asymptotes are all real, and we have the redundant hyperbola; in the latter case the real point at infinity gives rise to a real asymptote, the imaginary points to imaginary asymptotes: we have in this case the defective hyperbola. It is to be noticed that the imaginary asymptotes meet in a real point, called the asymptote-point; and that such point, if we regard it as an indefinitely small ellipse given as to the position and ratio of its axes, determines the imaginary asymptotes. Combining the division with the Δ, $\odot$, we have four subdivisions of the Hyperbola.

28. For a Hyperbola Δ redundant the three asymptotes form a triangle, and for a Hyperbola $\odot$ redundant they meet in a point. For a Hyperbola Δ defective, the asymptote-point does not lie on the real asymptote; for a Hyperbola $\odot$ defective it does lie on the real asymptote.

29. For a Parabolic Hyperbola: the onefold point and the twofold point at infinity are of necessity real, as are also the asymptote and the asymptotic parabola. If the asymptote meets the asymptotic parabola in two onefold points, these may be both real or both imaginary: if it meets it in a twofold point, this is real. We have thus three subdivisions of the Parabolic Hyperbola. For the Central Hyperbolism, the onefold point, and the node or twofold point at infinity, are both real; the asymptote is also real. But the node may be a crunode or an acnode; that is, the tangents at the node, or parallel asymptotes, may be both real, or both imaginary: we have thus two subdivisions, viz. the Hyperbolism of the hyperbola, and the Hyperbolism of the ellipse.

30. For the Parabolic Hyperbolism, the onefold point and the cusp or twofold point at infinity, and also the onefold asymptote and the twofold asymptote are all real.

31. For the Divergent Parabola, the inflexion or threefold point at infinity is real.

32. For the Trident Curve the node or threefold point at infinity is real, and inasmuch as one of the tangents is the line infinity, the node is a crunode, and the other tangent, or asymptote of the curve, is also real.

33. For the Cubical Parabola, the cusp or threefold point at infinity and the tangent at this point are each real.

Reckoning the hyperbolas as 4, the parabolic hyperbolas as 3, the central hyperbolisms as 2, and the parabolic hyperbolisms, the divergent parabolas, the trident curve, and the cubical parabola, each as 1, we have in all 13 divisions.

The Notion of a Group. Article No. 34.

34. I remark that the characters as well of the 7 divisions as of the 13 divisions have exclusive reference to the form of the asymptotic aggregate $V=0$; we have an ulterior division depending on the relation of the satellite line to the asymptotic aggregate, and which I regard as the proper origin of Plücker's Groups: viz. for a given form of the asymptotic aggregate $V=0$, and corresponding to each characteristically distinct position in relation thereto of the satellite line $s=0$, we have a Group. The determination of the characteristically distinct positions of the satellite line cannot be completely effected *à priori*; for instance, in the case of the Hyperbolas Δ redundant, the distinctions which immediately present themselves are that the satellite line cuts the three sides produced, or two sides and the third side produced, of the triangle formed by the asymptotes, or passes through an angle of the triangle, &c.; but these are not *all* the distinctions which have to be made; to determine them, taking the satellite line as given, we discuss the series of curves represented by the equation $V+\mu z^2 s=0$; for instance (and it is on this that the discussion chiefly turns), we see that the parameter μ may be so determined that the curve shall have a node, but the reality or non-reality of the roots of the equation in μ, and therefore the existence of a real nodal curve or curves will depend on the position of the satellite line $s=0$; and it is thus only by the discussion of the group that we arrive at an enumeration of the different groups.

Osculating Asymptotes and other Specialities. Article Nos. 35 to 41.

35. But Plücker nevertheless, prior to the establishment of his groups, introduces certain intermediate divisions as to osculating asymptotes, &c., which have really reference to the position of the satellite line; an osculating asymptote gives rise to a 'diameter,' and the diameter is a distinctive character in the Newtonian genera; to explain how all this is, I proceed as follows.

36. The parallel asymptotes of a Central Hyperbolism, the twofold asymptote of a Parabolic Hyperbolism and the asymptote of the Trident Curve are singular asymptotes, that is, each of them touches the curve at a node or a cusp, and is thus an asymptote of three-pointic intersection. Excluding these, and using the term asymptote to denote

a non-singular asymptote, an asymptote is in general an ordinary tangent or asymptote of two-pointic intersection; if, however, the point of contact is an inflexion, then the asymptote is an asymptote of three-pointic intersection, or osculating asymptote. In particular for the Hyperbolas, the asymptotes may be all ordinary, or they may be two ordinary and one osculating, or all three osculating; but they cannot be only two of them osculating; for the line through two inflexions meets a cubic curve in a third point which is also an inflexion; that is, if two asymptotes are osculating, the third is also an osculating asymptote. The foregoing remarks apply as well to the defective as the redundant Hyperbolas; it is to be noticed, however, as regards the defective Hyperbolas that the osculating asymptote, when there is only one, is necessarily the real asymptote, and consequently that the cases are—asymptotes ordinary; the real asymptote alone osculating; three osculating asymptotes. For the Parabolic Hyperbolas the asymptote, and for the Central Hyperbolisms and the Parabolic Hyperbolisms the onefold asymptote, may be ordinary or osculating.

37. The distinction of ordinary and osculating asymptotes has reference to the position of the satellite line; viz. for the Hyperbolas, when there is a single osculating asymptote, the satellite line passes through the point at infinity of the osculating asymptote, or what is the same thing, the satellite line is parallel to the osculating asymptote: and when there are three osculating asymptotes, the satellite line coincides with the line infinity. And, conversely, when the satellite line is parallel to an asymptote such asymptote is an osculating one, and when the satellite line is at infinity the three asymptotes are osculating. For the Parabolic Hyperbolas the asymptote, and for the Hyperbolisms the onefold asymptote, is an osculating asymptote when the satellite line is at infinity; and conversely.

38. There is in regard to the Divergent Parabolas a distinction which may be mentioned here; viz. the satellite line may disappear altogether ($\mu = 0$), and the curve thus coincide with the asymptotic semicubical parabola. Or, what is the general case, the satellite line may be distinct from the line infinity,—and it may cut in two real points, touch, or cut in two imaginary points the asymptotic semicubical parabola: or the satellite line may coincide with the line infinity, the asymptotic semicubical parabola being in this case of nine-pointic intersection.

39. The term "diameter" is used by Newton in the *Enumeratio* in two different senses; viz. for any given direction of the ordinates there exists a right line or "diameter," such that measuring the ordinates from this line the sum $y + y' + y''$ of the three ordinates is $= 0$. Such diameter is in fact the second or line polar in regard to the cubic of an arbitrary point on the line infinity. But the term diameter is afterwards and will be here used to denote a diameter *absolutè dictum*, viz. for a direction of the ordinates parallel to a non-singular asymptote there may exist a right line or "diameter" such that the ordinates measured from this point are equal and opposite to each other, or what is the same thing, such that the sum $y + y'$ of the two ordinates is $= 0$; this implies that the asymptote is an osculating asymptote. In fact, the first or conic polar of any inflexion of the cubic breaks up into a pair of lines, one of which is the tangent at the inflexion, the other of them, the 'polar' of

the inflexion, a line which cuts harmonically the chords through the inflexion, and which when the inflexion is at infinity becomes a diameter. The remarks previously made as to osculating asymptotes apply therefore to diameters, viz. the Hyperbolas may have no diameter, a single diameter, or three diameters, &c.

40. Newton speaks also of the "centre" of a cubic curve; viz. there may be a point on the curve such that for any line through this point the two radius vectors are equal and opposite to each other, or that the sum $r + r'$ of the two radius vectors is $= 0$. The centre is in fact a point of inflexion which has for its polar the line infinity. The curves which may have a centre are the Hyperbolas (redundant or defective), the Central Hyperbolisms (of the hyperbola or ellipse) and the Cubical Parabola. For the hyperbolas, the three asymptotes and the satellite line must meet in a point of the curve, which point is then the centre; for the central hyperbolisms the onefold asymptote and the satellite line must meet in a point of the curve, which point is then a centre; and for the cubical parabola no condition is required, but the inflexion is a centre. I remark here, in passing, that the notion of a centre as just explained has no place in Plücker's Classification, and that the two Newtonian species 58 and 59 (hyperbolisms of the hyperbola) and the two Newtonian species 61 and 62 (hyperbolisms of the ellipse) which differ, the two of a pair from each other, according as there is no centre or a single centre, form each pair a single species with Plücker; viz. they are 198 and 201 respectively.

41. It has been already remarked that the three asymptotes of a Hyperbola may meet in a point. As to this it is to be noticed that from any point we may draw six tangents to a cubic, the points of contact lie on a conic, the conic polar of the point: if, however the point lie on the Hessian of the cubic, then the conic breaks up into a pair of lines, each of which is a tangent to the Pippian; the two lines meet in a point of the Hessian, which point forms with the first mentioned point a pair of conjugate poles of the cubic[1].

Conversely, any tangent of the Pippian meets the cubic in three points, the tangents at which meet in a point of the Hessian; and from this point we may draw to the cubic three other tangents the points of contact of which lie on a line which is also a tangent of the Pippian, and the two tangents of the Pippian meet in a point of the Hessian; the two points of the Hessian being conjugate poles of the cubic. In particular, if the line infinity is a tangent of the Pippian, then the three asymptotes meet in a point of the Hessian, and the three tangents from this point to the cubic touch the cubic in three points lying on a line which is a tangent of the Pippian, and which meets the line infinity in a point forming with the first mentioned point a pair of conjugate poles of the cubic.

I proceed now to explain the classification of Newton so far as relates to the division into genera, and the classification of Plücker so far as relates to the divisions immediately superior to the groups.

[1] See as to this theory my *Memoir on Curves of the Third Order. Phil. Trans.* p. 147 (1856), [145].

Newton's Classification. Article Nos. 42 to 46.

42. Newton establishes in the first instance the following four cases; viz. the equation of a cubic curve is one of the forms

$$\begin{array}{lll} \text{I.} & xy^2 + ey & = ax^3 + bx^2 + cx + d, \\ \text{II.} & xy & = ax^3 + bx^2 + cx + d, \\ \text{III.} & y^2 & = ax^3 + bx^2 + cx + d, \\ \text{IV.} & y & = ax^3 + bx^2 + cx + d. \end{array}$$

It is not, I think, necessary to reproduce here the very interesting reasoning by means of which this most important step in the classification was effected.

43. Starting from the four cases, Newton obtains his 14 genera, viz. Case I gives 11 genera, and Cases II, III, IV give each a single genus. But these genera group themselves as follows, viz. 1, 2, 3, 4, 5, 6 are Hyperbolas; 7 and 8, Parabolic Hyperbolas; 9 and 10, Central Hyperbolisms; 11, Parabolic Hyperbolisms; 12, the Trident Curve; 13, the Divergent Parabolas; and 14, the Cubical Parabola. And the equations are as follows:

$$\begin{array}{lll} \text{the Hyperbolas} & xy^2 + ey & = ax^3 + bx^2 + cx + d, \\ \text{the Parabolic Hyperbolas} & xy^2 + ey & = \phantom{ax^3 +{}} bx^2 + cx + d, \\ \text{the Central Hyperbolisms} & xy^2 + ey & = \phantom{ax^3 + bx^2 +{}} cx + d, \\ \text{the Parabolic Hyperbolisms} & xy^2 + ey & = \phantom{ax^3 + bx^2 + cx +{}} d, \\ \text{the Divergent Parabolas} & y^2 & = ax^3 + bx^2 + cx + d, \\ \text{the Trident Curve} & xy & = ax^3 + bx^2 + cx + d, \\ \text{the Cubical Parabola} & y & = ax^3 + bx^2 + cx + d, \end{array}$$

where it is to be understood that the highest expressed power on the right-hand side of each equation does not vanish.

