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## NOTE ON LOBATSCHEWSKY'S IMAGINARY GEOMETRY.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 231—233.]

WRITING down the equations

$$\frac{1}{\cos a'} = \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

where $A$, $B$, $C$ are real positive angles each $< \frac{1}{2}\pi$ : *first*, if $A + B + C > \pi$, then $a$, $b$, $c$ are real positive angles each less than $\frac{1}{2}\pi$ (this is in fact the case of a real acute-angled spherical triangle), but $a'$, $b'$, $c'$ are pure imaginaries of the form $p'i$, $q'i$, $r'i$ (where $p'$, $q'$, $r'$ are real positive quantities; and *secondly*, if $A + B + C < \pi$, then $a$, $b$, $c$ are pure imaginaries of the form $pi$, $qi$, $ri$ (where $p$, $q$, $r$ are real positive quantities), but $a'$, $b'$, $c'$ are real positive angles each less than $\frac{1}{2}\pi$. Hence assuming $A + B + C < \pi$ and writing $ai$, $bi$, $ci$ in place of $a$, $b$, $c$, the system is

$$\frac{1}{\cos a'} = \cos ai = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \cos bi = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \cos ci = \frac{\cos C + \cos A \cos B}{\sin A \sin B},$$

which equations (if only we write therein $\frac{1}{2}\pi - a'$, $\frac{1}{2}\pi - b'$, $\frac{1}{2}\pi - c'$ in place of $a'$, $b'$, $c'$ respectively) are in fact the equations given under a less symmetrical form in the curious paper "Géométrie Imaginaire" by N. Lobatschewsky, Rector of the University of Kasan, *Crelle*, vol. XVII. (1837), pp. 295—320. The view taken of them by the author is hard to be understood. He mentions that in a paper published five years previously in a scientific journal at Kasan, after developing a new theory of parallels, he had endeavoured to prove that it is only experience which obliges us to assume that in a rectilinear triangle the sum of the angles is equal to two right angles, and that a geometry may exist, if not in nature at least in analysis, on the hypothesis that the sum of the angles is less than two right angles; and he accordingly attempts to establish such a geometry, viz. $a$, $b$, $c$ being the sides of a rectilinear triangle, wherein the sum of the angles $A + B + C$ is $< \pi$, and the angles $a'$, $b'$, $c'$ being calculated from the sides by the formulæ

$$\cos a' = \frac{1}{\cos ai}, \quad \cos b' = \frac{1}{\cos bi}, \quad \cos c' = \frac{1}{\cos ci}$$

(I have, as mentioned above, replaced Lobatschewsky's $a'$, $b'$, $c'$ by their complements): the relation between the angles $A$, $B$, $C$ and the subsidiary quantities $a'$, $b'$, $c'$ which replace the sides, is given by the formulæ

$$\frac{1}{\cos a'} = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\frac{1}{\cos b'} = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\frac{1}{\cos c'} = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

I do not understand this; but it would be very interesting to find a *real* geometrical interpretation of the last-mentioned system of equations, which (if only $A$, $B$, $C$ are positive real quantities such that $A + B + C < \pi$; for the condition, $A$, $B$, $C$ each $< \frac{1}{2}\pi$, may be omitted) contains only the *real* quantities $A$, $B$, $C$, $a'$, $b'$, $c'$; and is a system correlative to the equations of ordinary Spherical Trigonometry.

It is hardly necessary to remark that the equation

$$\frac{1}{\cos a'} = \cos ai$$

is Jacobi's imaginary transformation in the Theory of Elliptic Functions. See, as to this, my paper "On the Transcendent $\text{gd}\,.\,u = \frac{1}{i}\log\tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$," *Phil. Mag.* vol. XXIV. (1862), pp. 19—22, [320].

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