351.

ON CUBIC CONES AND CURVES.

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THERE is contained in Sir Isaac's Newton's *Enumeratio Linearum tertii Ordinis* (1706), under the heading *Genesis Curvarum per Umbras*, the remarkable theorem that, in the same way as the several curves of the second order may be considered as the shadows of a circle, that is, the sections of a cone having a circular base, so the several curves of the third order, or cubic curves, may be considered as the shadows of the five Divergent Parabolas. It was remarked by Chasles, Note XX. to the *Aperçu Historique* (1837), that they may also be considered as the shadows of the five curves having a centre (the Newtonian Species 27, 38, 59, 62, 72), and that the theorem may be stated as follows, viz. (in the same way that all the curves of the second order are the sections of a single kind of cone of the second order, so) all the curves of the third order may be considered as the sections of five kinds of cones of the third order—and that cutting these in one way we have the five Divergent Parabolas, cutting them in another way the five curves with a centre. The nature of these five kinds of cones, or, what is the same thing, that of the spherical curves in which they are intersected by a concentric sphere, was first pointed out by Möbius in his most interesting Memoir, "Grundformen der Linien dritter Ordnung," *Abh. der K. Sächs. Ges. zu Leipzig*, 1853. I reproduce in the present memoir the characterisation of these five kinds of cones—which I call the *simplex*, the *complex*, the *acnodal*, the *crunodal*, and the *cuspidal*—and I further develope the geometrical and analytical theory; in particular I arrive at a division of the simplex cones into three subkinds, the simplex *trilateral*, *neutral*, and *quadrilateral*. I have throughout spoken of cones rather than of plane curves, using however, as far as may be, language which is also applicable to a plane curve, thus, instead of lines of inflexion, tangent planes, of a cone, I say inflexions, tangents, &c. But the theory of the cone is of course that of the projective properties of the curves which are the sections of such cone; it appears to me that the true classification of curves is to divide them according to the cones which give rise to them; and I consider the present memoir as affording in part the materials for such a classification of cubic curves, viz. it seems to me that after, in the first instance, dividing them into the simplex, the complex, the acnodal, the crunodal, and the cuspidal kinds, the simplex kind should be further divided in the above-mentioned manner; and that we should establish, lastly, the divisions which relate to the particular mode in which the cone is to be cut, in order to obtain such and such a curve: in effect, that the principle of classification, according to the nature of the infinite branches adopted by Newton in the work above referred to, and by Plücker in his *System der Analytischen Geometrie* (Berlin, 1835), and to which has reference my Memoir *On the Classification of Cubic Curves*, [350], should be not the primary, but a secondary principle of classification. I remark that as regards the division into five kinds, Murdoch, in his highly interesting work, *Newtoni Genesis Curvarum per Umbras*, [Lond. 1746], has not only distinguished the Newtonian species which arise from each of the Divergent Parabolas, or, what is the same thing, from each of the five kinds of cones (it will presently appear how the mere inspection of Newton's figures is sufficient to enable this), but that he has also shown how the cone must in each case be cut in order to obtain the particular cubic curve. Murdoch also distinguishes the three forms ampullate, campaniform and intermediate, of the simplex Divergent Parabola, which correspond to the simplex quadrilateral, trilateral, and neutral.

I remark also that Plücker in his work above referred to, *Dritter Abschnitt*, 98, has considered the equation of a cubic curve in the form $pqr + \mu s^3 = 0$, which is in fact equivalent to the form $(X+Y+Z)^3 + 6kXYZ = 0$ used in the sequel, but without arriving at the results obtained in the present Memoir.

The five kinds of Cubic Cones. Nos. 1 to 7.

1. A cone of any order may comprise two distinct forms of sheet, viz. (1) a twin-pair sheet, or sheet which meets a concentric sphere in a pair of closed curves such that each point of the one curve is opposite to a point of the other curve (a cone of the second order affords an example of such a sheet); the twin-pair sheet may be considered as consisting of two sheets, each of which may be called a twin sheet: and (2) a single sheet, viz. a sheet which meets a concentric sphere in a closed curve such that each point of the curve is opposite to another point of the curve: the plane affords an example of such a curve. We have five kinds of cubic cones, viz. the simplex, the complex, the acnodal, the crunodal, and the cuspidal. The cone may consist of a single sheet; it is then of the *simplex* kind. Or it may consist of a single sheet and a twin-pair sheet, it is then of the *complex* kind: these are the non-singular kinds. The remaining kinds are singular ones, which are most easily explained by considering them as limiting forms of the complex kind; the twin-pair sheet may come to unite itself with the single sheet giving rise to a crunodal line, or say a crunode; the cone is then of the *crunodal* kind. Or the twin-pair sheet may dwindle into a

mere line which is an acnodal line, or say an acnode; the cone is then of the *acnodal* kind. Or the two things may happen together, viz. the twin-pair sheet at the instant that it unites itself with the single sheet may dwindle into a mere line, which is then a cuspidal line, or say a cusp; and the cone is then of the *cuspidal* kind. I remark, as regards the crunodal kind, that the cone may be considered as consisting of two portions, one of them corresponding to the single sheet of a complex cone, and which I call the quasi-single sheet; the other of them corresponding to the twin-pair sheet, and which I call the loop-sheet.

