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ON THE THEORY OF THE EVOLUTE.

[From the *Philosophical Magazine*, vol. XXIX. (1865), pp. 344—350.]

ACCORDING to the generalized notion of geometrical magnitude, two lines are said to be at right angles to each other when they are harmonics in regard to a certaiu conic called the Absolute; this being so, the normal at any point of a curve is the line at right angles to the tangent, and the Evolute is the envelope of the normals.

Let the equation of the absolute be

$$\Theta = (a, b, c, f, g, h \between x, y, z)^2 = 0,$$

and suppose, as usual, that the inverse coefficients are (A, B, C, F, G, H). Consider a given curve $U = (*\between x, y, z)^m = 0$, and suppose, for shortness, that the first differential coefficients of U are denoted by L, M, N. Then we have to find the equation of the normal at the point (x, y, z) of the curve $U = 0$.

The condition that any two lines are harmonics in regard to the absolute, is equivalent to this, viz. each line passes through the pole of the other line in regard to the absolute. Hence the normal at the point (x, y, z) is the line joining this point with the pole of the tangent. Now, taking (X, Y, Z) as current coordinates, the equation of the tangent is

$$LX + MY + NZ = 0,$$

the coordinates of the pole of the tangent are therefore

$$(A, H, G \between L, M, N) : (H, B, F \between L, M, N) : (G, F, C \between L, M, N),$$

and the equation of the normal is

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ (A, H, G \between L, M, N), & (H, B, F \between L, M, N), & (G, F, C \between L, M, N) \end{vmatrix} = 0.$$

The formula in this form will be convenient in the sequel; but there is no real loss of generality in taking the equation of the absolute to be $x^2+y^2+z^2=0$; the values of (A, B, C, F, G, H) are then $(1, 1, 1, 0, 0, 0)$, and the formula becomes

$$\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ L, & M, & N \end{vmatrix} = 0;$$

where it will be remembered that (L, M, N) denote the derived functions $(\partial_x U, \partial_y U, \partial_z U)$.

The evolute is therefore the envelope of the line represented by the foregoing equation, say the equation $\Omega = 0$, considering therein (x, y, z) as variable parameters connected by the equation $U = 0$.

As an example, let it be required to find the evolute of a conic; since the axes are arbitrary, we may without loss of generality assume that the equation of the conic is $xz - y^2 = 0$. The values of (L, M, N) here are $(z, -2y, x)$. Moreover the equation is satisfied by writing therein $x : y : z = 1 : \theta : \theta^2$; the values of (L, M, N) then become $(\theta^2, -2\theta, 1)$ and the equation is

$$\begin{vmatrix} 1 & , & \theta & , & \theta^2 \\ X & , & Y & , & Z \\ (A, H, G\between\theta, -1)^2, & & (H, B, F\between\theta, -1)^2, & & (G, F, C\between\theta, -1)^2 \end{vmatrix} = 0;$$

or, developing, this is

$$\begin{aligned} & X\left(\begin{matrix} G\theta^3 - 2F\theta^2 + C\theta \\ -H\theta^4 + 2B\theta^3 - F\theta^2 \end{matrix}\right) \\ +\, & Y\left(\begin{matrix} A\theta^4 - 2H\theta^3 + G\theta^2 \\ - G\theta^2 + 2F\theta - C \end{matrix}\right) \\ +\, & Z\left(\begin{matrix} H\theta^2 - 2B\theta + F \\ -A\theta^3 + 2H\theta^2 - G\theta \end{matrix}\right) = 0, \end{aligned}$$

which I leave in this form in order to show the origin of the different terms, and in particular in order to exhibit the destruction of the term θ^2 in the coefficient of Y. But the equation is, it will be observed, a quartic equation in θ, with coefficients which are linear functions of the current coordinates (X, Y, Z).

