375.

NOTES ON POLYHEDRA.

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Axial Properties. Article 1 to 18.

1. A POLYHEDRON may have a q-axis, viz. a line about which if it is made to rotate through an angle $=\frac{2\pi}{q}$ (but not through any sub-multiple of this angle), it will occupy the same portion of space. It is then clear that when the rotation is repeated any number of times the body will still occupy the same portion of space; or if Θ denote the rotation through the angle $\frac{2\pi}{q}$, then we have the rotations $1, \Theta, \Theta^2, \ldots \Theta^{q-1}$, and finally $\Theta^q = 1$, that is, when the rotation is q-times repeated, the body will resume its original position. Similarly for any number of axes ($\Theta^q = 1$, $\Theta'^{q'} = 1, \ldots$, where the indices $q, q', \ldots$ may be the same or different) we have the rotations $1, \Theta, \Theta^2, \ldots \Theta^{q-1}$, $\Theta', \Theta'^2, \ldots \Theta'^{q'-1}, \ldots$; and if $\Theta, \Theta', \ldots$ be the entire system of the axes of the body, these rotations will form a *group*. The rotations in question are in fact the entire series of those which leave unaltered the portion of space occupied by the body, and since any two rotations combine together into a single rotation, any two of the rotations in question must combine together into some one of these rotations, that is, the rotations in question form a group. Some analytical consequences of this theorem will be obtained in the sequel.

2. The number of axes may be denoted by $\Sigma 1$ and the number of rotations by $1 + \Sigma(q-1)$; we may say that $\Sigma 1$ is the number, and $1 + \Sigma(q-1)$ the efficiency or *weight*, of the axes.

3. For any one of the regular polyhedra, E being the number of edges, then the number of axes or $\Sigma 1$ is $= E + 1$, and their weight or $1 + \Sigma (q-1)$ is $= 2E$. In fact, if as usual S denote the number of summits, F the number of faces, and if there be m edges to a face, and n edges to a summit, then $S + F = E + 2$, $mF = nS = 2E$. Now in all the polyhedra except the tetrahedron, we have a number $\frac{1}{2}F$ of m-axes passing through the centres of opposite faces (amphihedral axes as Mr Kirkman has termed them) and a number $\frac{1}{2}S$ of n-axes passing through opposite summits (amphigonal axes); and we have besides a number $\frac{1}{2}E$ of 2-axes passing through the mid-points of opposite edges (amphigrammic axes): the entire number of axes is thus $\frac{1}{2}(S + F + E)$, which is $= E + 1$: and the weight is $1 + \frac{1}{2}F(m-1) + \frac{1}{2}S(n-1) + \frac{1}{2}E$, which is $= 1 + \frac{1}{2}mF + \frac{1}{2}nS - \frac{1}{2}(F + S - E)$, $= 1 + E + E - 1$, $= 2E$. In the case of the tetrahedron $S = F = 4$, $m = n = 3$, and the only difference is that instead of the $\frac{1}{2}F$ amphihedral m-axes and the $\frac{1}{2}S$ amphigonal n-axes, we have a number $(F = S =) \frac{1}{2}(F + S)$ of $(n =) m$-gonal axes each through a summit and the centre of an opposite face (gonohedral axes).

4. The theorem that the weight $1 + \Sigma(q-1) = 2E$, or say $1 + \Sigma(q-1) = mF$, may be extended so as to apply to any polyhedron whatever. In fact considering any face A of the polyhedron, let F be the number of faces homologous to (and inclusive of) A; and, taking a any edge of the face A, let m be the number of edges of A homologous to (and inclusive of) a: then we have $1 + \Sigma(q-1) = mF$. This is almost a truism when the signification of the term "homologous" is explained. Imagine the polyhedron placed on a plane, say the table, and draw on the table a polygon equal to the polygonal face A, and in this polygon select some one edge corresponding to the edge a. The polyhedron may be placed on the table with the face A coinciding with the polygon, or say the face A may be superimposed on the polygon, and that in m different ways, viz. any one of the edges homologous to a may be made to coincide with the assumed edge: and in like manner there are F different faces (viz. the faces homologous to A) which may be superimposed on the polygon, each of them in m different ways; that is there are in all mF different positions of the polyhedron for each of which it occupies the same portion of space. And we have thus the required theorem $1 + \Sigma(q-1) = mF$.