2. It is to be added that a cubic cone has in general 9 lines of inflexion, or say inflexions, but of these 6 are always imaginary; the remaining 3, which are real, belong to the single sheet. The plane through any two inflexions meets the cone in a line which is also an inflexion. In particular the three real inflexions lie in a plane.

3. When the cone is acnodal the six imaginary inflexions unite at the acnode; and the single sheet has still 3 real inflexions lying in a plane. But if the cone is crunodal, then 4 imaginary inflexions and 2 real inflexions unite in the crunode; and the cone has 1 real inflexion; there are besides 2 imaginary inflexions, the 3 inflexions lie in a real plane. Finally, if the cone is cuspidal, then 2 of the real inflexions, and the 6 imaginary inflexions unite together in the cusp; the cone has besides 1 real inflexion, but there are not any imaginary inflexions.

4. Suppose that the cone is of one of the non-singular kinds; that is, let it be simplex or complex. From any line of the cone we may draw four tangent planes to the cone—the anharmonic ratio of the four planes is the same whatever may be the line on the cone. As regards reality, the following distinction exists, viz. for the complex kind of cone, the planes are all real or all imaginary; for the simplex kind they are two real and two imaginary. First, as regards the complex kind, if the line be taken on the twin-pair sheet, the four tangent planes are all imaginary; but if it be taken on the single sheet, then there are two real tangent planes to the single sheet and two real tangent planes to the twin-pair sheet, together four real tangent planes. Secondly, as regards the simplex kind, there is only the single sheet, and the line being taken on it, there are two real tangent planes and no more.

5. As regards the singular kinds, assuming always that the line on the cone does not coincide with the node or the cusp (for when it does there are no tangent planes), it may be noticed that for the crunodal kind there are two tangent planes which are real or imaginary according as the line lies on the part corresponding to the single sheet or on the part corresponding to the twin-pair sheet. For the acnodal kind there are two tangent planes which are always real; and for the cuspidal kind there is a single tangent plane which is always real.

6. The foregoing properties of cubic cones apply to the curves which are the sections of these cones; thus a cubic curve is of the simplex, the complex, the crunodal, the acnodal, or the cuspidal kind. As regards the last-mentioned three kinds, or singular kinds, it is of course to be borne in mind that the crunode, acnode, or cusp, may be at infinity; and consequently that all the curves in Newton's genus 9 (the hyperbolisms of the hyperbola) and the curve in his genus 12 (the trident curve) belong to the crunodal kind; the curves in genus 10 (the hyperbolisms of the ellipse) to the

acnodal kind; and those of genus 11 (the hyperbolisms of the parabola) and the curve in genus 14 (the cubical parabola) to the cuspidal kind.

7. In the other genera such of the species as are without a node or a cusp, belong to the simplex or the complex kind: and the mere inspection of the figure (Newton's or Plücker's) is sufficient to show to which of the two kinds the curve belongs; in fact, when from any point of the curve there are four real tangents, or there is else no real tangent, the curve is of the complex kind, but if there are two and only two real tangents the curve is of the simplex kind. And in the former case we see whether a branch arises from the single sheet or the twin-pair sheet of the cone, viz. if from a point on the branch there can be drawn four real tangents to the curve, the branch arises from the single sheet, but if no real tangent can be drawn, the branch arises from the twin-pair sheet. And in the crunodal kind we see which part of the curve arises from the quasi-single sheet, and which part from the loop sheet.

Ulterior Theory leading to the Subkinds of the Simplex Cones. Nos. 8 to 35.

(*Several Subheadings.*)

8. But the division of cubic cones may be carried further: we may in fact subdivide the simplex kind. To show how this is, I consider a cone complex or simplex, but I attend for the moment only to the single sheet. The cone has on the single sheet three (real) inflexions lying in a plane. I call this the equator, and I call the tangent planes at the inflexions simply the tangents; the three tangents do not in general meet in a line, and they divide space into eight regions; of these there are two not divided by the equator, and which remain trilateral; the other six regions are divided by the equator each of them into a trilateral and a quadrilateral region, this gives six trilateral regions and six quadrilateral regions; there are thus in all $2+6=8$ trilateral regions (I distinguish them as the 2 and the 6 such regions) and 6 quadrilateral regions.