44. In these equations the axes $x=0$, $y=0$ are not for the most part lines precisely determined in relation to the curve, but it is easy to see as well analytically as geometrically how by a proper transformation of the equations they may be brought into forms such as those previously obtained, in which the several lines $p=0$, &c., stand in a determinate relation to the curve. Thus, taking the equation of the Cubical Parabola, this may be written $y = a\left(x + \dfrac{b}{3a}\right)^3 + c'x + d'$; or, what is the same thing, $y' = ax'^3$. Or geometrically, we see that $x=0$ is a line completely determined as to its direction, it is in fact a line through the cusp at infinity; but that $y=0$ is an arbitrary line in regard to the curve; taking for $y=0$ the tangent at the inflexion, and for $x=0$ the line from the inflexion to the cusp at infinity, then the curve must pass through the point ($x=0$, $y=0$), and $y=0$ must give a threefold value of x; the equation thus is $y = ax^3$. And so in other cases.

45. In the division into genera, Newton distinguishes the Hyperbolas into the redundant and defective, and the redundant hyperbolas into those for which the asymptotes form a triangle, and those for which the asymptotes meet in a point. The redundant hyperbolas with asymptotes forming a triangle are distinguished according as they have no diameter, a single diameter, or three diameters. The like distinction might have been, but is not, made as to the redundant hyperbolas with asymptotes meeting in a point; these are in fact included in a single genus; but the distinction presents itself in the species of that genus. As to the defective hyperbolas, Newton attends only to the real asymptote; and the only distinction is according as they have no diameter or a real diameter. The Parabolic Hyperbolas are in like manner divided according as they have no diameter or a diameter. The Central Hyperbolisms, according as the parallel asymptotes are real or imaginary, are the hyperbolisms of the hyperbola or of the ellipse. The hyperbolisms of the hyperbola form a single genus. Each of the Hyperbolisms might have been distinguished according as there is no diameter or a single diameter; this distinction appears in the species. The Trident Curve, the Divergent Parabolas, and the Cubical Parabola, form each a single genus.

46. We have thus the following Table of the Newtonian genera: I show in it the species in each genus, retaining Newton's numbers, and distinguishing by the numbers 10′, 13′, 22′, 22″ the four species added by Stirling, and by 56′ and 56″ the two species added by Murdoch or Cramer: I show also the division of genus 4, according to the number of diameters; and I also show the five species of curves having a centre.

Table of the Newtonian Genera.

1. Redundant Hyperbolas with asymptotes forming a triangle, and without a diameter.

Sp. 1, 2, 3, 4, 5, 6, 7, 8, 9.

2. Redundant Hyperbolas with asymptotes forming a triangle, and with a single diameter.

Sp. 10, 10′, 11, 12, 13, 13′, 14, 15, 16, 17, 18, 19, 20, 21.

3. Redundant Hyperbolas with asymptotes forming a triangle, and with three diameters.

Sp. 22, 22′, 22″, 23.

4. Redundant Hyperbola with asymptotes meeting in a point.

Without a diameter, Sp. 24, 25, 26, 27. With one diameter, Sp. 28, 29, 30, 31. With three diameters, Sp. 32. Sp. 27 has a centre.

5. Defective Hyperbolas without a diameter.

Sp. 33, 34, 35, 36, 37, 38. Sp. 38 has a centre.

6. Defective Hyperbolas with a diameter.

Sp. 39, 40, 41, 42, 43, 44, 45.

7. Parabolic Hyperbolas without a diameter.

Sp. 46, 47, 48, 49, 50, 51, 52.

8. Parabolic Hyperbolas with a diameter.

Sp. 53, 54, 55, 56, 56′, 56″.

9. Hyperbolisms of the hyperbola.

Without a diameter, Sp. 57, 58, 59. With a diameter, Sp. 60. Sp. 59 has a centre.

10. Hyperbolisms of the ellipse.

Without a diameter, Sp. 61, 62. With a diameter, Sp. 63. Sp. 62 has a centre.

11. Hyperbolisms of the parabola.

Without a diameter, Sp. 64. With a diameter, Sp. 65.

12. Trident Curve, Sp. 66.

13. Divergent Parabolas. Sp. 67, 68, 69, 70, 71.

14. Cubical Parabola. Sp. 72.

Plücker's Classification. Article Nos. 47 to 49.

47. Plücker in the first instance obtains by analytical considerations his six head divisions, corresponding to the seven divisions in the present Memoir, the Central Hyperbolisms and the Parabolic Hyperbolisms forming with him a single division. The equations are obtained in the form already mentioned, the only difference being that he writes $z=1$; his six head divisions with their equations thus are

Hyperbolas	$pqr + \mu s = 0,$
Parabolic Hyperbolas	$(p+\kappa)(q^2+\lambda p) + \mu(q+\sigma) = 0,$
Hyperbolisms	$p(q^2+\kappa) + \mu(q+\sigma) = 0,$
Divergent Parabolas	$p^3 + \lambda q^2 + \mu(p+\sigma) = 0,$
Trident Curve	$p(p^2+\lambda q) + \mu = 0,$
Cubical Parabola	$p^3 + \mu s = 0.$

48. He then divides the Hyperbolas into the redundant and defective. The redundant hyperbolas are then divided as they have no osculating asymptote, one osculating asymptote, or three osculating asymptotes; and each of these according as the asymptotes form a triangle or meet in a point. As regards the defective hyperbolas he attends to the imaginary asymptotes, represented by means of their real point of intersection, the "asymptote-point," and the division is thus similar to that of the redundant hyperbolas, viz. the defective hyperbolas are distinguished according as they have no osculating asymptote, a real osculating asymptote, or three osculating asymp-

totes; and each of these according as the asymptotes form a triangle or meet in a point; that is, according as the asymptote-point does not, or does, lie on the real asymptote.

The Parabolic Hyperbolas are distinguished according as the asymptote is ordinary and the asymptotic parabola one of five-pointic intersection; or, as the asymptote is osculating and the parabola one of six-pointic intersection; and each of these according as the asymptote cuts, touches, or does not cut, the parabola. The Hyperbolisms are distinguished into those of the hyperbola, ellipse, and parabola, and each of these according as the asymptote is ordinary or osculating. The Divergent Parabolas are distinguished in the manner already mentioned; viz. according as the curve is the semicubical parabola; or, as there is a satellite line not at infinity, and an asymptotic semicubical parabola of seven-pointic intersection, and which is cut, touched, or not cut, by the satellite line; or, as the satellite line is at infinity, and the semicubical parabola is of nine-pointic intersection. The Trident Curve and the Cubical Parabola are not divided.

49. I annex the following Table showing the Groups included in each division: for shortness I use the before-mentioned symbols Δ, ⊙ to denote that the asymptotes form a triangle or meet in a point respectively.

Table of the Plückerian Divisions.

Hyperbolas:

Redundant;

No osculating asymptote,

Δ I, II, III, IV, V, VI

⊙ VII, VIII

One osculating asymptote,

Δ IX, X, XI, XII, XIII, XIV

⊙ XV

Three osculating asymptotes,

Δ XVI

⊙ XVII

Defective;

No osculating asymptote,

Δ XVIII, XIX, XX, XXI, XXII, XXIII

⊙ XXIV, XXV, XXVI, XXVII

Real osculating asymptote,

Δ XXVIII, XXIX, XXX, XXXI, XXXII, XXXIII

⊙ XXXIV

Three osculating asymptotes,	
Δ	XXXV
$\odot$	XXXVI
Parabolic Hyperbolas:	
Ordinary asymptote and five-pointic parabola.	
asymptote cuts parabola	XXXVII, XXXVIII, XXXIX, XL, XLI
does not cut „	XLII
touches „	XLIII, XLIV
Osculating asymptote and six-pointic parabola.	
asymptote cuts parabola	XLV
does not cut „	XLVI
touches „	XLVII
Hyperbolisms:	
Of hyperbola;	
asymptote ordinary	XLVIII, XLIX
„ osculating	L
Of ellipse;	
asymptote ordinary	LI
„ osculating	LII
Of parabola;	
asymptote ordinary	LIII
„ osculating	LIV
Divergent Parabolas:	
Semicubical Parabola	LV
Asymptotic ditto, seven-pointic.	
satellite line cuts semic. parab.	LVI
does not cut „	LVII
touches (i.e. passes through the vertex of)	LVIII
Asymptotic ditto, nine-pointic	LIX
Trident Curve	LX
Cubical Parabola	LXI

As to the Theory of Groups. Article No. 50.

50. It has been already remarked that the division into groups depends on the different positions of the satellite line in regard to the asymptotic aggregate, and that the investigation relates in a great measure to the discussion of the values of the

parameter μ in the equation $V+\mu z^2 s=0$, which are such as to give rise to a nodal curve. It is to be noticed that except for the Hyperbolas and for the Cubical Parabola, the satellite $s=0$ is either a line passing through a given point at infinity (determinate, that is, as regards its direction), or in the case of the Trident Curve it is the given line infinity; there is at most only a *single* series of positions to be considered, and the theory is a short and easy one; and for the Cubical Parabola, although the satellite line $s=0$ is here an arbitrary line, yet on account of the cusp at infinity, there is not any critic value of μ, or in fact any distinction of cases. The only other case where the satellite line $s=0$ is an arbitrary line, admitting therefore of a double series of positions, is that of the Hyperbolas; and the division into groups constitutes an extensive and interesting theory, which is insufficiently discussed by Plücker; and it was with a view to the development of this theory that my Memoir, *On a Case of the Involution of two Cubic Curves*, (*ante*, pp. 39 to 81), referred to in the sequel as *Memoir on Involution*, [349], was written. I remark that of the three curves there established as material to the theory, and which are further spoken of in the sequel of the present memoir, viz. the envelope, the twofold centre locus, and the one-with-twofold centre locus, Plücker considers only the twofold centre locus. I proceed to apply the results of that Memoir to the present theory.

As to the Groups of the Hyperbolas Δ. Article Nos. 51 to 53.

51. The assumed form of equation was $pqr+\mu z^2 s=0$, but using now

$$x=0,\ y=0,\ z=0$$

(instead of $p=0$, $q=0$, $r=0$) for the equations of the asymptotes, we may imagine the implicit constants so determined that the line infinity (before represented by $z=0$) shall have for its equation $x+y+z=0$; writing moreover $\lambda x+\mu y+\nu z=0$ for the satellite line $s=0$, and k in the place of μ, the equation becomes

$$xyz+k\,(x+y+z)^2\,(\lambda x+\mu y+\nu z)=0,$$

which is the form considered in the Memoir just referred to.