5. As an example, take the regular pyramid on a square base; there is here a single axis, viz. a 4-axis, and we have $1 + \Sigma(q-1) = 1 + 3 = 4$. If for the face A we take the square base, then there is no other face homologous thereto and therefore $F = 1$; but the four sides are homologous to each other or $m = 4$, and we have $mF = 4$. Similarly taking for A one of the triangular faces, since these are homologous to each other, then $F = 4$; and if we take for the side a the base of the triangle, then there is no other side homologous to this, or $m = 1$; and therefore $mF = 4$. It might at first sight appear that the two equal sides of the triangle were homologous to each other, and therefore that taking for the edge a one of these sides we should have $m = 2$; but in fact although the two sides in question are homologously related to the pyramid, yet according to the definition they are not homologous sides of the triangular face, and we still have $m = 1$, and therefore $mF = 4$.

6. Of course in the case of a regular polyhedron the faces are all homologous, and the edges of a face are all homologous, that is F will denote the entire number of faces, and m the number of edges to a face: so that for this case the theorem gives $1 + \Sigma(q - 1) = 2E$ as above.

7. Returning to the regular polyhedra the axial systems are

Tetrahedron	$4L^3$, $3L^2$.
Cube and Octahedron .	$3L^4$, $4L^3$, $6L^2$.
Dodecahedron and Icosahedron	$6L^5$, $10L^3$, $15L^2$,

where L^3 denotes a 3-axis, &c.; this is in accordance with the notation of M. Bravais in the memoir subsequently referred to.

8. The regular polyhedra may be exhibited in connexion with each other as follows: Imagine the polyhedron projected on a concentric sphere by lines through the centre; so that the summits become points on the sphere, the edges arcs of great circles, and the faces spherical polygons. Starting from the dodecahedron, the centres of the pentagonal faces are the summits of the icosahedron, and conversely for the icosahedron the centres of the triangular faces are the summits of the dodecahedron: moreover each edge of the dodecahedron cuts at right angles an edge of the icosahedron and the two edges have the same mid-point. Again if in any face of the dodecahedron we draw one of the five diagonals (arcs through two non-adjacent summits) there is in the face a single edge not met by this diagonal; and in the other face through this edge a single diagonal not met by the edge; joining the extremities of the two diagonals we have a spherical square, the face of the cube; it is to be observed that the summits of the cube are eight out of the twenty summits of the dodecahedron, and that the centres of the faces of the cube are the mid-points of six out of the thirty edges of the dodecahedron or the icosahedron. The cube given by the foregoing construction is of course one out of five different cubes. The centres of the faces of the cube are the summits of the octahedron; and conversely the centres of the faces of the octahedron are the summits of the cube; moreover each edge of the cube cuts at right angles an edge of the octahedron; and the two edges have the same mid-point. Finally, taking four non-adjacent summits of the cube (which can be done in two different ways), these are the summits of the tetrahedron, and the mid-points of the edges of the tetrahedron are the summits of the octahedron.

9. Considering the polyhedra in the foregoing mutual connexion, all the axes of the tetrahedron are axes of the cube and octahedron, viz. the 2-axes of the tetrahedron are the 4-axes of the cube and octahedron; and the 3-axes of the tetrahedron are the 3-axes of the cube and octahedron; moreover the 3-axes of the cube and octahedron are included among the 3-axes of the dodecahedron and icosahedron and the 4-axes of the cube and octahedron are included among the 2-axes of the dodecahedron and icosahedron; but the 2-axes of the cube and octahedron are not included among the axes of the dodecahedron and icosahedron. The 4-axes of the cube and