The equation shows at once that the evolute is of the class 4; in fact treating the coordinates (X, Y, Z) as given quantities, we have for the determination of θ an equation of the order 4, that is, the number of normals through a given point (X, Y, Z), or, what is the same thing, the class of the evolute, is $=4$.

The equation of the evolute is obtained by equating to zero the discriminant of the foregoing quartic function of θ; the order of the evolute is thus $=6$. There are no inflexions, and the diminution of the order from $4 . 3, = 12$, to 6 is caused by three double tangents.

I consider the particular case where the conic touches the absolute. There is no loss of generality in assuming that the contact takes place at the point $(y=0,\ z=0)$, the common tangent being therefore $z=0$; the conditions for this are $a=0$, $h=0$, and we have thence $C=0$, $F=0$. Substituting these values, the equation contains the factor θ; and, throwing this out, it is

$$\begin{aligned} &X\,(-H\theta^3+(B+2G)\,\theta^2 \qquad\qquad\qquad\qquad)\\ &Y\,(\quad A\theta^3 - \qquad 2H\,\theta^2 \qquad\qquad\qquad\qquad)\\ +&Z\,(\qquad\qquad -\ A\,\theta^2+3H\theta-(B+2G))=0, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} &\theta^3\,(\quad -H\,X+AY \qquad\qquad)\\ +&\theta^2\,((B+2G)\,X-2HY- \qquad AZ)\\ +&\theta\,(\qquad\qquad\qquad\qquad\quad 3HZ)\\ +&\ (\qquad\qquad\qquad -(B+2G)\,Z)=0, \end{aligned}$$

where it will be observed that the constant term and the coefficient of θ have the same variable factor Z, where $Z=0$ is the equation of the common tangent of the conic and the absolute. The evolute is in this case of the class 3. It at once appears that the line $Z=0$ is a stationary tangent of the evolute, the point of contact (or inflexion on the evolute) being given by the equations $Z=0$, $(B+2G)\,X-2HY=0$. The equation of the evolute is found by equating to zero the discriminant of the cubic function; the equation so obtained has the factor Z, and throwing this out the order is $=3$. The evolute is thus a curve of the class 3 and order 3, the reduction in the order from $3\,.\,2$, $=6$, to 3 being caused by the existence of an inflexion. Comparing with the former case, we see that the effect of the contact of the conic with the absolute is to give rise to an inflexion of the evolute, and to cause a reduction $=1$ in the class, and a reduction $=3$ in the order.

I return now to the general case of a curve

$$U=(*\between x,\ y,\ z)^m=0.$$

Using, for greater simplicity, the equation $x^2+y^2+z^2=0$ for the absolute, the equation of the normal is

$$\Omega=\begin{vmatrix} X, & Y, & Z \\ x, & y, & z \\ \partial_x U, & \partial_y U, & \partial_z U \end{vmatrix}=0;$$

we may at once find the class of the evolute; in fact, treating $(X,\ Y,\ Z)$ as the coordinates of a given point, the two equations $U=0$, $\Omega=0$ determine the values $(x,\ y,\ z)$ of the coordinates of a point such that the normal thereof passes through

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the point (X, Y, Z); the number of such points is the number of normals which can be drawn through a given point (X, Y, Z), viz. it is equal to the class of the evolute. The points in question are given as the intersections of the two curves $U=0$, $\Omega=0$, which are respectively curves of the order m, hence the number of intersections is $=m^2$. It is to be observed, however, that if the curve $U=0$ has nodes or cusps, then the curve $\Omega=0$ passes through each node of the curve $U=0$, and through each cusp, the two curves having at the cusp a common tangent; that is, each node reckons for two intersections, and each cusp for three intersections. Hence, if the curve $U=0$ has δ nodes and κ cusps, the number of the remaining points of intersection is $=m^2-2\delta-3\kappa$. The class of the evolute is thus $=m^2-2\delta-3\kappa$. The number of inflexions is in general $=0$. If, however, the given curve touches the absolute, then it has been seen in a particular case that the effect is to diminish the class by 1, and to give rise to an inflexion, the stationary tangent being in fact the common tangent of the curve and the absolute: I assume that this is the case generally. Suppose that there are θ contacts, then there will be a diminution $=\theta$ in the class, or this will be $=m^2-2\delta-3\kappa-\theta$; and there will be θ inflexions; there may however be special circumstances giving rise to fresh inflexions, and I will therefore assume that the number of inflexions is $=\iota'$.