52. It is there shown that for an arbitrary position of the satellite line, the parameter k, or, what is the same thing, the auxiliary parameter θ, may be determined by a cubic equation in such manner that the curve shall have a node; the node, or rather the site of the node is termed a critic centre; and there are consequently three critic centres (all real or else one real, two imaginary). If however the satellite line touches a certain curve called the envelope, then two of the critic centres unite together, forming a twofold centre which is (not a mere node but) a cusp on the corresponding cubic curve; the other critic centre is termed a one-with-twofold centre;

and the loci of the twofold centre and of the one-with-twofold centre respectively are determinate curves, the former a conic, the latter a cubic. The critic centres corresponding to the entire series of satellite lines which pass through a certain fixed point lie on a cubic, which when the fixed point lies on the line $x+y+z=0$, here the line infinity, degenerates into a conic called the "Harmonic Conic," and when one of the critic centres is known the other two are determined as the intersections of the harmonic conic by the polar of the given critic centre in regard to the twofold centre conic.

53. For establishing the theory of the groups of the hyperbolas Δ, it is necessary to consider the geometrical forms of the several curves which have been just referred to; viz. this is to be done, on the assumption always that the line $x+y+z=0$ is the line infinity, for the Redundant Hyperbolas taking the lines $x=0$, $y=0$, $z=0$, to be real lines and for the Defective Hyperbolas, taking them to be one real and the other two imaginary. The formulæ of the memoir above referred to, are in their actual form adapted to the former case, but they can of course be transformed so as to adapt them to the latter case. I proceed to examine the two cases separately.

The hyperbolas Δ *Redundant* (*See fig.* 1). Article Nos. 54 to 71.

54. Every thing is symmetrical with respect to the three asymptotes, and to fix the ideas and without any real loss of generality we may consider the asymptotes as forming an equilateral triangle. Taking the perpendicular distance of a vertex from the opposite side as unity, the absolute magnitudes of the coordinates may be fixed by assuming $x+y+z=1$; x, y, z will then denote the perpendicular distances of a point from the three sides respectively. If the coordinates of a point are proportional to α, β, γ, then the absolute magnitudes are of course

$$\frac{\alpha}{\alpha+\beta+\gamma},\ \frac{\beta}{\alpha+\beta+\gamma},\ \frac{\gamma}{\alpha+\beta+\gamma};$$

the point may be spoken of indifferently as the point (α, β, γ) or the point

$$\left(\frac{\alpha}{\alpha+\beta+\gamma},\ \frac{\beta}{\alpha+\beta+\gamma},\ \frac{\gamma}{\alpha+\beta+\gamma}\right),$$

and it is sometimes convenient to use both of the two notations; thus the Harmonic point (that is, the point the harmonic of infinity in regard to the triangle) is the point $(1, 1, 1)$ or $(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})$. The lines $y-z=0$, $z-x=0$, $x-y=0$, which are the lines joining the harmonic point with the three vertices respectively, are in the case of the equilateral triangle the perpendiculars from the vertices on the three sides respectively, and they may be spoken of simply as the perpendiculars, but it is to be borne in mind that the former is the proper construction of these lines.

55. The equation of the envelope is

$$\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0,$$

or, what is the same thing,

$$x^4+y^4+z^4-4\,(yz^3+y^3z+zx^3+z^3x+xy^3+x^3y)$$
$$+6\,(y^2z^2+z^2x^2+x^2y^2)-124\,(x^2yz+xy^2z+xyz^2)=0.$$

The curve consists as shown in the figure of a trigonoid branch inscribed in the triangle and of three acnodes outside the triangle.

56. The side $x=0$ touches the curve in the point $(0, 1, 1)$ or $(0, \frac{1}{2}, \frac{1}{2})$, which is its intersection with the perpendicular $y-z=0$; the side $x=0$ has with the curve at the point in question a four-pointic intersection. The last-mentioned line $y-z=0$ meets the curve in the point $(-4, 1, 1)$ or $(2, -\frac{1}{2}, -\frac{1}{2})$, which is one of the acnodes, and therefore a point of twofold intersection; then again in the point $(16, 1, 1)$ or $(\frac{8}{9}, \frac{1}{18}, \frac{1}{18})$ which may be considered as a vertex of the trigonoid branch, and finally in the before-mentioned point $(0, 1, 1)$ or $(0, \frac{1}{2}, \frac{1}{2})$, which is the point of contact with the side $x=0$.

57. The equation of the twofold centre locus is

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0,$$

or in a rational form

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

which is in the case of an equilateral triangle, a *circle* inscribed in the triangle and touching the sides at their midpoints respectively. The circle is shown in the figure.

58. The equation of the one-with-twofold centre locus is

$$-(-x+y+z)\,(x-y+z)\,(x+y-z)+xyz=0,$$

or, what is the same thing,

$$x^3+y^3+z^3-(yz^2+y^2z+zx^2+z^2x+xy^2+x^2y)+3xyz=0.$$

It is a cubic having the harmonic point $(1, 1, 1)$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, for an acnode, touching the sides of the triangle externally at their midpoints respectively, and having the three asymptotes

$$5x-4y-4z=0,\ -4x+5y-4z=0,\ 4x-4y+5z=0,$$

or, what is the same thing, $x=\frac{4}{9}$, $y=\frac{4}{9}$, $z=\frac{4}{9}$; the form of the curve is shown in the figure.

59. The equation of the harmonic conic corresponding to the satellite line $\lambda x + \mu y + \nu z = 0$, is

$$\frac{\mu - \nu}{x} + \frac{\nu - \lambda}{y} + \frac{\lambda - \mu}{z} = 0,$$

or say

$$2fyz + 2gzx + 2hxy = 0,$$

where $f + g + h = 0$.

It is a hyperbola passing through the angles of the triangle, and through the harmonic point (1, 1, 1) or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Observing that the points in question are the intersections of the two rectangular hyperbolas (pairs of lines) $x(y-z) = 0$, $y(z-x) = 0$, it follows that the harmonic conic is a *rectangular* hyperbola.

60. The coordinates of the centre are f^2, g^2, h^2 and the centre is consequently a point on the circle which is the twofold centre locus.

The asymptotes of course meet in the centre, and they again meet the circle in two points which are the intersections of the circle with the line $fx + gy + hz = 0$.

61. The Harmonic Conic is the same for the satellite lines which have a given direction, and we may to determine it take a satellite line which touches the envelope. If the constants α, β, γ satisfy the condition $\alpha + \beta + \gamma = 0$; then the equation of a satellite line tangent to the envelope is $\frac{x}{\alpha^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3} = 0$: the coordinates of the point of contact with the envelope are as $\alpha^4 : \beta^4 : \gamma^4$; the coordinates of the twofold centre as $\alpha^2 : \beta^2 : \gamma^2$; the coordinates of the one with twofold centre as

$$\alpha^2(\beta - \gamma) : \beta^2(\gamma - \alpha) : \gamma^2(\alpha - \beta).$$

The values of f, g, h are as $\alpha^3(\beta^3 - \gamma^3) : \beta^3(\gamma^3 - \alpha^3) : \gamma^3(\alpha^3 - \beta^3)$, or what is the same thing, as $\alpha^3(\beta - \gamma) : \beta^3(\gamma - \alpha) : \gamma^3(\alpha - \beta)$, or as $\alpha^2(\beta^2 - \gamma^2) : \beta^2(\gamma^2 - \alpha^2) : \gamma^2(\alpha^2 - \beta^2)$: the last-mentioned values show that the line $fx + gy + hz = 0$ passes through the harmonic point (1, 1, 1) or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and also through the point $\left(\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}\right)$, the inverse of the point $(\alpha^2, \beta^2, \gamma^2)$ which is the twofold centre.

62. On account of the symmetry of the figure in regard to the three asymptotes, it is sufficient to construct the harmonic conic for a direction of the satellite line inclined to the base at an angle not $> 30°$, and this is what is accordingly done: it may however be remarked that for the limiting inclination $= 0°$, that is, when the satellite line is parallel to the base, the harmonic conic becomes a pair of right lines, the base and perpendicular; but that for the other limiting inclination $= 30°$, that is, when the satellite line is perpendicular to one of the legs of the triangle, the harmonic conic is still a proper hyperbola, and is situate symmetrically in regard to the leg in question; the two limiting cases will be readily understood by means of the general case shown in the figure.

63. Imagine now the satellite line moving parallel to itself through the series of positions $ABCMDEA'$; to simplify the figure these are not delineated in their proper positions (but they are merely indicated according to their order of succession), and it is to be understood that they have the following positions, viz.

A, at infinity[1],

B, through the vertex B,

C, touching the envelope,

M, through the vertex M_3,

D, through the vertex D_1,

E, through node X of the envelope,

A', at infinity[1];

then the corresponding positions of the critic centres are

On one branch of the hyperbola,	On the other branch,
A_3, at infinity,	A_1, at infinity: A_2, the harmonic point,
B_3,	B_1, B_2,
C_3, a one-with-twofold centre,	C_{12}, a twofold centre,
M_3,	M_1, M_2 are imaginary,
C_3', a one-with-twofold centre,	C_{12}', a twofold centre,
D_3,	D_1, D_2,
E_3,	E_1, E_2,
A_3', at infinity.	A_1', at infinity; A_2, the harmonic point.

64. For the further explanation of the figure it is to be observed that B_2, B_3 lie on the line joining the midpoints of two sides; and in like manner D_2, D_3 on the line joining the midpoints of two sides; (the imaginary points M_1, M_2 are in like manner on the line joining the midpoints of two sides): these relations depend on the theorem, No. 81, of the Memoir on Involution, viz. that for the satellite lines which pass through a vertex (1, 0, 0) of the triangle, one of the critic centres is the vertex (1, 0, 0), and the other two critic centres are points on the line $-x+y+z=0$, or, what is the same thing, $x=\frac{1}{2}$.

65. Again, the point E_3 is on the line $(x=1)$ through the vertex D, parallel to the base, and the points E_1, E_2 are on the hyperbola (indicated by a dotted line in the figure) $(y+\frac{1}{4})(z+\frac{1}{4})=\frac{5}{16}$; this depends on the theorem Nos. 73 and 74 of the

[1] Strictly speaking a line at infinity is the line infinity, and as such has no definite direction; but we may of course consider a line which moves parallel to itself in opposite senses as having for its limit the line infinity.

Memoir on Involution, viz. the critic centres corresponding to the satellite lines through the point

$$(-4,\ 1,\ 1) \text{ or } (2,\ -\tfrac{1}{2},\ -\tfrac{1}{2})$$

lie one of them on the line $y+z=0$, and the other two on the conic $x(x+y+z)-4yz=0$; reducing by the condition $x+y+z=1$, these equations become respectively $x=1$, and

$$(y+\tfrac{1}{4})(z+\tfrac{1}{4})-\tfrac{5}{16}=0.$$

66. The foregoing positions of the satellite line, and the critic centres, as exhibited in the figure, were selected partly for facility of delineation; I wished however to examine the effect of the passage of the satellite line through a node of the envelope; and it appears that such passage does not give rise to any marked peculiarity in regard to the critic centres. The selected positions are sufficient to indicate the circumstances of the critic centres as the satellite line passes from the position A at infinity continuously to the position A' at infinity; in particular we see that as the line passes from A to C, or from C' to A', there are three real centres; but that as the line passes from C to C' there is only one real centre.