octahedron form thus a system of rectangular axes common to all the polyhedra, and representing these axes (or say the summits of the corresponding rectangular spherical triangle) by X, Y, Z, we have a convenient system of coordinate axes to which to refer all the other axes of the polyhedron, viz. if P be the extremity (chosen at pleasure) of the axis in question, then the position of the axis may be determined by its distance PZ and azimuth XPZ (measured in the direction from X to Y), or by its distances PX, PY, PZ, or say X, Y, Z from the three rectangular axes (we have, it is clear, $\cos X = \sin \text{dist.} \cos \text{azim.}$, $\cos Y = \sin \text{dist.} \sin \text{azim.}$, $\cos Z = \cos \text{dist.}$). The rotation angle of a q-axis is $= \dfrac{2\pi}{q}$ (i.e. this is the angle through which if the body be turned about the axis, it still occupies the same portion of space) and the half-rotation angle is therefore $= \dfrac{\pi}{q}$. Moreover if i, j, k are Sir W. R. Hamilton's quaternion symbols, then the "rotation symbol" of the axis is

$$\cos\frac{\pi}{q} + \sin\frac{\pi}{q}(i\cos X + j\cos Y + k\cos Z),$$

the application of which will be presently explained.

10. The angular coordinates of the different axes may be found by spherical trigonometry without much difficulty; and we are then able to form the following axial tables of the several polyhedra: the extremity of each axis is chosen in such manner that the distance PZ is not $> 90°$.

Axial System of the Tetrahedron.

Distances angle	Distances cos	Distances sin	Azimuths angle	Azimuths cos	Azimuths sin	cos X	cos Y	cos Z	Rot. Symbols
			4 3-axes, $\frac{1}{2}$ Rot. angle $= 60°$, $\cos = \frac{1}{2}$, $\sin = \frac{1}{2}\sqrt{3}$.						
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{2}(1+i+j+k)$
"	"	"	135°	− "	+ "	− "	+ "	+ "	$\frac{1}{2}(1-i+j+k)$
"	"	"	225°	− "	− "	− "	− "	+ "	$\frac{1}{2}(1-i-j+k)$
"	"	"	315°	+ "	− "	+ "	− "	+ "	$\frac{1}{2}(1+i-j+k)$
			3 2-axes, $\frac{1}{2}$ Rot. angle $= 90°$, $\cos = 0$, $\sin = 1$.						
0°	1	0	*	*	*	0	0	1	k
90°	0	1	0°	1	0	1	0	0	i
"	"	"	90°	0	1	0	1	0	j

Axial System of the Cube and Octahedron.

Distances angle	Distances cos	Distances sin	Azimuths angle	Azimuths cos	Azimuths sin	cos X	cos Y	cos Z	Rot. Symbols
4 3-axes, $\frac{1}{2}$ Rot. angle $=60°$, $\cos=\frac{1}{2}$, $\sin=\frac{1}{2}\sqrt{3}$.									
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{2}(1+i+j+k)$
,,	,,	,,	135°	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$\frac{1}{2}(1-i+j+k)$
,,	,,	,,	225°	− ,,	− ,,	− ,,	− ,,	+ ,,	$\frac{1}{2}(1-i-j+k)$
,,	,,	,,	315°	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}(1+i-j+k)$
3 4-axes, $\frac{1}{2}$ Rot. angle $=45°$, $\cos=\frac{1}{\sqrt{2}}$, $\sin=\frac{1}{\sqrt{2}}$.									
0°	1	0	*	*	*	0	0	1	$\frac{1}{\sqrt{2}}(1+k)$
90°	0	1	0°	1	0	1	0	0	$\frac{1}{\sqrt{2}}(1+i)$
,,	,,	,,	90°	0	1	0	1	0	$\frac{1}{\sqrt{2}}(1+j)$
6 2-axes, $\frac{1}{2}$ Rot. angle $=90°$, $\cos=0$, $\sin=1$.									
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0°	1	0	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(i+k)$
,,	,,	,,	90°	0	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(j+k)$
,,	,,	,,	180°	−1	0	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(-i+k)$
,,	,,	,,	270°	0	−1	0	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}(-j+k)$
90°	0	1	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}(i+j)$
,,	,,	,,	135°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}(-i+j)$

Axial System of the Dodecahedron and Icosahedron.