Suppose in general that for any curve we have

m, the order,
n, „ class,
δ, „ number of nodes,
κ, „ „ cusps,
τ, „ „ double tangents,
ι, „ „ inflexions.

Then Plücker's equations give

$$\iota-\kappa=3(n-m),\quad \tau-\delta=\tfrac{1}{2}(n-m)(n+m-9);$$

and we thence have

$$\iota-\kappa+\tau-\delta=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}(m-1)(m-2),$$

or, what is the same thing,

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa=\tfrac{1}{2}(n-1)(n-2)-\tau-\iota.$$

Now M. Clebsch in his recent paper "Ueber die Singularitäten algebraischer Curven," *Crelle*, vol. LXIV. (1864), pp. 98—100, has remarked (as a consequence of the investigations of Riemann in the Integral Calculus) that whenever from a given curve another curve is derived in such manner that to each point (or tangent) of the given curve there corresponds a *single* tangent (or point) of the derived curve, then the expression

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa,\quad =\tfrac{1}{2}(n-1)(n-2)-\tau-\iota,$$

has the same value in the two curves respectively, or that, writing m', n', δ', κ', τ', ι' for the corresponding quantities in the second curve, then we have

$$\tfrac{1}{2}(m-1)(m-2)-\delta-\kappa \;=\tfrac{1}{2}(n-1)(n-2)-\tau-\iota,$$
$$=\tfrac{1}{2}(m'-1)(m'-2)-\delta'-\kappa',=\tfrac{1}{2}(n'-1)(n'-2)-\tau'-\iota';$$

and consequently that, knowing any *two* of the quantities m', n', δ', κ', τ', ι', the remainder of them can be determined by means of this relation and of Plücker's equations. The theorem is applicable to the evolute according to the foregoing generalized definition[1]; and starting from the values

$$n'=m^2-2\delta-3\kappa-\theta,$$
$$\iota'=\iota,$$

we find in the first instance

$$\tau=\tfrac{1}{2}(n'-1)(n'-2)-\tfrac{1}{2}(m-1)(m-2)+\delta+\kappa-\iota';$$

and substituting in the equation

$$m'=n'(n'-1)-2\tau'-3\iota',$$

we find

$$m'=2(n'-1)+(m-1)(m-2)-2\delta-2\kappa-\iota';$$

and the equation $\iota'-\kappa'=3(n'-m')$ gives also

$$\kappa'=-3(n'-m')+\iota',$$

whence, attending to the value of n', we find the following system of equations for the singularities of the evolute, viz.

$$\begin{aligned}
n'&=m^2 &&-2\delta-3\kappa-\theta,\\
m'&=3m(m-1)&&-6\delta-8\kappa-2\theta-\iota',\\
\iota'&= &&\iota',\\
\kappa'&=3m(2m-3)&&-12\delta-15\kappa-3\theta-2\iota',
\end{aligned}$$

and the values of τ' and δ' may then also be found from the equations

$$m'=n'(n'-1)-2\tau'-3\iota',$$
$$n'=m'(m'-1)-2\delta'-3\kappa'.$$

I have given the system in the foregoing form, as better exhibiting the effect of the inflexions; but as each of the θ contacts with the absolute gives an inflexion, we

[1] M. Clebsch in fact applies it to the evolute in the ordinary sense of the term, but by inadvertently assuming $\iota'=\kappa$ instead of $\iota'=0$ he is led to some incorrect results.