67. The case of the satellite line parallel to the asymptote $x=0$, is included (as already mentioned) as a limiting case in the foregoing one; the harmonic conic is here the pair of lines $x(y-z)=0$; and we have two critic centres on the line $y-z=0$, (the perpendicular), and the third (not properly a critic centre) at infinity on the asymptote $x=0$; in fact, starting with a critic centre on the line $y-z=0$, the polar of the centre in regard to the twofold centre conic or circle is a line parallel to the asymptote $x=0$, and which therefore meets the harmonic conic $x(y-z)=0$ in a second centre on the line $y-z=0$, and in the point at infinity on the line $x=0$. But the analytical theory of the case is peculiar and may be specially considered.

68. Writing $\mu=\nu$, the equation of the satellite line is $\lambda x+\mu(y+z)=0$, or putting $x+y+z=0$ this is $x=\dfrac{\mu}{\mu-\lambda}$. The equation in θ (see Memoir on Involution, No. 20) becomes

$$(\theta+\mu)(\theta^2-\theta\mu-2\lambda\mu)=0;$$

or, disregarding the factor $\theta+\mu=0$, which corresponds to the centre at infinity, the equation is

$$\theta^2-\theta\mu-2\lambda\mu=0,$$

which is a quadratic equation, giving therefore two values of θ, and the corresponding critic centres lie on the perpendicular $y-z=0$, the x coordinate being given by the equation

$$x=\frac{1}{\theta+\lambda}\div\left(\frac{1}{\theta+\lambda}+\frac{2}{\theta+\mu}\right)=\frac{1}{\theta+\lambda}\div\frac{2}{\theta}=\tfrac{1}{2}\,\theta\div(\theta+\lambda).$$

We have therefore conversely

$$\theta=\frac{\lambda x}{\frac{1}{2}-x},$$

and thence

$$\lambda x^2 - \mu x\left(\tfrac{1}{2} - x\right) - 2\mu\left(\tfrac{1}{2} - x\right)^2 = 0,$$

or, what is the same thing,

$$(\lambda - \mu)\, x^2 + \tfrac{3}{2}\mu x - \tfrac{1}{2}\mu = 0,$$

so that putting for shortness $\dfrac{\mu}{\mu - \lambda} = \varpi$, ($\varpi$ denotes the distance of the satellite line from the asymptote $x = 0$) then the equation which determines the distance of the critic centres from the asymptote is

$$x^2 - \tfrac{3}{2}\varpi x + \tfrac{1}{2}\varpi = 0,$$

or we have

$$x = \tfrac{1}{4}\left(3\varpi \pm \sqrt{9\varpi^2 - 8\varpi}\right).$$

The condition for a twofold centre is ($\varpi = 0$, which may be disregarded, or else) $\varpi = \frac{8}{9}$, or, what is the same thing, $8\lambda + \mu = 0$.

69. If x_1, x_2 are the coordinates of the two centres, we have

$$2x_1{}^2 = \varpi\,(3x_1 - 1),$$

$$2x_2{}^2 = \varpi\,(3x_2 - 1),$$

and thence

$$\frac{x_1{}^2}{x_2} = \frac{3x_1 - 1}{3x_2 - 1},$$

or, reducing,

$$x_1 + x_2 - 3x_1x_2 = 0,$$

a relation connecting the two values x_1, x_2; this equation however only expresses the known relation that the two centres are harmonics of each other in regard to the twofold centre conic or circle.

70. The foregoing examination of the form of the envelope shows very readily what are the positions of the satellite line which give rise to Plücker's groups for the Hyperbolas Δ Redundant.

We have in fact first,

Hyperbolas Δ Redundant, no osculating asymptote.

The satellite line is not parallel to a side of the triangle; and the different positions give the following six of Plücker's groups, viz.

I. Satellite line cuts three sides produced.

II. „ passes through a vertex and cuts opposite side produced.

III. „ passes through a vertex and cuts opposite side.

IV. „ cuts two sides and a side produced, but does not cut the envelope.

V. „ touches the envelope.

VI. „ cuts the envelope.

Next,

Hyperbolas Δ Redundant, one osculating asymptote.

The satellite line is parallel to the osculating asymptote, say to the base of the triangle; and the different positions give the following six of Plücker's groups, viz.

IX. Satellite line above the vertex.

X. „ through the vertex.

XI. „ below the vertex, but not cutting envelope.

XII. „ touches envelope.

XIII. „ cuts envelope.

XIV. „ lies below the base.

And finally,

Hyperbolas Δ Redundant, three osculating asymptotes.

The position of the satellite line is here completely determined, giving one of Plücker's groups, viz.

XVI. Satellite line at infinity.

71. It may be remarked that in this enumeration no account is taken of the nodes of the envelope: the enumeration was in fact made by Plücker by considerations relating to the critic centres, but without arriving at or making use of the envelope at all: if account were taken of the nodes of the envelope several of the foregoing groups would have to be subdivided according to the different positions of the satellite line in regard to these nodes: but the effect produced by the passage of the satellite line through a node of the envelope is so slight, that I am inclined to think that the enumeration may be properly effected in the foregoing manner, without any account being taken of these nodes.

The Hyperbolas Δ Defective (*See fig.* 2). Article Nos. 72 to 101.

72. If in the formulæ for the Hyperbolas Δ Redundant we write

$$\begin{aligned}\tfrac{1}{2}(x+yi) &\text{ for } x,\\ \tfrac{1}{2}(x-yi) &\text{ ,, } y,\\ \lambda-\mu i &\text{ ,, } \lambda,\\ \lambda+\mu i &\text{ ,, } \mu,\end{aligned}$$

then the equation of the satellite line is

$$\tfrac{1}{2}(\lambda-\mu i)(x+yi)+\tfrac{1}{2}(\lambda+\mu i)(x-yi)+\nu z=0,$$

which is $\lambda x+\mu y+\nu z=0$ as before; the equation of the line infinity is $x+z=0$, and the equation of the curve is

$$\tfrac{1}{4}(x^2+y^2)z+k(x+z)^2(\lambda x+\mu y+\nu z)=0.$$

We may fix the absolute magnitudes of the coordinates by writing $x+z=1$; and the equation then becomes

$$(x^2+y^2)(1-x)+4k\{(\lambda-\nu)x+\mu y+\nu\}=0.$$

The origin is at the asymptote-point, or intersection of the imaginary asymptotes; the equation of the real asymptote is $x=1$; that of the imaginary asymptotes is $x^2+y^2=0$.

If x and y are ordinary rectangular coordinates then the pair of lines represented by this equation will be an indefinitely small circle, and conversely, if the Asymptote-Point be an indefinitely small circle, then x and y will be rectangular coordinates; and we may without loss of generality assume that this is so.

73. The equation of the envelope is

$$\sqrt[4]{\tfrac{1}{2}(x+yi)}+\sqrt[4]{\tfrac{1}{2}(x-yi)}+\sqrt[4]{z}=0;$$

this gives successively

$$\sqrt{\tfrac{1}{2}(x+yi)}+\sqrt{\tfrac{1}{2}(x-yi)}-\sqrt{z}=-\sqrt{2}\sqrt[4]{x^2+y^2},$$

$$x+z-\sqrt{x^2+y^2}=2\sqrt{z}\{\sqrt{\tfrac{1}{2}(x+iy)}+\sqrt{\tfrac{1}{2}(x-iy)}\},$$

$$(x+z)^2+x^2+y^2-2(x+z)\sqrt{x^2+y^2}=4z(x+\sqrt{x^2+y^2}),$$

$$(x-z)^2+x^2+y^2=2(x+3z)\sqrt{x^2+y^2},$$

$$\{(x-z)^2+x^2+y^2\}^2=4(x+3z)^2(x^2+y^2);$$

that is

$$(x-z)^4+2(x^2+y^2)\{(x-z)^2-2(x+3z)^2\}+(x^2+y^2)^2=0.$$

Putting $z=1-x$, and therefore $x-z=2x-1$, and $x+3z=3-2x$, this becomes

$$(2x-1)^4+2(x^2+y^2)\{(2x-1)^2-2(2x-3)^2\}+(x^2+y^2)^2=0,$$

that is

$$(x^2+y^2)^2-2(x^2+y^2)(4x^2-20x+17)+(2x-1)^4=0,$$

which may also be written

$$y^4-2y^2(3x^2-20x+17)+9x^4+8x^3-10x^2-8x+1=0,$$

or, what is the same thing,

$$y^4-2y^2(x-1)(3x-17)+(x-1)(9x-1)(x+1)^2=0,$$

for the equation of the envelope. The solution of the equation in y gives

$$y^2=(x-1)(3x-17)\pm(2x-3)\sqrt{-32(x-1)}.$$

74. The original curve $\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}=0$ had the three nodes

$$(2,\ -\tfrac{1}{2},\ -\tfrac{1}{2}),\quad (-\tfrac{1}{2},\ 2,\ -\tfrac{1}{2}),\quad (-\tfrac{1}{2},\ -\tfrac{1}{2},\ 2);$$

and thence writing $\frac{1}{2}(x+yi),\ \frac{1}{2}(x-yi),\ z$, for x, y, z,

we find the nodes

$$(x=\tfrac{3}{2},\ y=\tfrac{5}{2}i),\quad (x=\tfrac{3}{2},\ y=-\tfrac{5}{2}i),\quad (x=-1,\ y=0);$$

the first and second of these are acnodes, or the curve has a pair of imaginary acnodes; the third is a crunode; and to find the directions at this point, if in the equation of the curve we write $x-1$ for x, the equation becomes

$$y^4-2y^2(x-2)(3x-20)+(x-2)(9x-10)x^2=0;$$

the lowest terms therefore are $20(-4y^2+x^2)=0$; and we have $y=\pm\frac{1}{2}(x+1)$ for the equation of the tangents at the crunode.

$y=0$ gives $x=1$, $x=\frac{1}{9}$ and (as a twofold value) $x=-1$, which belongs to the crunode.

$x=1$ gives $y^4=0$, or the line $x=1$ is a tangent of four-pointic intersection.

$x=0$ gives $y^4-34y^2+1=0$, that is $y^2=17\pm 12\sqrt{2}$, or $y=\pm(3\pm 2\sqrt{2})$.

75. The curve has a pair of asymptotic parabolas, and taking for the equation of one of them

$$(y-x\sqrt{3}+\beta)^2=-\tfrac{32}{3}(x+\alpha),$$

this gives

$$y=x\sqrt{3}-\beta+\sqrt{-\tfrac{32}{3}(x+\alpha)},$$

and thence

$$\begin{aligned} y^2 &= 3x^2-2\beta x\sqrt{3}+\beta^2 \\ &\quad -\tfrac{32}{3}(x+\alpha) \\ &\quad +2\left(x-\frac{\beta}{\sqrt{3}}\right)\sqrt{-32(x+\alpha)}, \end{aligned}$$

which agrees with the value of y^2 in the envelope as to the terms x^2, $x^{\frac{3}{2}}$, and by properly determining α, β, it may be made to agree as to the terms x and $x^{\frac{1}{2}}$.