9. It is easy to see that for a complex cone the single sheet lies wholly in the 6 trilateral regions, and the twin-pair sheet wholly in the 2 trilateral regions. Imagine the twin-pair sheet to dwindle into a line and then disappear, that is, let the cone pass from the complex, through the acnodal, into the simplex kind; the single sheet of the simplex cone will lie wholly in the 6 trilateral regions; this is one form of the simplex cone; I call it the *simplex trilateral.* But there is a different form, viz. the cone may lie wholly in the 6 quadrilateral regions; this is the *simplex quadrilateral.* And there is an intermediate form, viz. the three tangent planes at the inflexions may meet in a line, the 2 trilateral regions then disappear, and there are only 12 regions, all of them trilateral, which may be considered as forming two systems, each of 6 regions, viz. each system consists of three non-contiguous regions on one side of the equator, and (alternating therewith) three non-contiguous regions on the other side of the equator: the cone lies wholly in the one 6 regions or in the other 6 regions and I say that the cone is *simplex neutral.*

10. A non-singular cubic cone (simplex or complex) may be represented by an equation of the canonical form

$$x^3 + y^3 + z^3 + 6lxyz = 0,$$

where the coordinates x, y, z and the parameter l are all real; the invariants of the form are $S = -l + l^4$, $T = 1 - 20l^3 - 8l^6$. It is to be noticed that the form in question cannot represent a singular cone; we find as the condition that it may do so

$$R = 64S^3 - T^2 = -(1 + 8l^3)^3 = 0,$$

but when this condition is satisfied, the cone breaks up into a system of three planes; thus for the real root $l = -\frac{1}{2}$, we have

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

where ω is an imaginary cube root of unity; and by merely writing ωx, $\omega^2 x$ successively in place of x, we see that the like decomposition occurs from the imaginary roots

$$l = -\tfrac{1}{2}\omega, \quad l = -\tfrac{1}{2}\omega^2.$$

11. The equation $x^3 + y^3 + z^3 + 6lxyz = 0$ is in general transformable into the form

$$(X + Y + Z)^3 + 6kXYZ = 0,$$

where X, Y, Z are linear functions of the original coordinates, such that $X = 0$, $Y = 0$, $Z = 0$ are the equations of the tangent planes at three inflexions in the plane $X + Y + Z = 0$; if however the three tangent planes meet in a line, then X, Y, Z will satisfy identically a certain linear equation, and it is clear *à priori* that the transformation must fail. The condition for the three tangent planes meeting in a line is $S = -l + l^4 = 0$, that is, we have

$$l = 0,\ 1,\ \omega, \text{ or } \omega^2;$$

and attending only to the real roots $l = 0$, $l = 1$, it will be presently seen that for $l = 0$ the tangent planes at the three real inflexions do not, for $l = 1$, they do meet in a line. Hence the simplex neutral cone corresponds to the value $l = 1$, that is, the equation is

$$x^3 + y^3 + z^3 + 6xyz = 0,$$

and this equation is not transformable into the form $(X + Y + Z)^3 + 6kXYZ = 0$, which is that employed in the sequel for the general discussion of the simplex and complex cones. The theory on which the foregoing conclusion depends is as follows.

On the condition $S = 0$. Nos. 12 to 17.

12. A cubic has in general nine inflexions, which lie by threes on twelve planes, viz. denoting the inflexions by 1, 2, 3, 4, 5, 6, 7, 8, 9, the planes may be taken to be

123,	456,	789,
147,	258,	369,
159,	267,	348,
168,	249,	357,

that is, we have four systems, each of three planes passing through the nine inflexions.

The tangent planes, or, say the tangents at the inflexions *in plano*, for instance, at the inflexions 1, 2, 3, form a trilateral, and we have thus corresponding to each of the three planes a trilateral formed by the tangents at the inflexions on such plane; and there are of course four systems, each of three trilaterals formed by the tangents at the nine inflexions.

13. I say that if $S=0$, then in one of the four systems the trilaterals become each of them a line, that is, the tangents at the nine inflexions meet by threes in three lines.

14. This may be shown by means of the before-mentioned canonical form

$$x^3+y^3+z^3+6lxyz=0$$

of the equation of a cubic cone, for then the notation of the inflexions being in accordance with the foregoing scheme, the coordinates may be taken to be

$$\begin{array}{llll} (1)\ \ x=0,\ \ y+z=0, & (4)\ \ x=0,\ \ y+\omega z=0, & (7)\ \ x=0,\ \ y+\omega^2 z=0, \\ (2)\ \ y=0,\ \ z+x=0, & (5)\ \ y=0,\ \ z+\omega x=0, & (8)\ \ y=0,\ \ z+\omega^2 x=0, \\ (3)\ \ z=0,\ \ x+y=0, & (6)\ \ z=0,\ \ x+\omega y=1, & (9)\ \ z=0,\ \ x+\omega^2 y=0, \end{array}$$