Distances angle	Distances cos	Distances sin	Azimuths angle	Azimuths cos	Azimuths sin	$\cos X$	$\cos Y$	$\cos Z$	Rot. Symbols
				6 5-axes, $\frac{1}{2}$ Rot. angle $=36°$, $\cos=\frac{\sqrt{5}+1}{4}$, $\sin=\frac{\sqrt{(10-2\sqrt{5})}}{4}$.					
31°44′	$\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	0°	1	0	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	0	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$\frac{\sqrt{5}+1}{4}+\frac{\sqrt{5}-1}{4}i+\frac{1}{2}k$
,,	,,	,,	180°	−1	0	− ,,	,,	+ ,,	$\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}i+\frac{1}{2}k$
58°17′	$\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	90°	0	1	0	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$\frac{\sqrt{5}+1}{4}+\frac{1}{2}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	270°	0	−1	,,	− ,,	+ ,,	$\frac{\sqrt{5}+1}{4}-\frac{1}{2}j+\frac{\sqrt{5}-1}{4}k$
90°	0	1	31°43′	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10+2\sqrt{5})}}{2\sqrt{5}}$	$+\frac{\sqrt{(10-2\sqrt{5})}}{2\sqrt{5}}$	0	$\frac{\sqrt{5}+1}{4}+\frac{1}{2}i+\frac{\sqrt{5}-1}{4}j$
,,	,,	,,	148°17′	− ,,	+ ,,	− ,,	+ ,,	,,	$\frac{\sqrt{5}+1}{4}-\frac{1}{2}i+\frac{\sqrt{5}-1}{4}j$
				10 3-axes, $\frac{1}{2}$ Rot. angle $=60°$, $\cos=\frac{1}{2}$, $\sin=\frac{1}{2}\sqrt{3}$.					
54°44′	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{2}}{\sqrt{3}}$	45°	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{2}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	$\frac{1}{2}(1+i+j+k)$
,,	,,	,,	135°	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$\frac{1}{2}(1-i+j+k)$
,,	,,	,,	225°	− ,,	− ,,	− ,,	− ,,	+ ,,	$\frac{1}{2}(1-i-j+k)$
,,	,,	,,	315°	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}(1+i-j+k)$
20°55′	$\frac{\sqrt{5}+1}{2\sqrt{3}}$	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	90°	0	1	0	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$\frac{1}{2}+\frac{\sqrt{5}-1}{4}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	270°	0	− 1	0	− ,,	+ ,,	$\frac{1}{2}-\frac{\sqrt{5}-1}{4}j+\frac{\sqrt{5}+1}{4}k$
69°5′	$\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{\sqrt{5}+1}{2\sqrt{3}}$	0°	1	0	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	0	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$\frac{1}{2}+\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	180°	− 1	0	− ,,	0	+ ,,	$\frac{1}{2}-\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}k$
90°	0	1	69°5′	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	0	$\frac{1}{2}+\frac{\sqrt{5}-1}{4}i+\frac{\sqrt{5}+1}{4}j$
,,	,,	,,	110°55′	− ,,	+ ,,	− ,,	+ ,,	0	$\frac{1}{2}-\frac{\sqrt{5}-1}{4}i+\frac{\sqrt{5}+1}{4}j$

Axial System of the Dodecahedron and Icosahedron (concluded).

