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## ADDITION TO THE MEMOIR ON TSCHIRNHAUSEN'S TRANSFORMATION.

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In the memoir "On Tschirnhausen's Transformation," *Philosophical Transactions*, vol. CLII. (1862), pp. 561—568, [275], I considered the case of a quartic equation: viz. it was shown that the equation

$$(a,\ b,\ c,\ d,\ e \between x,\ 1)^4 = 0$$

is, by the substitution

$$y = (ax+b)\,B + (ax^2+4bx+3c)\,C + (ax^3+4bx^2+6cx+3d)\,D,$$

transformed into

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4 = 0$$

where $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ have certain given values. It was further remarked that $(\mathfrak{C}, \mathfrak{D}, \mathfrak{E})$ were expressible in terms of $U'$, $H'$, $\Phi'$, invariants of the two forms $(a, b, c, d, e \between X, Y)^4$, $(B, C, D \between Y, -X)^2$, of $I$, $J$, the invariants of the first, and of $\Theta'$, $=BD-C^2$, the invariant of the second of these two forms, viz. that we have

$$\mathfrak{C} = 6H' - 2I\Theta',$$

$$\mathfrak{D} = 4\Phi',$$

$$\mathfrak{E} = IU'^3 - 3H'^2 + I^2\Theta'^2 + 12J'\Theta'U' + 2I'\Theta'H';$$

and by means of these I obtained an expression for the quadrinvariant of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4;$$

viz. this was found to be

$$= IU'^2 + \tfrac{4}{3}I^2\Theta'^2 + 12J\Theta'U'.$$

But I did not obtain an expression for the cubinvariant of the same function: such expression, it was remarked, would contain the square of the invariant $\Phi'$; it was probable that there existed an identical equation,

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2,$$

which would serve to express $\Phi'^2$ in terms of the other invariants; but, assuming that such an equation existed, the form of the factor $M$ remained to be ascertained; and until this was done, the expression for the cubinvariant could not be obtained in its most simple form. I have recently verified the existence of the identical equation just referred to, and have obtained the expression for the factor $M$; and with the assistance of this identical equation I have obtained the expression for the cubinvariant of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4.$$

The expression for the quadrinvariant was, as already mentioned, given in the former memoir: I find that the two invariants are in fact the invariants of a certain linear function of $U$, $H$; viz. the linear function is $= U'U + \frac{2}{3}\Theta'H$; so that, denoting by $I^*$, $J^*$, the quadrinvariant and the cubinvariant respectively of the form

$$(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4,$$

we have

$$I^* = \tilde{I}\,(U'U + 4\Theta'H),$$

$$J^* = \tilde{J}\,(U'U + 4\Theta'H),$$

where $\tilde{I}$, $\tilde{J}$ signify the functional operations of forming the two invariants respectively. The function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E}\between y,\ 1)^4$, obtained by the application of Tschirnhausen's transformation to the equation

$$(a,\ b,\ c,\ d,\ e\between x,\ 1)^4 = 0,$$

has thus the *same invariants* with the function

$$U'U + 4\Theta'H = U'(a, b, c, d, e\between x, 1)^4 + 4\Theta'(ac - b^2, ad - bc, ae + 2bd - 3c^2, be - cd, ce - d^2\between x, 1)^4,$$

and it is consequently a linear transformation of the last-mentioned function; so that the application of Tschirnhausen's transformation to the equation $U = 0$ gives an equation linearly transformable into, and thus virtually equivalent to, the equation

$$U'U + 4\Theta'H = 0,$$

which is an equation involving the single parameter $\frac{4\Theta'}{U'}$: this appears to me a result of considerable interest. It is to be remarked that Tschirnhausen's transformation, wherein $y$ is put equal to a rational and integral function of the order $n-1$ (if $n$ be the order of the equation in $x$), is not really less general than the transformation wherein $y$ is put equal to any rational function $\frac{V}{W}$ whatever of $x$; such rational function may, in fact, by means of the given equation in $x$, be reduced to a rational and integral function of the order $n-1$; hence in the present case, taking $V$, $W$ to be respectively of the order $n-1, =3$, it follows that the equation in $y$ obtained by the elimination of $x$ from the equations

$$(a, b, c, d, e \between x, 1)^4 = 0,$$

$$y = \frac{(\alpha, \beta, \gamma, \delta \between x, 1)^3}{(\alpha', \beta', \gamma', \delta' \between x, 1)^3},$$

is a mere linear transformation of the equation $AU + BH = 0$, where $A$, $B$ are functions (not as yet calculated) of $(a, b, c, d, e, \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta')$.