where ω denotes an imaginary cubic root of unity, and the equations of the tangents are

$$\begin{array}{lll} (1)\ \ -2lx+y+z=0, & (4)\ \ -2lx+\omega y+\omega^2 z=0, & (7)\ \ -2lx+\omega^2 y+\omega z=0, \\ (2)\ \ x-2ly+z=0, & (5)\ \ \omega^2 x-2ly+\omega z=0, & (8)\ \ \omega x-2ly+\omega^2 z=0, \\ (3)\ \ x+y-2lz=0, & (6)\ \ \omega x+\omega^2 y-2lz=0, & (9)\ \ \omega^2 x+\omega y-2lz=0. \end{array}$$

15. The value of S is $=-l+l^4$, and for each of the values 1, 0, ω, ω^2, of l, which give $S=0$, we have a system of the nine tangents meeting by threes in three lines, viz. the systems are

$$\begin{array}{ll} \text{for } l=1\,, & 123,\ 456,\ 789, \\ \ ,,\ \ l=0\,, & 147,\ 258,\ 369, \\ \ ,,\ \ l=\omega\,, & 159,\ 267,\ 348, \\ \ ,,\ \ l=\omega^2, & 168,\ 249,\ 357. \end{array}$$

16. It is proper to notice that starting with the systems in question, or what is the same thing, with a single set of each system, say the sets 123, 147, 159, 168, we obtain as the condition to be satisfied by l, the equation

$$l\,(4l^3-3l-1)\,(4l^3-3l\omega+1)\,(4l^3-3l\omega^2+1)=0,$$

or, as it may otherwise be written,

$$l\,(2l-1)^2\,(l-1)\,(2l\omega-1)^2\,(l\omega-1)\,(2l\omega^2-1)^2\,(l\omega^2-1)=0, \text{ that is, } (-l+l^4)\,(1+8l^3)^2=0\,;$$

it is clear that a factor has dropped out, and that the true form of the condition is

$$(-l+l^4)\,(1+8l^3)^3=0\,;$$

that is, $S(64S^3 - T^2) = 0$; where the equation $64S^3 - T^2 = 0$ would denote the existence of a nodal line, and consequent coincidence therewith of 6 of the 9 inflexions; the equation $S = 0$ being left as the proper condition for the intersection by threes of the tangents at the inflexions in three lines.

17. The investigation shows that the four systems correspond respectively to the four values of l which give $S = 0$; and that (reality being disregarded) there is no distinction between the four systems, or the corresponding values of l; if however we assume that x, y, z, l are all of them real, then the cone has only the three real inflexions 1, 2, 3, lying in the real plane 123; and there is an essential distinction between the real roots $l = 1$, $l = 0$ of the equation $S = -l + l^4 = 0$, viz. for $l = 1$, the tangents at the three real inflexions meet in a line; whereas for $l = 0$ there is not any relation between the tangents at the real inflexions, and there is consequently no visible peculiarity in the form of the cone.

18. I return to the analytical theory of the general case, as depending on the representation of the equation of the cone in the form

$$(X + Y + Z)^3 + 6kXYZ = 0,$$

where the coordinates are real, viz. $X = 0$, $Y = 0$, $Z = 0$ represent the tangent planes at the three (real) inflexions, or, as they have before been called, the tangents; and $X + Y + Z = 0$ represents the plane through the three inflexions, or, as it has before been called, the equator. And we may assume the signs to be such that in one of the 2 trilateral regions the coordinates X, Y, Z shall be each of them positive: this being so the 14 regions will correspond to the following combinations of signs

X	Y	Z	$X+Y+Z$	
+	+	+	+	the 2 trilateral regions,
−	−	−	−	
+	+	−	−	the 6 trilateral regions,
+	−	+	−	
+	−	−	+	
−	+	+	−	
−	+	−	+	
−	−	+	+	
−	+	+	+	the 6 quadrilateral regions.
+	−	+	+	
+	+	−	+	
+	−	−	−	
−	+	−	−	
−	−	+	−	

where it may be noted that the equator $X+Y+Z=0$ does not cut the two trilateral regions $(+++ +)$ and $(- - - -)$; and further that the line $X=Y=Z$ which is the harmonic of the equator $X+Y+Z=0$ in regard to the system of the three tangents $XYZ=0$, lies wholly in the two trilateral regions $(++++)$ and $(- - - -)$.