Distances			Azimuths						
angle	cos	sin	angle	cos	sin	cos X	cos Y	cos Z	Rot. Symbols
15 2-axes, $\frac{1}{2}$ Rot. angle $=90^\circ$, cos $=0$, sin $=1$.									
0°	1	0	*	*	*	0	0	1	k
90°	0	1	0°	1	0	1	0	0	i
,,	,,	,,	90°	0	1	0	1	0	j
36°	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{(10-2\sqrt{5})}}{4}$	$62^\circ 40'$	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{\sqrt{5}-1}{4}$	$+\frac{1}{2}$	$+\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{5}-1}{4}i+\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	$297^\circ 20'$	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{\sqrt{5}-1}{4}i-\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	$242^\circ 40'$	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{\sqrt{5}-1}{4}i-\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
,,	,,	,,	$117^\circ 20'$	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{\sqrt{5}-1}{4}i+\frac{1}{2}j+\frac{\sqrt{5}+1}{4}k$
60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$20^\circ 55'$	$+\frac{\sqrt{5}+1}{2\sqrt{3}}$	$+\frac{\sqrt{5}-1}{2\sqrt{3}}$	$+\frac{\sqrt{5}+1}{4}$	$+\frac{\sqrt{5}-1}{4}$	$+\frac{1}{2}$	$\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	$339^\circ 5'$	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{\sqrt{5}+1}{4}i-\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	$200^\circ 55'$	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{\sqrt{5}+1}{4}i-\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
,,	,,	,,	$159^\circ 5'$	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{\sqrt{5}+1}{4}i+\frac{\sqrt{5}-1}{4}j+\frac{1}{2}k$
72°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{(10+2\sqrt{5})}}{4}$	$62^\circ 40'$	$+\frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{2}{\sqrt{(10-2\sqrt{5})}}$	$+\frac{1}{2}$	$+\frac{\sqrt{5}+1}{4}$	$+\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}i+\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	$297^\circ 20'$	+ ,,	− ,,	+ ,,	− ,,	+ ,,	$\frac{1}{2}i-\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	$242^\circ 40'$	− ,,	− ,,	− ,,	− ,,	+ ,,	$-\frac{1}{2}i-\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$
,,	,,	,,	$117^\circ 20'$	− ,,	+ ,,	− ,,	+ ,,	+ ,,	$-\frac{1}{2}i+\frac{\sqrt{5}+1}{4}j+\frac{\sqrt{5}-1}{4}k$

11. Before proceeding further I remark that exclusively of the foregoing axial systems of the regular polyhedra the only cases are as follows:

A. A polyhedron may have a single q-axis, say Λ^q: taking this as the axis of Z the table is

X	Y	Z	Rot. Symbol
90°	90°	0°	$\cos\frac{\pi}{q}+\sin\frac{\pi}{q}\,.\,k$

B. It may have a single q-axis, and (symmetrically arranged in a plane at right angles thereto) q 2-axes, say Λ^q, qL^2. Taking the q-axis as the axis of Z and some one of the 2-axes as the axis of X, the table is

X	Y	Z	Rot. Symbols
One q-axis, $\frac{1}{2}$ Rot. angle $=\frac{\pi}{q}$.			
90°	90°	0°	$\cos\frac{\pi}{q}+\sin\frac{\pi}{q}\,.\,k$
q 2-axes, $\frac{1}{2}$ Rot. angle $=90°$.			
0°	90°	90°	i
$\frac{\pi}{q}$	$90°-\frac{\pi}{q}$	90°	$i\cos\frac{\pi}{q}+j\sin\frac{\pi}{q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(q-1)\frac{\pi}{q}$	$90°-\frac{(q-1)\pi}{q}$	90°	$i\cos\frac{(q-1)\pi}{q}+j\sin\frac{(q-1)\pi}{q}$

and in particular if $q=2$, the axes are $3L^2$ and the table is

X	Y	Z	Rot. Symbols
3 2-axes, $\frac{1}{2}$ Rot. angle $=90°$			
90°	90°	0°	k
0°	90°	90°	i
90°	0°	90°	j

This in fact appears, Bravais, "Mémoire sur les polyèdres de forme symétrique," *Liouville*, t. XIV., pp. 141—180 (1843), observing that for the present purpose there is no distinction between his three cases

$$\Lambda^{2q+1},\ (2q+1)\,L^2\ ;\quad \Lambda^{2q},\ qL^2,\ qL'^2\ ;\quad \Lambda^{2q},\ 2qL^2.$$

12. The meaning of the rotation symbol is as follows: viz. if in general we have a rotation θ about an axis inclined at the angles X, Y, Z to any three rectangular axes, and if Π be the rotation symbol,

$$\Pi = \cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta\,(i \cos X + j \cos Y + k \cos Z),$$

then if x, y, z are the original coordinates of any point of the body, and x', y', z' the coordinates of the same point after the rotation; the values of x', y', z' are given in terms of x, y, z by the formula

$$ix' + jy' + kz' = \Pi\,(ix + jy + kz)\,\Pi^{-1}.$$

This is in fact the form under which, in the paper "On certain results relating to Quaternions," *Phil. Mag.*, vol. XXVI. (1845), p. 141, [20], I exhibited the rotation formulæ of Euler and Rodrigues. See also my paper "On the application of Quaternions to the Theory of Rotation," *Phil. Mag.*, vol. XXXIII. (1848), p. 196, [68].