Article Nos. 1, 2, 3. *Investigation of the identical equation*

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2.$$

1. It is only necessary to show that we have such an equation, $M$ being an invariant, in the particular case $a = e = 1$, $b = d = 0$, $c = \theta$, that is for the quartic function $(1, 0, \theta, 0, 1 \between x, 1)^4$; for, this being so, the equation will be true in general. Writing the equation in the form

$$-M\Theta' = U'^2(JU' - IH') + 4H'^3 + \Phi'^2,$$

and observing that we have

$$\begin{aligned}
U' &= (B^2 + D^2) + 2\theta BD + 4\theta C^2,\\
H' &= \theta(B^2 + D^2) + (1 + \theta^2) BD - 4\theta^2 C^2,\\
\Theta' &= BD - C^2,\\
\Phi' &= (1 - 9\theta^2) C (B^2 - D^2),\\
I &= 1 + 3\theta^2,\\
J &= \theta - \theta^3,
\end{aligned}$$

and thence

$$JU' - IH' = -4\theta^3(b^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4) BD + (8\theta^2 + 8\theta^4) C^2,$$

the equation becomes

$$\begin{aligned}
-(BD - C^2) M = &\\
&\{-4\theta^3(B^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4) BD + (8\theta^2 + 8\theta^4) C^2\} \times \{B^2 + D^2 + 2\theta BD + 4\theta C^2\}^2\\
&+ 4\{\theta(B^2 + D^2) + (1 + \theta^2) BD - 4\theta^2 C^2\}^2\\
&+ (1 - 9\theta^2)^2 C^2 \{(B^2 + D^2)^2 - 4B^2D^2\}.
\end{aligned}$$

2. It is found by developing that the right-hand side is in fact divisible by $BD - C^2$, and that the quotient is

$$\begin{aligned}
= \quad & (-1 + 10\theta^2 - 9\theta^4)(B^2 + D^2)^2 \\
& + (8\theta + 16\theta^3 - 24\theta^5)(B^2 + D^2)\,BD \\
& + (4 + 8\theta^2 + 4\theta^4 - 16\theta^6)\,B^2D^2 \\
& + (-64\theta^3 - 192\theta^5)(B^2 + D^2)\,C^2 \\
& + (16\theta^2 - 416\theta^4 - 112\theta^6)\,BDC^2 \\
& + (-128\theta^4 + 128\theta^6)\,C^4.
\end{aligned}$$

3. This is found to be

$$\begin{aligned}
= & -I^2U'^2 + 12JU'H' + 4IH'^2 \\
& -\ 8IJU'\Theta' \\
& - 16J^2\Theta'^2,
\end{aligned}$$

which is consequently the value of $-M$. We have therefore

$$\begin{aligned}
-\Phi'^2 = \quad & JU'^3 - IU'^2H' + 4H'^3 \\
& + (I^2U'^2 - 12JU'H' - 4IH'^2)\,\Theta' \\
& +\ 8IJU'\Theta'^2 \\
& + 16J^2\Theta'^3,
\end{aligned}$$

which is the required identical equation.

## Article No. 4. *Calculation of the Cubinvariant.*

4. We have

$$\begin{aligned}
J^* = & \tfrac{1}{6}\,\mathfrak{C}\,.\,\mathfrak{C} - (\tfrac{1}{6}\,\mathfrak{C})^3 - (\tfrac{1}{4}\mathfrak{D})^2 \\
= \quad & (H - \tfrac{1}{3}I\Theta')\,\{IU'^2 - 3H'^2 + (12JU' + 2IH')\,\Theta' + I^2\Theta'^2\} \\
& - (H - \tfrac{1}{3}I\Theta')^3 \\
& -\ \Phi'^2,
\end{aligned}$$

whence, substituting for $-\Phi'^2$ its value and reducing, we find

$$J^* = JU'^3 + \Theta'\,.\,\tfrac{2}{3}\,I^2U'^2 + \Theta'^2\,(4IJU') + \Theta'^3\,(16J^2 - \tfrac{8}{27}\,I^3).$$

## Article No. 5. *Final expressions of the two Invariants.*

The value of $I^*$ has been already mentioned to be $I^* = IU'^2 + \Theta' 12\,JU' + \Theta'^2\,.\,\tfrac{4}{3}I^2$, and it hence appears that the values of the two invariants may be written

$$\begin{aligned}
I^* &= (I,\ 18J,\ 3I^2 \between U',\ \tfrac{2}{3}\Theta')^2, \\
J^* &= (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2 \between U',\ \tfrac{2}{3}\Theta')^3.
\end{aligned}$$

But we have (see Table No. 72 in my "Seventh Memoir on Quantics," *Philosophical Transactions*, vol. CLI. (1861), pp. 277—292, [269])

$$\tilde{I}(\alpha U + 6\beta H) = (I,\ 18J,\ 3I^2 \between \alpha,\ \beta)^2,$$
$$\tilde{J}(\alpha U + 6\beta H) = (J,\ I^2,\ 9IJ,\ -I^3 + 54J^2 \between \alpha,\ \beta)^3;$$

so that, writing $\alpha = U'$, $\beta = \frac{2}{3}\Theta'$, we have

$$I^* = \tilde{I}(U'U + 4\Theta' H),$$
$$J^* = \tilde{J}(U'U + 4\Theta' H);$$

or the function $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4$ obtained from Tschirnhausen's transformation of the equation $U = 0$ has the same invariants with the function $U'U + 4\Theta' H$; or, what is the same thing, the equation $(1,\ 0,\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E} \between y,\ 1)^4 = 0$ is a mere linear transformation of the equation $U'U + 4\Theta H = 0$; which is the above-mentioned theorem.