We have in the parabola

$$\begin{aligned} y^2 &= 3x^2-2\beta x\sqrt{3}+\beta^2 \\ &\quad -\tfrac{32}{3}x-\tfrac{32}{3}\alpha \\ &\quad +\sqrt{-32}\left\{2x\sqrt{x}+\left(\alpha-\frac{2\beta}{\sqrt{3}}\right)\sqrt{x}\right\}, \end{aligned}$$

and in the curve

$$\begin{aligned} y^2 &= 3x^2-20x+17 \\ &\quad +\sqrt{-32}\,\{2x\sqrt{x}-4\sqrt{x}\}. \end{aligned}$$

We have therefore

$$-2\beta\sqrt{3}-\tfrac{32}{3}=-20,\ \alpha-\frac{2\beta}{\sqrt{3}}=-4,$$

giving

$$\beta=\frac{14}{3\sqrt{3}}=\frac{14\sqrt{3}}{9},\ \alpha=-\tfrac{8}{9},$$

so that the equation of the parabola is

$$\{y-(x-\tfrac{14}{9})\sqrt{3}\}^2=-\tfrac{32}{3}(x-\tfrac{8}{9});$$

that of the other parabola is of course obtained by merely changing the sign of $\sqrt{3}$.

76. From the foregoing results we may trace the curve, but this may be done somewhat more easily by means of polar coordinates, viz. writing $x=r\cos\theta$, $y=r\sin\theta$, and therefore $z=1-r\cos\theta$, we have

$$\sqrt[4]{\tfrac{1}{2}}\,r\,.\,2\cos\tfrac{1}{4}\theta+\sqrt[4]{1-r\cos\theta}=0,$$

that is

$$r(\cos\theta+8\cos^2\tfrac{1}{4}\theta)=1\ ;$$

or since

$$\begin{aligned}\cos\theta&=1-8\sin^2\tfrac{1}{4}\theta\cos^2\tfrac{1}{4}\theta,\\&=1-8\cos^2\tfrac{1}{4}\theta+8\cos^4\tfrac{1}{4}\theta,\end{aligned}$$

we have

$$\begin{aligned}\cos\theta+8\cos^4\tfrac{1}{4}\theta&=(1-4\cos^2\tfrac{1}{4}\theta)^2\\&=(-1-2\cos\tfrac{1}{2}\theta)^2,\end{aligned}$$

and the equation is

$$r=\frac{1}{(1+2\cos\frac{1}{2}\theta)^2}\,;$$

$\theta=0^\circ$ gives $r=\frac{1}{9}$, $\theta=180^\circ$ gives $r=1$, $\theta=240^\circ$ gives $r=\infty$, $\theta=360^\circ$ gives $r=1$,

values which agree with the results obtained by rectangular coordinates.

77. The form is shown in the figure; we see that the curve consists of a lower branch without any singularity; and of an upper branch which cuts itself in the crunode.

78. The equation of the twofold centre locus (making in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$, the foregoing transformation) is

$$\sqrt{\tfrac{1}{2}(x+yi)}+\sqrt{\tfrac{1}{2}(x-yi)}+\sqrt{1-x}=0,$$

that is

$$x+\sqrt{x^2+y^2}=1-x,$$

or

$$\sqrt{x^2+y^2}=1-2x,$$

which is

$$3x^2 - 4x - y^2 = -1,$$

or

$$3\left(x - \tfrac{2}{3}\right)^2 - y^2 = \tfrac{1}{3},$$

or finally

$$9\left(x - \tfrac{2}{3}\right)^2 - 3y^2 = 1,$$

which is a hyperbola having its centre at the harmonic point $x = \frac{2}{3}$, $y = 0$; having $x = \frac{1}{3}$, $x = 1$ for the extremities of the transverse axis, and such that the asymptotes are inclined to the axis of x at an angle $= 60°$; this curve is also shown in the figure.

79. Similarly making the transformation in the equation of the one-with-twofold centre locus written under the form $-(-x+y+z)(x-y+z)(x+y-z) + xyz = 0$, this becomes

$$-(-yi + z)(yi + z)(x - z) + \tfrac{1}{4}(x + yi)(x - yi)z = 0,$$

that is

$$-4(y^2 + z^2)(x - z) + (x^2 + y^2)z = 0,$$

which is

$$-4z^2(x - z) + x^2 z + y^2(5z - 4x) = 0,$$

or, what is the same thing,

$$z(x - 2z)^2 + y^2(5z - 4x) = 0,$$

or, putting for z its value $= 1 - x$, this is

$$(1 - x)(3x - 2)^2 + y^2(5 - 9x) = 0,$$

that is

$$y^2 = -\frac{(x-1)(3x-2)^2}{9x-5},$$

which is the equation of the one-with-twofold centre locus.

80. The curve is symmetrical in regard to the axis of x. And moreover

$x < \frac{5}{9}$, y is impossible,

$x = \frac{5}{9}$, $y^2 = \infty$, or the line $x = \frac{5}{9}$ is an asymptote,

$x = \frac{2}{3}$, $y^2 = 0$, which is a crunode,

$x = 1$, $y^2 = 0$,

and

$x > 1$, y is impossible.

The equation of the tangents at the crunode are $y^2 = 3\left(x - \frac{2}{3}\right)$, or the tangents are inclined to the axis of x at angles $= 60°$. The curve consists, as shown in the figure, of a single branch cutting itself in the crunode, and tending on each side towards the asymptote.

81. The equation of the harmonic conic, making the foregoing transformation in the equation $\frac{\mu-\nu}{x}+\frac{\nu-\lambda}{y}+\frac{\lambda-\mu}{z}=0$, becomes

$$\frac{\lambda-\nu+\mu i}{x+yi}-\frac{\lambda-\nu-\mu i}{x-yi}-\frac{\mu i}{z}=0,$$

that is

$$-2(\lambda-\nu)yi+2\mu i\,x-\mu i\frac{x^2+y^2}{z}=0,$$

which is

$$\mu(x^2+y^2)-2z\{\mu x-(\lambda-\nu)y\}=0,$$

or, what is the same thing,

$$\mu(x^2+y^2-2zx)+2(\lambda-\nu)yz=0,$$

which, putting for z its value $=1-x$, is

$$\mu(3x^2+y^2-2x)+2(\lambda-\nu)y(1-x)=0,$$

or developing,

$$3\mu x^2-2(\lambda-\nu)xy+\mu y^2-2\mu x+2(\lambda-\nu)y=0,$$

either of which is the equation of the harmonic conic corresponding to the satellite line $(\lambda-\nu)x+\mu y+\nu=0$, or, since the direction is alone material, to the satellite line $(\lambda-\nu)x+\mu y=0$. The second form shows that the conic is

$$\begin{array}{ll} \text{an ellipse for} & 3\mu^2>(\lambda-\nu)^2, \\ \text{a parabola}\ ,, & 3\mu^2=(\lambda-\nu)^2, \\ \text{a hyperbola}\ ,, & 3\mu^2<(\lambda-\nu)^2. \end{array}$$

82. The first form shows that the conic passes through the four points which are the intersection of the ellipse $3x^2+y^2-2x=0$, (or, as the equation may also be written, $9(x-\frac{1}{3})^2+3y^2=1$), with the pair of lines $(x-1)y=0$: this is right, for the points in question are the three points $(x=0,\ y=0)$, $(x=1,\ y=i)$, $(x=1,\ y=-i)$, which are the vertices of the triangle formed by the asymptotes $(x^2+y^2)(x-1)=0$; and the point $x=\frac{2}{3}$, $y=0$, which is the harmonic point.

83. Putting for shortness $\lambda-\nu=\kappa$, so that the equation of the satellite line is $\kappa x+\mu y+\nu=0$, and that of the corresponding harmonic conic is

$$\mu(3x^2+y^2-2x)+2\kappa y(1-x)=0,$$

the coordinates of the centre are found from the formulæ

$$\tfrac{1}{2}(x+yi)\ :\ \tfrac{1}{2}(x-yi)\ :\ z=(\kappa+\mu i)^2\ :\ (\kappa-\mu i)^2\ :\ -4\mu^2,$$

whence, since $x+z=1$, we have for the coordinates of the centre

$$x=\frac{\mu^2-\kappa^2}{3\mu^2-\kappa^2},\quad y=\frac{-2\mu\kappa}{3\mu^2-\kappa^2},$$

and it is easy to verify that these belong to a point on the twofold centre conic

$$3x^2-4x-y^2+1=0.$$

84. The asymptotes of the harmonic conic meet at the centre; and they again cut the twofold centre conic in two points, the intersections of the last-mentioned conic with the line

$$(\kappa + \mu i)\,\tfrac{1}{2}(x + yi) + (-\kappa + \mu i)\,\tfrac{1}{2}(x - yi) - 2\mu iz = 0$$

that is

$$\mu x + \kappa y - 2\mu z = 0,$$

or, what is the same thing,

$$3\mu x + \kappa y - 2\mu = 0.$$

85. I remark that the equation of the asymptotes of the harmonic conic is

$$(3\mu^2 - \kappa^2)\,[\mu\,(3x^2 + y^2 - 2x) + 2\kappa y\,(1 - x)] + \mu\,(\mu^2 + \kappa^2) = 0,$$

and that the theorem for the construction of the asymptotes depends on the identical equation

$$(3\mu^2 - \kappa^2)\,[\mu\,(3x^2 + y^2 - 2x) + 2\kappa y\,(1 - x)] + \mu\,(\mu^2 + \kappa^2) + 3\mu\,(\mu^2 + \kappa^2)\,(3x^2 - y^2 - 4x + 1)$$

$$= -2\,[-(3\mu^2 - \kappa^2)\,x + 2\mu\kappa\,y + \mu^2 + \kappa^2]\,(3\mu x + \kappa y - 2\mu),$$

which is easily verified; and where

$$-(3\mu^2 - \kappa^2)\,x + 2\mu\kappa\,y + \mu^2 + \kappa^2 = 0$$

is the equation of the tangent of the twofold centre conic $3x^2 - y^2 - 4x + 1 = 0$ at the centre of the harmonic conic.

86. On account of the symmetry in regard to the axis of x, it will be sufficient to attend to the series of curves corresponding to a direction $y = -\dfrac{\kappa}{\mu}x$ of the satellite line, for which the ratio $-\dfrac{\kappa}{\mu}$ has a given sign; and the inclination of the satellite line to the asymptote will pass from 90° to 0° according as the value of the ratio $-\dfrac{\kappa}{\mu}$ passes from 0 to ∞.

87. First, if $-\dfrac{\kappa}{\mu}$ is $=0$, that is, if the satellite line be perpendicular to the asymptote, then the harmonic conic is the ellipse

$$3x^2 + y^2 - 2x = 0,$$

or, as it may also be written,

$$9\,(x - \tfrac{1}{3})^2 + 3y^2 = 1.$$

As $-\dfrac{\kappa}{\mu}$ increases from 0 to $\sqrt{3}$, that is, as the inclination of the satellite line diminishes from 90° to 30°, the harmonic conic becomes an ellipse of greater and greater excentricity, and ultimately a parabola.

88. I notice the particular case $-\frac{\kappa}{\mu}=\frac{1}{2}$, which corresponds to the direction parallel to one of the nodal tangents of the envelope: the harmonic conic is in this case the ellipse

$$3x^2+y^2-2x-y+xy=0,$$

which, it will be observed, passes through the point ($x=0$, $y=1$) which is one of the points of intersection of the twofold centre locus or hyperbola $3x^2-4x-y^2=-1$ by the line $x=0$.