19. The equation in question,

$$(X+Y+Z)^3+6kXYZ=0,$$

shows that, as above stated, the cone lies wholly in the 8 trilateral regions, or in the 6 quadrilateral regions, viz. if k be negative, it lies wholly in the 8 trilateral regions, and if k be positive, it lies wholly in the 6 quadrilateral regions. Let k be negative, then the positive quantity $-\frac{1}{6k}$, which is

$$=\frac{XYZ}{(X+Y+Z)^3},$$

if we attend only to the values of X, Y, Z which have the same sign (that is, to points in one of the two trilateral regions), has a maximum value $=\frac{1}{27}$ corresponding to $X=Y=Z$. And if $-\frac{1}{6k}$ exceeds this value, that is, if $-k<\frac{9}{2}$, or, what is the same thing, if k lie between the values 0 and $-\frac{9}{2}$, then the equation $-\frac{1}{6k}=\frac{XYZ}{(X+Y+Z)^3}$ cannot be satisfied in the assumed manner, that is, by values of X, Y, Z having the same sign; and thus no portion of the cone lies in the two trilateral regions: in the contrary case, that is, if k lie between the values $-\infty$, $-\frac{9}{2}$, the equation can be so satisfied, and a portion of the cone lies in the two trilateral regions.

Hence k being negative, we have as follows:

k between $-\infty$ and $-\frac{9}{2}$, the cone is complex,

$k=-\frac{9}{2}$, the cone is acnodal,

k between $-\frac{9}{2}$ and 0, the cone is simplex trilateral;

and k being positive, or say

k between 0, ∞, the cone is simplex quadrilateral.

20. It is to be remarked that for $k=0$, the cone as represented by the foregoing equation degenerates into the threefold plane $(X+Y+Z)^3=0$. The value $k=0$ corresponds however to the value $l=1$ of the parameter l in the equation $x^3+y^3+z^3+6lxyz=0$, that is, it corresponds to the *simplex neutral* cone, represented by the equation

$$x^3+y^3+z^3+6xyz=0,$$

which, as already remarked, is not transformable into the form $(X+Y+Z)^3+6kXYZ=0$: this leads to the consideration of the transformation in question.

On the relation of the two forms $x^3+y^3+z^3+6lxyz=0$, *and* $(X+Y+Z)^3+6kXYZ=0$.

Nos. 21 to 24.

21. Starting with the form $x^3+y^3+z^3+6lxyz=0$, and writing

$$\begin{aligned} X &= -2lx + y + z, \\ Y &= x - 2ly + z, \\ Z &= x + y - 2lz, \end{aligned}$$

then we have

$$\begin{aligned} XYZ = &-2l(x^3+y^3+z^3) \\ &+(1-2l+4l^2)(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ &+2(1-3l-4l^3)xyz, \end{aligned}$$

$$X+Y+Z = 2(1-l)(x+y+z),$$

and thence

$$\begin{aligned} (X+Y+Z)^3 = &\ 8(1-l)^3(x^3+y^3+z^3) \\ &+24(1-l)^3(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ &+48(1-l)^3 xyz, \end{aligned}$$

and we thus obtain

$$\begin{aligned} (1-2l+4l^2)(X+Y+Z)^3 &+ 24(l-1)^3 XYZ \\ &= 8(2l+1)^2(l-1)^3(x^3+y^3+z^3+6lxyz); \end{aligned}$$

or, what is the same thing,

$$(X+Y+Z)^3+6kXYZ = \frac{8(2l+1)^2(l-1)^3}{1-2l+4l^2}(x^3+y^3+z^3+6lxyz),$$

if

$$k = \frac{4(l-1)^3}{1-2l+4l^2}.$$

22. For the form

$$(X+Y+Z)^3+6kXYZ=0,$$

we find

$$\begin{aligned} S = &\ (1+k)^4, & T = &- 8(1+k)^6 \\ &-6(1+k)^2 & &+72(1+k)^4 \\ &+8(1+k) & &-128(1+k)^3 \\ &-3 & &+72(1+k)^2 \\ & & &-8 \\ = &\ k^3(4+k) & = &-8k^4(6+6k+k^2), \end{aligned}$$

and thence

$$R = 64S^3 - T^2 = \quad 64k^8\{k(k+4)^3 - (k^2+6k+6)^2\},$$
$$= \quad 64k^8(-8k-36),$$
$$= -256k^8(2k+9).$$