13. We have, it is clear,

$$\Pi^s = \cos s\theta + \sin s\theta\,(i \cos X + j \cos Y + k \cos Z)$$

which shows that Π^s is the symbol for the rotation Π repeated s times: (more generally performing on the body, first the rotation Π and then the rotation Φ about any axis, the same or different, the symbol of the resultant rotation is $=\Phi\Pi$). If Π be the symbol for a rotation through the angle $\dfrac{2\pi}{q}$, then the rotation which corresponds to the symbol Π^q is a rotation through 360°, that is the body returns to its original position; it might at first sight appear that we ought to have $\Pi^q = 1$, and that the symbols $1, \Pi, \Pi^2, \ldots \Pi^{q-1}$ would form a *group;* this however is not so, for we have not $\Pi^q = 1$, but $\Pi^q = -1$; in fact, it is to be observed that to pass from $ix + jy + kz$ to $ix' + jy' + kz'$, we have to multiply by $\Pi(\quad)\Pi^{-1}$, so that the symbol of the rotation is indifferently $\pm\Pi$, and that the rotation symbol -1 is thus equivalent to the rotation symbol $+1$. But as regards the formation of the group, the only difference is that it is not $1, \Pi, \Pi^2, \ldots \Pi^{q-1}$ which form a group of q symbols, but $\pm 1, \pm\Pi, \pm\Pi^2, \ldots \pm\Pi^{q-1}$ which form a group of $2q$ symbols. And so in the axial system of any polyhedron, if Π be the rotation symbol of any q-axis, then taking for each axis of the polyhedron the set of symbols $\pm\Pi, \pm\Pi^2, \ldots \pm\Pi^{q-1}$, and besides the two symbols ± 1, the whole series of symbols form together a group.

14. Thus in the before-mentioned case $B\,(q=2)$ we have the eight symbols

$$\pm 1,\ \pm i,\ \pm j,\ \pm k$$

forming (as they obviously do) a group. In the general case B, putting for shortness

$$\Theta = \cos\frac{\pi}{q} + \sin\frac{\pi}{q}\,.\,k \ \text{ and } \ \Phi_s = i \cos\frac{s\pi}{q} + j \sin\frac{s\pi}{q},$$

the group consists of the $4q$ symbols

$$\pm 1,\ \pm\Theta,\ \ldots \pm\Theta^{q-1};\quad \pm\Phi_1,\ \pm\Phi_2,\ \ldots \pm\Phi_{q-1},$$

(to verify that this is so, it is only necessary to form the equations

$$\Theta^r\Theta^s = \Theta^{r+s},\quad \Phi_s{}^2 = -1,\quad \Theta^r\Phi_s = \Phi_{s+r},\quad \Phi^s\Theta^r = \Phi_{s-r},\quad \Phi_r\Phi_s = -\Theta^{r-s},$$

which are at once seen to be true).

15. The $\pm$ general case A gives merely the group of the $2q$ symbols

$$\pm 1,\ \pm\Theta,\ \ldots \pm\Theta^{q-1},$$

which has been already mentioned.

16. The tetrahedron gives the group of 24 symbols,

$\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$	16 cube roots of ± 1
$\pm i,\ \pm j,\ \pm k$	6 square " " "
± 1	2 terms
	24

(the signs $\pm$ being all independent).