89. For the value $-\frac{\kappa}{\mu}=\sqrt{3}$ corresponding to a direction inclined at an angle $=30°$ to the asymptote, the harmonic conic becomes the parabola $(x\sqrt{3}-y)^2-2(x-y\sqrt{3})=0$: this equation may also be written $(x-\frac{1}{2})^2+y^2=\frac{1}{4}(x+y\sqrt{3}-1)^2$, a form which puts in evidence the focus and directrix of the parabola.

90. For a value $-\frac{\kappa}{\mu}>\sqrt{3}$, that is, when the satellite line is inclined to the asymptote at an angle $<30°$, the harmonic conic is a hyperbola, and ultimately when $-\frac{\kappa}{\mu}=\infty$, or the satellite line is parallel to the asymptote, the harmonic conic becomes the pair of lines $y(1-x)=0$.

91. I have in the figure shown the following forms of the harmonic conic; viz.

hyperbola, corresponding to inclination $<30°$ of satellite line to asymptote.

parabola, to inclination $=30°$.

$$\left.\begin{array}{l}\text{ellipse}\\\text{ellipse}\\\text{ellipse}\end{array}\right\}\text{to inclination}\left\{\begin{array}{l}<\\=\\>\end{array}\right.\text{inclination } (=\tan^{-1}2) \text{ of a nodal tangent of the envelope,}$$

and for these forms respectively the successive positions of the satellite line are indicated as follows:

92. For the inclination $<30°$, the positions are $ACMC'DEA'$, viz.

A, at infinity,

C, touching lower branch of envelope,

M, between C and C',

C', touching upper branch of envelope,

D, passing through asymptote point D_1,

E, passing through crunode of envelope,

A', at infinity.

The corresponding positions of the critic centres on the hyperbola are

On one branch of hyperbola.	On the other branch.
A_1, A_3, each at infinity,	A_2, the harmonic point,
B_1, B_3,	B_2,
C_{13}, a twofold centre,	C_2, a one-with-twofold centre,
	M_2, a real centre,
	C'_3, a one-with-twofold centre and
	C'_{12}, a twofold centre,
	D_3 and
	D_1 (Asymptote-Point), D_2,
	E_3 and E_1, E_2,
	A'_3, at infinity, and
	A'_1, at infinity, A_2, harmonic point.

93. For inclination $=30^\circ$ the positions are $(AC)MC'DE(A'C)$, viz.

AC, at infinity, touches envelope at infinity,
M, between (AC) and C',
C', touching upper branch of envelope,
D, passing through asymptote point D_1,
E, passing through crunode of envelope,
$A'C$, at infinity touches envelope at infinity.

The corresponding positions of the critic centres on the parabola are

A_2, (harmonic point) a one-with-twofold centre; $A_{13} = A'_{13}$ at infinity, a twofold centre,

M_2, real centre, the other two centres being imaginary,

C'_{12}, a twofold centre, C'_3, a one-with-twofold centre,

D_1, (asymptote point), D_3; D_2,

E_1, E_3; E_2.

A_{13}, $= A'_{13}$, at infinity, a one-with-twofold centre; A_2 (harmonic point) a twofold centre.

94. For inclination $=\tan^{-1}2$, the positions are $AMC'D(EC'')NA'$, viz.

A, at infinity,
M, between A and C',
C', touching upper branch of envelope,
D, through asymptote point D_1,
EC''_1, touching upper branch of envelope at crunode N between EC'' and A',
A', at infinity.

And the corresponding positions of the critic centres on the ellipse are

A_2, (harmonic point), the other two centres imaginary,

M_2, real centre, the other two centres imaginary,

C'_{12}, a twofold centre, C'_3 a one-with-twofold centre,

D_1, (asymptote point), $D_2 : D_3$,

$C''E_{13}$, twofold centre, $C''E_2$ one-with-twofold centre,

N_2, real centre, the other two centres imaginary,

A_2, harmonic point; the other two centres imaginary.

95. And for the inclinations < and > $\tan^{-1}2$, the only difference is that the positions are $AC'DEC''A'$, viz.

A, at infinity,

M, between A and C'',

C', touching upper branch of envelope,

D, through asymptote point D_1,

E, through crunode,

C'', touching upper branch of envelope,

N, between C'' and A',

A', at infinity,

and that instead of the points $C''E_{13}$ and $C''\ E_2$ we have separately the points C''_{13}, C''_2 and E_1, E_3, E_2 as shown in the figure.

96. For the better understanding of the figure it is to be observed that the points D_2 and D_3 lie on the line $x=\frac{1}{2}$: this depends on the theorem No. 81 of the Memoir on Involution, viz. of the critic centres which belong to a satellite line through the vertex (0, 0, 1), one is the vertex itself, the other two lie on the line $x+y-z=0$; or making the transformation $\frac{1}{2}(x+iy)$, $\frac{1}{2}(x-iy)$, z for x, y, z and writing $x+z=1$, of the critic centres which pass through the vertex (0, 0, 1) (the asymptote point), one is this point itself, the other two lie on the line $x-z=0$, that is $x=\frac{1}{2}$.

97. Again it is to be observed that the centre E_3 lies on the line $x=0$, and the centres E_1, E_2 on the circle (indicated by a dotted line in the figure) $(x+\frac{1}{2})^2+y^2=\frac{5}{4}$: this depends on the theorem Nos. 73 and 74 of the Memoir on Involution, viz. of the critic centres for satellite lines through the node (acnode) (1, 1, −4), one lies on the line $x+y=0$, and the other two lie on the conic $z(x+y+z)-4xy=0$; making the substitution

$$\tfrac{1}{2}(x+yi),\quad \tfrac{1}{2}(x-yi),\quad z\quad \text{for } x,\ y,\ z,$$

and writing also $x+z=1$, we find that, for the satellite lines which pass through the crunode (1, 0, 2), the critic centres lie one on the line $x=0$, the other two on the conic

$$z(x+z)-x^2-y^2=0,$$

that is, on the circle $x^2+y^2+x-1=0$, or $(x+\frac{1}{2})^2+y^2=\frac{5}{4}$.

98. The circle in question cuts the twofold centre conic $3x^2-4x-y^2=-1$ at its intersections with the line $x=0$, viz. in the points $x=0$, $y=\pm 1$; and it moreover touches the one-with-twofold centre locus $y^2=\dfrac{-(x-1)(3x-2)}{9x-5}$ at a point where this same circle meets the ellipse $3x^2+xy+y^2-2x-y=0$, which is the harmonic conic corresponding to the inclination $\tan^{-1}2$. In fact, writing down the three equations,

$$x^2+y^2+x-1=0,$$

$$3x^2+xy+y^2-2x-y=0,$$

$$y^2=-\frac{(x-1)(3x-2)^2}{9x-5},$$

the first and third equations give

$$-\frac{(x-1)(3x-2)^2}{9x-5}=1-x-x^2,$$

that is

$$-(x-1)(3x-2)^2+(9x-5)(x^2+x-1)=0,$$

or, reducing,

$$(5x-3)^2=0,$$

that is $x=\frac{3}{5}$, and then from the first or third equation $y^2=\frac{1}{25}$, or $y=\pm\frac{1}{5}$; hence the circle *touches* the one-with-twofold centre locus at the points

$$x=\tfrac{3}{5},\quad y=-\tfrac{1}{5};\quad x=\tfrac{3}{5},\quad y=+\tfrac{1}{5};$$

and by means of the second equation we see that the first of these points, viz. the point $x=\frac{3}{5}$, $y=-\frac{1}{5}$, is a point of the ellipse or harmonic conic $3x^2+xy+y^2-2x-y=0$.

99. I consider the analytical theory of the case where the satellite line is parallel to the asymptote; this is in fact similar to the theory *ante* Nos. 67—69; writing

$$\tfrac{1}{2}(x+yi),\quad \tfrac{1}{2}(x-yi),\ z,\ \lambda-\mu i,\ \lambda+\mu i,\ \nu$$

in the place of x, y, z, λ, μ, ν, and putting afterwards $\mu=0$, that is, in the transformed equation $\lambda x+\mu y+\nu z=0$ writing $\mu=0$, we find for the satellite line $\lambda x+\nu(1-x)=0$; the equation in θ (the factor $\theta+\lambda=0$ being disregarded) is

$$\theta^2-\lambda\theta-2\lambda\nu=0,$$

and the corresponding critic centres lie on the line $y=0$, at the distances given by the equation

$$x:1-x=\frac{2}{\theta+\lambda}:\frac{1}{\theta+\nu},\quad \text{whence } x=\frac{\theta}{\theta+\lambda},\quad z=1-x=\frac{\lambda}{\theta+\lambda};$$

we have then

$$\theta = \frac{\lambda(1-z)}{z},$$

and the values of z are given by the equation

$$\lambda(1-z)^2 - \lambda z(1-z) - 2\nu z^2 = 0,$$

that is

$$2(\lambda-\nu)z^2 - 3\lambda z + \lambda = 0,$$

which, putting $\frac{\nu}{\nu-\lambda} = 1-\varpi$, or $\varpi = \frac{\lambda}{\lambda-\nu}$ (ϖ is the distance of the satellite line from the asymptote $z=0$), becomes

$$2z^2 - 3\varpi z + \varpi = 0,$$

or we have

$$z = \tfrac{1}{4}\{3\varpi \pm \sqrt{\varpi(9\varpi-8)}\}.$$

The condition for a twofold centre is ($\varpi = 0$ which may be disregarded, or else)

$$9\varpi - 8 = 0;$$

or, what is the same thing, $\lambda + 8\nu = 0$.

100. If z_1, z_2 are the coordinates of the two critic centres, then we have

$$2z_1^2 = \varpi(3z_1-1),$$

$$2z_2^2 = \varpi(3z_2-1),$$

and thence

$$\frac{z_1^2}{z_2^2} = \frac{3z_1-1}{3z_2-1};$$

or, reducing,

$$3z_1z_2 - z_1 - z_2 = 0,$$

or in terms of the x-coordinates

$$3x_1x_2 - 2(x_1+x_2) + 1 = 0,$$

which equation however merely expresses that the two centres are harmonics to each other in regard to the twofold centre conic $3x^2 - 4x - y^2 + 1 = 0$. It is right to remark that the formulæ, although referring to a different system of coordinates, are absolutely identical with those given Nos. 67—69, writing therein ν for μ, and z for x.

101. An inspection of the form of the envelope shows what are the positions of the satellite line which gives rise to Plücker's Groups for the Hyperbolas Δ Defective. We have in fact,

Hyperbolas Δ Defective, asymptote not osculating.

The satellite line is not parallel to the asymptote, and the different positions give the following six of Plücker's Groups, viz.

XVIII. Satellite line cuts upper, cuts lower, branch of envelope.

XX. " " " " and passes through asymptote point.

XIX. Satellite line touches upper, cuts lower, branch.

XXI. " does not cut upper, cuts lower, branch.

XXIII. " " " , touches lower branch.

XXII. " " " , does not cut lower branch.

Hyperbolas Δ Defective, asymptote osculating.