It may be right to remark that from the value $k = \dfrac{-4(1-l)^3}{1-2l+4l^2}$ we deduce

$$k+4 = \frac{4l(1+l+l^2)}{1-2l+4l^2},$$

$$k^2+6k+6 = \frac{-2(1-20l^3-8l^6)}{(1-2l+4l^2)^3},$$

and that thence

$$S = \alpha^4 S', \quad T = \alpha^6 T', \quad \frac{T^2}{S^3} = \frac{T'^2}{S'^3},$$

if

$$S' = -l + l^4,$$

$$T' = 1 - 20l^3 - 8l^6,$$

$$\alpha = \frac{4(1-l)^2}{1-2l+4l^2} = \frac{k}{l-1}.$$

23. The equation

$$k = \frac{4(l-1)^3}{4l^2-2l+1},$$

or as it may also be written

$$k = \frac{16(l-1)^3}{16(l-\frac{1}{4})^2+3},$$

gives without difficulty

$$k+\tfrac{9}{4} = \frac{16(l-\frac{1}{4})^3 + 27(l-\frac{1}{4})}{16(l-\frac{1}{4})^2+3},$$

and

$$k+\tfrac{9}{2} = \frac{16(l+\frac{1}{2})^2}{16(l-\frac{1}{4})^2+3}.$$

24. Hence treating l, k as coordinates, we see that the locus is a cubic curve, viz. a hyperbolism of the ellipse, having a centre (Newton's species 62), the coordinates of the centre being $l=\frac{1}{4}$, $k=-\frac{9}{4}$, and the equation of the asymptote being $k+\frac{9}{4}=l-\frac{1}{4}$, (that is the asymptote passes through the centre and is inclined at an angle $=45^\circ$ to the axis of l). The centre is of course an inflexion, the equation of the tangent at this point is $k+\frac{9}{4}=9(l-\frac{1}{4})$, and for the other two inflexions we have $l=1$, $k=0$, and $l=-\frac{1}{2}$, $k=-\frac{9}{2}$, the tangents at the two inflexions respectively being $k=0$ and $k=-\frac{9}{2}$, that is the tangents at the inflexions are parallel to the axis of l. The curve consists of a single branch lying below the asymptote for large negative values of l, k, crossing the asymptote at the centre and lying above it for large positive values of k, l. For each value of l there is consequently a single value of k and reciprocally; and l, k pass together from $-\infty$ to $+\infty$. There are certain critical values of k and l, the meaning of which will appear from the following article.

On the Anharmonic Property of a Cubic Cone. Nos. 25 to 29.

25. The property in question is the one already referred to, viz. the four tangent planes, or say the four tangents, to the cone from any line of the cone form a system the anharmonic ratios of which are constant. Taking the equations of the tangents to be

$$p - aq = 0,\ p - bq = 0,\ p - cq = 0,\ p - dq = 0,$$

and writing for shortness $m = 64 - \frac{T^2}{S^3}$, then the functions

$$(a-b)(c-d),\ (a-c)(d-b),\ (a-d)(b-c),$$

or say α, β, γ, on which the anharmonic ratios depend, are the roots of the equation $t^3 - 12t + 2\sqrt{m} = 0$. The anharmonic ratios are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}, \frac{\alpha}{\gamma}, \frac{\gamma}{\alpha}, \frac{\beta}{\gamma}, \frac{\gamma}{\beta}$; hence forming the equation $\left(\vartheta - \frac{\alpha}{\beta}\right)\left(\vartheta - \frac{\beta}{\alpha}\right) = 0$, and reducing by the conditions,

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \beta\gamma + \gamma\alpha + \alpha\beta &= -12, \\ \alpha\beta\gamma &= -2\sqrt{m}, \end{aligned}$$

this is found to be $(\vartheta^2 + \vartheta + 1) + \frac{6\gamma}{\sqrt{m}}\vartheta = 0$, or we have $\gamma = -\frac{\sqrt{m}}{6}\frac{\vartheta^2 + \vartheta + 1}{\vartheta}$, and substituting this value in the equation $\gamma^3 - 12\gamma + 2\sqrt{m} = 0$, we find

$$(\vartheta^2 + \vartheta + 1)^3 - (\vartheta^2 + \vartheta + 1)\frac{432\,\vartheta^2}{m} - \frac{432\,\vartheta^3}{m} = 0,$$

which is

$$m(\vartheta^2 + \vartheta + 1)^3 - 432\,\vartheta^2(\vartheta + 1)^2 = 0,$$

or, what is the same thing,

$$\frac{(\vartheta^2 + \vartheta + 1)^3}{\vartheta^2(\vartheta + 1)^2} = \frac{432}{m} = \frac{432}{64 - \frac{T^2}{S^3}},$$

that is

$$\frac{(\vartheta^2 + \vartheta + 1)^3}{\vartheta^2(\vartheta + 1)^2} = \frac{27}{4\left(1 - \frac{T^2}{64S^3}\right)};$$

and as a verification it may be remarked that, θ being a root, the six roots are

$$\theta,\ \frac{1}{\theta},\ -(1+\theta),\ \frac{-1}{1+\theta},\ -\frac{\theta}{1+\theta},\ -\frac{1+\theta}{\theta}:$$

of course the roots are all real or else all imaginary.