may write $\iota' = \theta + \iota''$, where, in the absence of special circumstances giving rise to any more inflexions, $\iota'' = 0$. The system thus becomes

$$\begin{aligned} n' &= m^2 - 2\delta - 3\kappa - \theta, \\ m' &= 3m(m-1) - 6\delta - 8\kappa - 3\theta - \iota'', \\ \iota' &= \theta + \iota'', \\ \kappa' &= 3m(2m-3) - 12\delta - 15\kappa - 5\theta - 2\iota'', \end{aligned}$$

so that each contact with the absolute diminishes the class by 1, the order by 3, and the number of cusps by 5.

I remark that when the absolute becomes a pair of points, a contact of the given curve m means one of two things: either the curve touches the line through the two points, or else it passes through one of the two points: the effect of a contact of either kind is as above stated. Suppose that the two points are the circular points at infinity, and let $m = 2$, the evolute in question is then the evolute of a conic, in the ordinary sense of the word evolute. We have, in general, class $= 4$, order $= 6$; but if the conic touches the line infinity (that is, in the case of the parabola), the reductions are 1 and 3, and we have class $= 3$, order $= 3$, which is right. If the conic passes through one of the circular points of infinity, then in like manner the reductions are 1 and 3; and therefore if the conic passes through each of the circular points at infinity (that is, in the case of a circle), the reductions are 2 and 6, and we have class $= 2$, order $= 0$, which is also right; for the evolute is in this case the centre, regarded as a pair of coincident points. That this is so, or that the class is to be taken to be (not $= 1$ but) $= 2$, appears by the consideration that the number of normals to the circle from a given point is in fact $= 2$, the two normals being, however, coincident in position.

To complete the theory in the general case where the absolute is a proper conic, I remark that, besides the inflexions which arise from contacts of the given curve with the absolute, there will be an inflexion, first, for each stationary tangent of the given curve which is also a tangent of the absolute; secondly, for each cusp of the given curve situate on the absolute. Hence, if the number of such stationary tangents be $= \lambda$, and the number of such cusps be $= \mu$, we may write $\iota'' = \lambda + \mu$, and therefore also $\iota' = \theta + \lambda + \mu$.

I remark also that we have

$$\begin{aligned} -2\delta - 3\kappa &= -m(m-1) + n, \\ -6\delta - 8\kappa &= -3m(m-2) + \iota, \end{aligned}$$

and therefore also

$$-12\delta - 15\kappa = -6m^2 + 15m - 3n + 3\iota.$$

The general formulæ thus become

$$\begin{aligned} n' &= m + n - \theta, \\ m' &= 3m + \iota - 2\theta - \iota', \\ \iota' &= \iota', \\ \kappa' &= 6m - 3n + 3\iota - 3\theta - 2\iota'. \end{aligned}$$

If instead of the given curve we consider its reciprocal in regard to the absolute, then

$$m,\ n,\ \delta,\ \kappa,\ \tau,\ \iota;\quad \theta,\ \lambda,\ \mu;\quad \iota',\ = \theta+\lambda+\mu$$

are changed into

$$n,\ m,\ \tau,\ \iota,\ \delta,\ \kappa:\quad \theta,\ \mu,\ \lambda;\quad \iota',\ = \theta+\mu+\lambda$$

respectively.

Hence for the evolute of the reciprocal curve we have

$$\begin{aligned} n' &= n + m \qquad - \theta \quad ; \\ m' &= 3n \qquad + \kappa - 2\theta - \iota', \\ \iota' &= \qquad\qquad\qquad \iota', \\ \kappa' &= 6n \quad 3m + 3\kappa - 3\theta - \iota', \end{aligned}$$

which, attending to the relation $\iota - \kappa = 3(n-m)$, are in fact the same as the former values; that is, the evolute of the given curve, and the evolute of the reciprocal curve are curves of the same class and order, and which have the same singularities.

Cambridge, February 22, 1865.