The satellite line is parallel to the asymptote, and we have the six groups,

XXVIII. Satellite line cuts upper branch, cuts lower branch of envelope, viz. it lies above the asymptote point.

XXIX. Do, Do, but it passes through the asymptote point.

XXX. Do, Do, but it lies below the asymptote point.

XXXI. Satellite line touches upper branch, cuts lower branch.

XXXII. " does not cut upper branch, cuts lower branch.

XXXIII. " " " , does not cut lower branch, viz. it lies below the asymptote.

And finally,

Hyperbolas Δ Defective, three osculating asymptotes.

Satellite line at infinity, giving the single group

XXXV.

But the division gives rise to a remark such as is made *ante* No. 71.

As to the Groups of the Hyperbolas ⊙. Article Nos. 102 to 104.

102. Taking $z=0$ as the equation of the line infinity, and $x=0$, $y=0$ as the equation of any two lines through the point of intersection of the asymptotes, or 'asymptote point,' then the equation of the cubic may be taken to be

$$\tfrac{1}{3}(a,\ b,\ c,\ d \between x,\ y)^3 + kz^2(\lambda x + \mu y + \nu z) = 0.$$

103. To determine the critic centres we have

$$(a,\ b,\ c \between x,\ y)^2 + kz^2\lambda = 0,$$

$$(b,\ c,\ d \between x,\ y)^2 + kz^2\mu = 0$$

$$2z(\lambda x + \mu y + \nu z) + z^2\nu = 0,$$

and thence

$$\mu (a,\ b,\ c \between x,\ y)^2 - \lambda (b,\ c,\ d \between x,\ y)^2 = 0,$$

or as it may also be written

$$(\mu a - \lambda b,\ \mu b - \lambda c,\ \mu c - \lambda d \between x,\ y)^2 = 0,$$

and also

$$2 (\lambda x + \mu y) - 3\nu z = 0,$$

which two equations determine the critic centres for a given position of the satellite line; the first of them gives a pair of lines through the asymptote point; the latter is a line parallel to the satellite line: there are thus two critic centres.

104. The condition for a twofold centre is

$$(ac - b^2,\ bc - ad,\ bd - c^2 \between \lambda,\ \mu)^2 = 0,$$

so that there are a pair of twofold centres which will be real if

$$(bc - ad)^2 - 4 (ac - b^2) (bd - c^2) = +,$$

imaginary if

$$(bc - ad)^2 - 4 (ac - b^2) (bd - c^2) = -,$$

that is, the twofold centres will be real or imaginary, according as the equation

$$(a,\ b,\ c,\ d \between x,\ y)^3 = 0$$

has its roots one real and two imaginary, or all three real; viz. the twofold centres are real for the Hyperbolas ⊙ Defective; imaginary for the Hyperbolas ⊙ Redundant. And we see also that for the Hyperbolas ⊙ Redundant the critic centres are always real; but that for the Hyperbolas ⊙ Defective, they may be both real, or both imaginary, or may coincide together, giving a twofold centre. But the two cases are best studied by assuming different special forms for the equation.

The Hyperbolas ⊙ Redundant. Article Nos. 105 to 107.

105. The equation may be taken to be

$$xy (x - y) + kz^2 (\lambda x + \mu y + \nu z) = 0,$$

or writing $z = 1$, then the equation is

$$xy (x - y) + k (\lambda x + \mu y + \nu z) = 0.$$

We may, to fix the ideas, consider the case where the three asymptotes are parallel to the sides of an equilateral triangle; x, y, and $x - y$ will then denote the perpendicular distances of the point from the three asymptotes respectively.

106. The critic centres are given by the equations

$$2(\lambda x + \mu y) - 3\nu = 0,$$
$$\lambda x^2 - 2(\lambda + \mu)xy + \mu y^2 = 0;$$

or, what is the same thing, they are the intersections of the line

$$2(\lambda x + \mu y) - 3\nu = 0$$

by the two real lines

$$\lambda x = [(\lambda + \mu) \pm \sqrt{\lambda^2 + \lambda\mu + \mu^2}]\, y.$$

107. The groups are

Hyperbolas ⊙ Redundant. No osculating asymptote.

The satellite line not parallel to any asymptote, that is $\lambda = 0$, $\mu = 0$, $\lambda + \mu = 0$. We have the two groups

VII. Satellite line does not pass through asymptote point (ν not $= 0$).

VIII. Satellite line passes through asymptote point ($\nu = 0$).

Hyperbolas ⊙ Redundant. One osculating asymptote. Satellite line is parallel to an asymptote, suppose to the asymptote $x = 0$; that is, $\mu = 0$, or the satellite line is $\lambda x + \nu = 0$. We have only the group

XV.

Hyperbolas ⊙ Redundant. Three osculating asymptotes. Satellite line lies at infinity, that is, $\lambda = 0$, $\mu = 0$. We have only the group

XVII.

The Hyperbolas ⊙ Defective. Article Nos. 108 to 110.

108. The equation may be taken to be

$$\tfrac{1}{3}x(x^2 + y^2) + kz^2(\lambda x + \mu y + \nu z) = 0,$$

or writing $z = 1$, then it is

$$\tfrac{1}{3}x(x^2 + y^2) + k(\lambda x + \mu y + \nu) = 0,$$

and if to fix the ideas we take the case where the two imaginary asymptotes are the asymptotes of a circle, then x, y will be ordinary rectangular coordinates.

109. The critic centres are given by the equations

$$2(\lambda x + \mu y) - 3\nu = 0,$$
$$3\mu x^2 - 2\lambda xy + \mu y^2 = 0,$$

that is they are the intersections of the line $2(\lambda x + \mu y) - 3\nu = 0$ (which is a line parallel to the satellite line, on the other side of the asymptote point and at a distance from it $= \frac{3}{2}$ distance of satellite line) by the pair of lines

$$3\mu x = (\lambda \pm \sqrt{\lambda^2 - 3\mu^2})\, y.$$

Hence the critic centres are real if $\lambda^2 > 3\mu^2$, that is, if the satellite line is inclined to the asymptote at an angle $> 60^\circ$; imaginary if $\lambda^2 < 3\mu^2$, that is, if the satellite line is inclined to the asymptote at an angle $< 60^\circ$: and there is a twofold centre if $\lambda^2 = 3\mu^2$, that is, if the inclination is $= 60^\circ$. This assumes, however, that ν is not $= 0$, that is that the satellite line does not pass through the asymptote point; when it does the distinction of the cases disappears. Hence the groups are

110. Hyperbolas ⊙ Defective. No osculating asymptote. The Satellite line is not parallel to the asymptote, and the groups are,

Satellite line not passing through asymptote point.

XXIV. Satellite line inclined to asymptote at angle $> 60^\circ$.

XXV. " " " " $= 60^\circ$.

XXVI. " " " " $< 60^\circ$.

Satellite line passes through asymptote point, the single group

XXVII.

Hyperbolas ⊙ Defective. Real osculating asymptote. The satellite line is parallel to the asymptote, and we have the single group

XXXIV.

Hyperbolas ⊙ Defective. Three osculating asymptotes. Satellite line is at infinity and we have the single group

XXXVI.

The foregoing theory of the hyperbolas Δ and ⊙ completes the enumeration of the groups I. to XXXVI.

As to the groups of the parabolic hyperbolas. Article Nos. 111 to 115.

111. I consider the equation in the form

$$\tfrac{1}{2}x(by^2 + cz^2 + 2gzx) + kz^2(\mu y + \nu z) = 0,$$

viz. the cubic $x(by^2 + cz^2 + 2gzx) = 0$ is made up of a conic $by^2 + cz^2 + 2gzx = 0$, and a line $x = 0$; the other cubic $z^2(\mu y + \nu z) = 0$ is made up of a tangent of the conic, regarded as a twofold line, $z^2 = 0$, and of a line $\mu y + \nu z = 0$ through the point of contact of such tangent.

112. To determine the critic centres we have

$$\begin{aligned} x \,.\, gz + \tfrac{1}{2}(by^2 + cz^2 + 2gzx) &= 0, \\ x \,.\, by + kz \,.\, \mu z &= 0, \\ x(cz + gz) + kz(2\mu y + 3\nu z) &= 0\,; \end{aligned}$$

eliminating k from the second and third equations

$$\mu z(cz+gx)-by(2\mu y+3\nu z)=0,$$

that is

$$-2b\mu y^2+c\mu z^2+g\mu zx-3b\nu yz=0\,;$$

or reducing by means of the first equation written under the form

$$by^2+cz^2+4gzx=0,$$

we find

$$3c\mu z^2+9g\mu zx-3b\nu yz=0,$$

that is $z=0$, which may be rejected, or else

$$c\mu z+3g\mu x-b\nu y=0,$$

or, as it may be written,

$$\mu(3gx+cz)-\nu by=0.$$

Hence the entire series of critic centres lie on the conic

$$by^2+cz^2+4gzx=0,$$

and corresponding to the satellite line $\mu y+\nu z=0$, we have the two critic centres which are the intersections of the conic by the line

$$\mu(3gx+cz)-\nu by=0,$$

the lines pass through the fixed point $3gx+cz=0$, $y=0$, and form a pencil homographic with the satellite lines $\mu y+\nu z=0$.

113. We have a twofold centre if the line touches the conic, the condition for this is

$$(bc,\ -4g^2,\ 0,\ 0,\ -2bg,\ 0\between 3g\mu,\ -b\nu,\ c\mu)^2=0,$$

that is $3bg^2(3c\mu^2+4b\nu^2)=0$, or simply,

$$3c\mu^2+4b\nu^2=0,$$

and from this and the equation $\mu y+\nu z=0$, eliminating μ and ν we find

$$4by^2+3cz^2=0,$$

for the equation of the satellite lines which respectively give rise to a twofold centre; the lines in question are real or imaginary according as the lines $by^2+cz^2=0$ are real or imaginary, that is, according as the line $x=0$ cuts the conic $by^2+cz^2+2gzx=0$ in two real, or in two imaginary, points.

114. Writing now $b=1$, $c=-mn$, $2g=n$ and $x+mz$ in the place of z, the equation is

$$(x+mz)(y^2+nzx)+2kz^2(\mu y+\nu z)=0,$$

and we may consider $z=0$ as the equation of the line infinity: writing in the formulæ $z=1$, the critic centres are given as the intersection of the parabola

$$y^2+2nx+mn=0,$$

with the line

$$\nu y=\tfrac{1}{2}\mu n(3x+m);$$

and the condition for a two-fold centre is

$$4\nu^2-3mn\mu^2=0;$$

the equation of the satellite lines corresponding respectively to a two-fold centre is

$$4y^2-3mn=0;$$

the lines are real or imaginary according as mn is positive or negative, or (observing that the equations $x+m=0$, $y^2+nx=0$ give $y^2-mn=0$), according as the line $x+m=0$ cuts or does not cut the parabola $y^2+nx=0$. Suppose for a moment that the line does cut the parabola and that y_1 is the corresponding value of y, then $y_1^2=mn$; and the equation $4y^2-3mn=0$ of the satellite lines which correspond respectively to the case of a two-fold centre is $y^2=\frac{3}{4}y_1^2$. We have thus $y=\pm y_1$ and $y=\pm\sqrt{\frac{3}{2}}\,y_1$ as special positions of the satellite line $\mu y+\nu=0$. In the case where the line $x+m=0$ touches the parabola $y^2+nx=0$, the value of y_1 is $=0$, and we have only the special position $y=0$; finally, when the line does not cut the parabola there is no special position.