26. If $T=0$, the equation becomes

$$(\vartheta^2+\vartheta+1)^3-\tfrac{27}{4}\vartheta^2(\vartheta+1)^2=0;$$

or reducing, this is

$$\{(\vartheta-1)(\vartheta+\tfrac{1}{2})(\vartheta+2)\}^2=0,$$

that is the six roots are $1, -\frac{1}{2}, -2$, each twice: and the four tangents form therefore a harmonic pencil, which is the geometrical interpretation of the condition $T=0$.

27. The function $\dfrac{(\vartheta^2+\vartheta+1)^3}{\vartheta^2(\vartheta+1)^2}$ is constantly positive and it has three equal minima values corresponding to the last-mentioned values $1, -\frac{1}{2}, -2$ of ϑ, viz. this minimum value is $=\frac{27}{4}$. Hence we see that the equation in ϑ will have its six roots all real if $1-\dfrac{T^2}{64S^3}$ is positive and less than unity, that is, if S and $64S^3-T^2$ are each of them positive: but when these conditions are not satisfied the six roots are imaginary: the limiting case $1-\dfrac{T^2}{64S^3}=1$ or $T=0$ gives, as already mentioned, the three roots $1, -\frac{1}{2}, -2$, each twice.

28. The quantities a, b, c, d which determine the four tangents may be all real, or two real and two imaginary, or all four imaginary; but the imaginary values appear as usual as a conjugate pair or conjugate pairs; and this being so, it is easy to see that in general if ϑ be real the quantities a, b, c, d are all real or else all imaginary; but if ϑ is imaginary then a, b, c, d are two of them real, two imaginary: in fact if a, b are real and c and d are conjugate imaginaries $\gamma\pm\delta i$, then we have for one of the six values of ϑ,

$$\vartheta=\frac{(a-b)\,.\,2\delta i}{-(a-\gamma-\delta i)(b-\gamma+\delta i)},$$

which is in general imaginary.

29. But, as might have been foreseen, the limiting values $\vartheta=1, -\frac{1}{2}, -2$, are an exception, viz. for these values a, b, c, d *may be* two of them real the other two imaginary: in fact the last-mentioned value of ϑ is real and $=\dfrac{(a-b)\,.\,2\delta i}{(a-b)\,.\,\delta i}$, $=2$, if $(a-\gamma)(b-\gamma)+\delta^2=0$, that is $ab+\gamma^2+\delta^2=\gamma(a+b)$, or, as the condition may also be written,

$$2ab+2(\gamma+\delta i)(\gamma-\delta i)=(\gamma+\delta i+\gamma-\delta i)(a+b),$$

that is $2(ab+cd)=(a+b)(c+d)$, or if a, b, c, d form a harmonic system.

The two forms $x^3+y^3+z^3+6lxyz=0$, $(X+Y+Z)^3+6kXYZ=0$; *Enumeration of the Cones comprised therein.* Nos. 30 and 31.

30. I form the following Table:

l	k	S	T	$64\,S^3-T^2$	$1-\dfrac{T^2}{64\,S^3}$,	
$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	0	complex,
$-\frac{1}{2}(1+\sqrt{3})$	$-3-\sqrt{3}$	$\frac{3}{4}(3+2\sqrt{3})$	0	$81(45+26\sqrt{3})$	1	
$-$	$-$	$+$	$+$	$+$	$+$	
$-\frac{1}{2}$	$-\frac{9}{2}$	$\frac{9}{16}$	$\frac{27}{8}$	0	0	acnodal,
$-$	$-$	$+$	$+$	$-$	$-$	simplex trilateral,
0	-4	0	1	-1	$\pm\infty$	
$\frac{1}{4}$	$-\frac{9}{4}$	$-\frac{63}{256}$	$\frac{351}{512}$	$-\frac{729}{512}$	$\frac{512}{343}$	
$\frac{1}{2}(\sqrt{3}-1)$	$-3+\sqrt{3}$	$-\frac{3}{4}(-3+2\sqrt{3})$	0	$-81(-45+26\sqrt{3})$	1	
$+$	$-$	$-$	$-$	$-$	$+$	
1	0	0	-27	-729	∞	simplex neutral,
$+$	$+$	$+$	$-$	$-$	$-$	simplex quadrilateral.
∞	∞	∞	$-\infty$	∞	0	

And I further describe as follows the nature of the cones which correspond to the several values of k and l.

31. l between $-\infty$ and $-\frac{1}{2}$, or k between $-\infty$ and $-\frac{9}{2}$.