17. The cube and octahedron give the group of 48 symbols

$\frac{1}{\sqrt{2}}(\pm 1 \pm i),\ \frac{1}{\sqrt{2}}(\pm 1 \pm j),\ \frac{1}{\sqrt{2}}(\pm 1 \pm k)$	12 fourth roots of ± 1
$\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$	16 cube " " "
$\pm i,\ \pm j,\ \pm k,\ \frac{1}{\sqrt{2}}(\pm j \pm k),\ \frac{1}{\sqrt{2}}(\pm k \pm i),\ \frac{1}{\sqrt{2}}(\pm i \pm j)$	18 square " " "
± 1	2 terms
	48

(the signs $\pm$ being all independent).

18. The dodecahedron and icosahedron give the group of 120 symbols

$$\left.\begin{array}{l} \pm\dfrac{\sqrt{5}\pm 1}{4} \pm \tfrac{1}{2}j \pm \dfrac{\sqrt{5}\mp 1}{4}k, \\ \pm\dfrac{\sqrt{5}\pm 1}{4} \pm \tfrac{1}{2}k \pm \dfrac{\sqrt{5}\mp 1}{4}i, \\ \pm\dfrac{\sqrt{5}\pm 1}{4} \pm \tfrac{1}{2}i \pm \dfrac{\sqrt{5}\mp 1}{4}j, \end{array}\right\}\ \text{48 fifth roots of } \pm 1$$

$$\tfrac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

$$
\begin{array}{ll}
\left.\begin{array}{l}
\pm\tfrac{1}{2} \pm \dfrac{\sqrt{5}-1}{4} j \pm \dfrac{\sqrt{5}+1}{4} k, \\
\pm\tfrac{1}{2} \pm \dfrac{\sqrt{5}-1}{4} k \pm \dfrac{\sqrt{5}+1}{4} i, \\
\pm\tfrac{1}{2} \pm \dfrac{\sqrt{5}-1}{4} i \pm \dfrac{\sqrt{5}+1}{4} j,
\end{array}\right\} & \text{40 cube roots of } \pm 1 \\
\pm i,\ \pm j,\ \pm k, & \\
\left.\begin{array}{l}
\pm\tfrac{1}{2} i \pm \dfrac{\sqrt{5}+1}{4} j \pm \dfrac{\sqrt{5}-1}{4} k, \\
\pm\tfrac{1}{2} j \pm \dfrac{\sqrt{5}+1}{4} k \pm \dfrac{\sqrt{5}-1}{4} i, \\
\pm\tfrac{1}{2} k \pm \dfrac{\sqrt{5}+1}{4} i \pm \dfrac{\sqrt{5}-1}{4} j,
\end{array}\right\} & \text{30 square } ,, \quad ,, \\
\pm 1 & \quad\ \underline{\text{2 terms}} \\
 & \quad\quad 120
\end{array}
$$

(The signs $\pm$ are all independent, except that in each of the three expressions in the top line the signs in $\sqrt{5} \pm 1$, $\sqrt{5} \mp 1$, are opposite to each other, so that each of the three expressions has 16 values.)

It is to be remarked that in the groups of 24 and 48, the group is not altered by any permutation whatever of the symbols i, j, k; whereas the group of 120 is not altered by the cyclical permutation of these symbols, but it is altered by the interchange of any two of them; the geometrical reason of this difference may be perceived without difficulty.

P.S. I found accidentally, *Gergonne*, t. xv., p. 40, (1824—25), the following problem: "De combien de manières m couleurs différentes les unes des autres peuvent-elles être appliquées sur les faces d'un polyèdre régulier; m représentant tour à tour les nombres 4, 6, 8, 12, 20?"

Instead of the m of the problem, writing as before F for the number of faces, and writing also E for the number of edges; then if different positions of the same polyhedron were reckoned as different polyhedra, the number of ways would of course be $\Pi(F)$ $(=1.2.3\ldots F)$; and since by what precedes the same polyhedron can be placed in $2E$ positions, the required number of ways is $\dfrac{1}{2E}\Pi(F)$.

Thus for the tetrahedron, if the colours are black, white, red, green, we may place it with the black face on the table and the white face in front; the only variation in the disposition of the colours, is according as the right hand and the left hand faces are coloured red and green or else green and red respectively; and the number of ways therefore is $=2$, which agrees with the formula.

2, *Stone Buildings, W.C.*, 30 *January*, 1863.