115. Plücker's groups are consequently as follows:

Parabolic Hyperbolas; ordinary asymptote and five-pointic asymptotic parabola, that is the line $\mu y+\nu=0$ is not at infinity.

Asymptote cuts parabola, $mn=+$.

XXXVII. Satellite line lies outside the lines $y=\pm y_1$ which belong to the points of intersection.

XXXVIII. Satellite line passes through a point of intersection, that is, coincides with one of the lines $y=\pm y_1$.

XXXIX. Satellite line lies between the lines $y=\pm y_1$ and $y=\pm\sqrt{\frac{3}{2}}\,y$.

XL. Satellite line coincides with one of the lines $y=\pm\sqrt{\frac{3}{2}}\,y_1$, which give respectively a two-fold centre.

XLI. Satellite line lies between the lines $y=\pm\sqrt{\frac{3}{2}}\,y_1$.

Asymptote touches parabola, viz. $m=0$.

XLIII. Satellite line does not pass through the point of contact.

XLIV. Satellite line passes through point of contact or its equation is $y=0$.

Asymptote does not cut parabola, viz. $mn=-$. This gives the single group

XLII.

Parabolic hyperbolas. Osculating asymptote and six-pointic asymptotic parabola. The satellite line is here at infinity, and there is no new distinction of groups. The groups therefore are

Asymptote cuts parabola,

XLV.

Asymptote does not cut parabola,

XLVI.

Asymptote touches parabola,

XLVII.

As to the Groups of the Central and Parabolic Hyperbolisms. Article No. 116.

116. For the Hyperbolisms, Central and Parabolic, since these have a node or a cusp at infinity, they cannot acquire a new node, and the theory of critic centres does not arise. There is, however, as regards the Hyperbolisms of the Hyperbola a distinction in the position of the satellite line, viz. this may lie outside, or between, the parallel asymptotes. The groups are

Hyperbolisms of the Hyperbola. Ordinary asymptote. The satellite line is not at infinity, and it may lie in either of the positions just mentioned. We have therefore

XLVIII. Satellite line lies between the parallel asymptotes.

XLIX. " " outside " " .

Osculating asymptote; the satellite line is at infinity. We have

L.

Hyperbolisms of the Ellipse. Ordinary asymptote. The satellite line is not at infinity, and we have

LI.

Osculating asymptote. Satellite line is at infinity,

LII.

Hyperbolisms of the Parabola. Ordinary asymptote. Satellite line is not at infinity we have

LIII.

Osculating asymptote: satellite line is at infinity: we have

LIV.

As to the Groups of the Divergent Parabolas. Article Nos. 117 and 118.

117. Taking the equation under the form

$$ax^3 + by^2z + kz^2(\lambda x + \nu z) = 0,$$

we find for a critic centre

$$\begin{aligned} 3ax^2 + kz \,.\, \lambda z &= 0, \\ 2byz &= 0, \\ by^2 + kz(2\lambda x + 3\nu z) &= 0\,; \end{aligned}$$

hence there is a single critic centre $y = 0$, $2\lambda x + 3\nu z = 0$, the critic value of k is $k = \dfrac{-27a\nu^2}{4\lambda^3}$, and with this value of k the equation in fact is

$$4\lambda^3(ax^3 + by^2z) - 27a\nu^2(\lambda x + \nu z)z^2 = 0,$$

that is

$$a(4\lambda^3x^3 - 27\lambda^2\nu x^2z - 27\nu^2z^2) + 4\lambda^3by^2z = 0,$$

or, as it may be written,

$$a(2\lambda x + 3\nu z)^2(\lambda x - 3\nu z) + 4\lambda^3by^2z = 0,$$

which puts in evidence the critic centre or node of the curve. But, as there is here only a single critic centre, there is of course no further theory of the two-fold centre, &c.

118. The groups are as given *à priori* by the relation of the satellite line $\lambda x + \nu z = 0$, to the semicubical parabola $ax^3 + by^2z = 0$, viz. writing $z = 1$ and changing the constants,

Divergent Parabolas, the semicubical parabola $y^2 = x^3$, which is

LV.

Divergent Parabolas, Asymptotic Semicubical Parabola of seven-pointic contact, viz. equation of the asymptotic parabola being $y^2 - x^3 = 0$, and writing for convenience $k(\lambda x + \nu) = -3ax + 2b = 0$ for the equation of the satellite line, the equation of the curve is $y^2 = x^3 - 3ax + 2b$. And the groups are

LVI. Satellite line cuts asymptotic parabola.

LVII. „ does not cut „ „

LVIII. „ passes through vertex of parabola.

And further

Divergent Parabolas. Asymptotic Semicubical Parabola of nine-pointic intersection. The satellite line is at infinity, and the equation is $y^2 = x^3 + 2b$. This is group

LIX.

As to the Trident Curve and the Cubical Parabola. Article Nos. 119 and 120.

119. For the Trident Curve, equation is $x(x^2 + \lambda y) + \mu = 0$, or the satellite line is at infinity, and there is no distinction of groups; we have only group

LX.

120. For the Cubical Parabola this is $x^3 + \mu y = 0$, there is no distinction of groups, and the curve is group

LXI.

As to the Division into Species: Comparison of Newton and Plücker.

Article Nos. 121 and 122.

121. The division into species is obtained without difficulty when the groups are once established; in fact it only remains to trace for each given form of V and s the series of curves $V+\mu s=0$, as μ passes from ∞ to $-\infty$ through the value 0 and the critic values which correspond to nodal curves: I have nothing to add to what has been done by Plücker, and it is unnecessary to reproduce the investigation. It may be remarked that the mere inspection of Plücker's figures is sufficient to show which of his species correspond to the same Newtonian species; the species which do so belong to the same Newtonian species in some instances closely resemble each other in form, although in others the difference of form is apparent enough: but the Plückerian species which correspond to the same Newtonian species belong for the most part to different groups and are thus distinguished from each other by the characters which distinguish the groups to which they respectively belong: thus for instance Newton's Species 1 (a hyperbola Δ Redundant) is characterised as consisting of three hyperbolic branches, one inscribed, one circumscribed, and one ambigene, with an oval. Such a curve may exist with three different positions of the satellite line in regard to the asymptotes, viz. the satellite line may cut the three sides produced, or it may pass through a vertex, cutting the opposite side produced, or it may cut two sides and the third side produced, not cutting the envelope—which are the characters of Plücker's groups I, II, IV, respectively, and there belongs to the Newtonian species 1, a species out of each of these groups, viz. they are I. 1: II. 9, and IV. 18.

122. The correspondence of the Plückerian Species with those of Newton is shown in the following Table.

Newton's Genus 1, contains 9 Species, viz.

	1	2	3	4	5	6	7	8	9
corresponding to Plücker's Species									
I.	1			2	3	8	7	4	5, 6
II.	9			10	11			12	13, 14
III.									15
IV.	18	17		19	16, 20			21	22, 23
V.			25		24, 26			27	28, 29
VI.					30			31	32, 33

Part of Newton's Genus 4, contains 3 Species, viz.

	24	25	26
corresponding to Plücker's Species			
VII.	34	35	36
VIII.		37	

Newton's Genus 2, contains 14 Species, viz.

corresponding to Plücker's Species

	10	10′	11	12	13	13′	14	15	16	17	18	19	20	21
IX.	38				39		40			43	42		41	
X.	44				45		46						47	
XI.	50		49		51		48, 52						53	
XII.				55			54, 56						57	
XIII.							58						59	
XIV.		60				61		62	65			63		64

Further part of Newton's Genus 4, contains 4 Species, viz.

corresponding to Plücker's Species

	28	29	30	31
XV.	69	66	67	68

Newton's Genus 3, contains 4 Species, viz.

corresponding to Plücker's Species

	22	22′	22″	23
XVI.	72	70	71	73

Residue of Newton's Genus 4, contains 1 Species, viz.

corresponding to Plücker's Species

	32
XVII.	74

Newton's Genus 5, contains 6 Species, viz.

corresponding to Plücker's Species

	33	34	35	36	37	38
XVIII.	77, 82	78, 81		76	75, 79, 80	
XIX.	88	87	84		83, 85, 86	
XX.	89	90			91	
XXI.	92	93			94, 95	
XXII.	96, 100	97, 101		99	98, 102, 103	
XXIII.	104	105	107		106, 108, 109	
XXIV.	112	113		111	110, 114, 115	
XXV.			117		116, 118, 119	
XXVI.					120	
XXVII.						121

Newton's Genus 6, contains 7 species, viz.

39 40 41 42 43 44 45

corresponding to Plücker's Species

	39	40	41	42	43	44	45
XXVIII.	122	125	123			126	124, 127
XXIX.	128	131	129				130
XXX.	134	137	135		133		132, 136
XXXI.		141		139			138, 140
XXXII.		143					142
XXXIII.	144	148	145				146, 149
XXXIV.		152					150, 153
XXXV.	154		155				156, 157
XXXVI.							158

Newton's Genus 7, contains 7 Species, viz.

46 47 48 49 50 51 52

corresponding to Plücker's Species

	46	47	48	49	50	51	52
XXXVII.	159			160	161, 162	163	164
XXXVIII.	165			166	167, 168		
XXXIX.	171	170		172	169, 173, 174		
XL.			176		175, 177, 178		
XLI.					179		
XLII.					182	181	180
XLIII.					185, 186	184	183
XLIV.					187		

Newton's Genus 8, contains 6 Species, viz.

53 54 55 56 56′ 56″

corresponding to Plücker's Species

	53	54	55	56	56′	56″
XLV.	190			191	188	189
XLVI.	194	193	192	195		
XLVII.	196			197		

Newton's Genus 9, contains 4 Species, viz.

57 $\underbrace{58 \quad 59}$ 60

corresponding to Plücker's Species

	57	58, 59	60
XLVIII.		198	
XLIX.	199		
L.			200

Newton's Genus 10, contains 3 Species, viz.

$\underbrace{61 \quad 62} \quad 63$

corresponding to Plücker's Species

LI.	201	
LII.		202

Newton's Genus 11, contains 2 Species, viz.

64 65

corresponding to Plücker's Species

LIII.	203	
LIV.		204

Newton's Genus 12, contains 1 Species, viz.

66

corresponding to Plücker's Species

LX.	218

Newton's Genus 13, contains 5 Species, viz.

67 68 69 70 71

corresponding to Plücker's Species

LV.				205	
LVI.	208	207			206, 209
LVII.	212		211		210, 213
LVIII.	214				215
LIX.					216, 217

Newton's Genus 14, contains 1 Species, viz.

72

corresponding to Plücker's Species

LXI.	219

It is to be noticed that (as appears by the Table) Plücker enumerates 13 species of the Divergent Parabola, viz. corresponding to the *Parabola Pura* of Newton we have five species, and to the *Parabola cum Ovali* three species; but to each of the other three Newtonian species (*Nodata, Punctata, Cuspidata*) only a single species. The difference in nowise affects Newton's before-mentioned theorem, that every cubic curve is the shadow of a Divergent Parabola; but (the characters of Plücker's species being unaffected by projection) the number of resulting kinds of cubic curves (or cones) will be five or thirteen according as the one or the other classification is adopted; but this is a subject which I do not enter upon in the present Memoir.

Cambridge, February 8, 1864.