The cone is complex. In the series, viz. corresponding to $l=-\frac{1}{2}(1+\sqrt{3})$ or $k=-3-\sqrt{3}$, there is a special form which may be called the complex harmonic, viz. the four tangents from any line of the cône form a harmonic system: but observe, *quà* complex cone, the tangents are all real or all imaginary. $l=-\frac{1}{2}$ (form fails), $k=-\frac{9}{2}$, the cone is acnodal. l between $-\frac{1}{2}$ and 1, or k between $-\frac{9}{2}$ and 0; the cone is simplex trilateral. In the series, viz. corresponding to $l=0$ or $k=-4$, there is a special form which might be called the quasi-neutral, the speciality having however reference to the imaginary inflexions, viz. corresponding to each real inflexion we have two imaginary inflexions such that the three tangents meet in a line.

There is also corresponding to $l=\frac{1}{4}$, or $k=-\frac{9}{4}$, a form which seems to be a special one, though I have not ascertained wherein that speciality consists.

And there is corresponding to $l=\frac{1}{2}(\sqrt{3}-1)$ or $k=-3+\sqrt{3}$, a special form which might be called the simplex harmonic, viz. the tangents from any line of the cone form a harmonic system. It is to be observed that, *quà* simplex cone, the four tangents are two of them real, two imaginary.

$l=1$; $k=0$ (form fails). The cone is simplex neutral.

l between 1 and ∞, or k between 0 and ∞; the cone is simplex quadrilateral.

32. It will be observed that the crunodal and cuspidal kinds of cones do not present themselves in the foregoing investigations; the reason is that the crunodal kind admits of no representation in the form $x^3+y^3+z^3+6lxyz=0$, and (inasmuch as there is only one real inflexion) it admits of no real representation in the other form $(X+Y+Z)^3+6kXYZ=0$; the cuspidal kind admits of no representation in either of the two forms.

I conclude with a discussion not absolutely required for the purpose of the memoir, but which is of interest in regard to the form $(X+Y+Z)^3+6kXYZ=0$.

Reduction of the Hessian to the form $(X'+Y'+Z')^3+6k'X'Y'Z'=0$.

33. The cubic cone

$$x^3+y^3+z^3+6lxyz=0,$$

has for its Hessian

$$-l^2(x^3+y^3+z^3)+(1+2l^3)\,xyz=0,$$

or say

$$x^3+y^3+z^3+6l'xyz=0,$$

if

$$l'=-\frac{1+2l^3}{6l^2}.$$

Hence writing

$$X'=-2l'x+y+z,$$

$$Y'=x-2l'y+z,$$

$$Z'=x+y-2l'z,$$

we have

$$(X'+Y'+Z')^3+6k'X'Y'Z'=\frac{8\,(2l'+1)^2\,(l'-1)^3}{1-2l'+4l'^2}\,(x^3+y^3+z^3+6l'xyz).$$

Hence the equation of the Hessian is

$$(X'+Y'+Z')^3+6k'X'Y'Z'=0,$$

where the value of k' is

$$k'=\frac{4\,(l'-1)^2}{1-2l'+4l'^2}.$$

34. But we have

$$l'-1=-\frac{1+2l^3}{6l^2}-1=-\frac{1}{6l^2}(1+6l^2+2l^3),$$

$$4(1-2l'+4l'^2)=(4l'-1)^2+3=\frac{1}{9l^4}\left\{(2+3l^2+4l^3)^2+27l^4\right\}=\frac{4}{9l^4}(1+l+l^2)^2(1-2l+4l^2),$$

and thence

$$k'=-\tfrac{1}{6}\frac{(1+6l^2+2l^3)^3}{l^2(1+l+l^2)^2(1-2l+4l^2)}.$$

But the equation

$$k=\frac{4(l-1)^3}{1-2l+4l^2},$$

gives

$$k+4=\frac{4l(1+l+l^2)}{1-2l+4l^2},\quad k+6=\frac{2(1+6l^2+2l^3)}{1-2l+4l^2},$$

and we thence have

$$k'=-\tfrac{1}{3}\frac{(k+6)^3}{(k+4)^2},$$

which determines k' in terms of k.

35. It may be observed that the value $k=-6$ corresponds to $l'=1$, that is, the Hessian is here $x^3+y^3+z^3+6xyz=0$, a simplex neutral form *not* transformable into $(X'+Y'+Z')^3+6k'X'Y'Z'=0$; the corresponding value of l is of course given by the equation $1+6l^2+2l^3=0$; the only speciality of the cone $x^3+y^3+z^3+6lxyz=0$, or $(X+Y+Z)^3\ 36XYZ=0$, consequently is that the Hessian is a simplex neutral cone.

The value $k=-4$ corresponds to $l=0$, $l'=\infty$, $k'=\infty$; hence $X':Y':Z'=x:y:z$ and the transformation of the Hessian $x^3+y^3+z^3+6l'xyz=0$ into the new form $(X'+Y'+Z')^3+6kX'Y'Z'=0$ degenerates into the mere identity $xyz=xyz$.

Cambridge, 19*th Feb.* 1865.