382.

NOTE ON THE TETRAHEDRON.

THE following simple properties of a tetrahedron seem worth noticing.

In the tetrahedron $ABCD$ if $AC = BD$ and $AD = BC$, then the line joining the middle points of $AB + CD$, or say the points $\frac{1}{2}AB$ and $\frac{1}{2}CD$, cuts at right angles these lines AB and CD.

If $AB = CD$, then the line joining the points $\frac{1}{2}AC$, $\frac{1}{2}BD$, and the line joining the points $\frac{1}{2}AD$, $\frac{1}{2}BC$ (lines which in any tetrahedron meet each other), cut each other at right angles.

In fact if A, B, C, D have for their coordinates $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$: then the coordinates of the point $\frac{1}{2}AB$ are $\frac{1}{2}(\alpha_1 + \alpha_2)$, $\frac{1}{2}(\beta_1 + \beta_2)$, and so for the points $\frac{1}{2}CD$, &c.: the equations of the line through the points $\frac{1}{2}AB$, $\frac{1}{2}CD$ therefore are

$$\frac{x - \frac{1}{2}(\alpha_1 + \alpha_2)}{\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4} = \frac{y - \frac{1}{2}(\beta_1 + \beta_2)}{\beta_1 + \beta_2 - \beta_3 - \beta_4} = \frac{z - \frac{1}{2}(\gamma_1 + \gamma_2)}{\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4},$$

and I observe in passing that this line passes through the point whose coordinates are

$$\tfrac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4),\ \tfrac{1}{4}(\beta_1 + \beta_2 + \beta_3 + \beta_4),\ \tfrac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4);$$

the other two similar lines pass through the same point, and the above-mentioned property of the general tetrahedron is thus proved.

The condition that the foregoing line may cut at right angles the line AB, the equations whereof are

$$\frac{x - \alpha_1}{\alpha_1 - \alpha_2} = \frac{y - \beta_1}{\beta_1 - \beta_2} = \frac{z - \gamma_1}{\gamma_1 - \gamma_2},$$

is at once seen to be

$$\Sigma\,(\alpha_1 - \alpha_2)\,(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) = 0,$$

where Σ denotes the sum of the corresponding terms in α, β, γ. And so the condition that the same line may cut at right angles the line CD is

$$\Sigma\,(\alpha_3 - \alpha_4)\,(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) = 0.$$

But the conditions $AC = BD$, $AD = BC$ give respectively

$$\Sigma\,\{(\alpha_1 - \alpha_3)^2 - (\alpha_2 - \alpha_4)^2\} = 0,\ \ \Sigma\,\{(\alpha_1 - \alpha_4)^2 - (\alpha_2 - \alpha_3)^2\} = 0,$$

or writing these in the form

$$\Sigma\,(\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)\,(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) = 0,$$
$$\Sigma\,(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)\,(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) = 0,$$

we obtain, by successively adding and subtracting, the two required equations.

The equations of the line through $\tfrac{1}{2}AC$, $\tfrac{1}{2}BD$ are

$$\frac{x - \tfrac{1}{2}(\alpha_1 + \alpha_3)}{\alpha_1 + \alpha_3 - \alpha_2 - \alpha_4} = \frac{y - \tfrac{1}{2}(\beta_1 + \beta_3)}{\beta_1 + \beta_3 - \beta_2 - \beta_4} = \frac{z - \tfrac{1}{2}(\gamma_1 + \gamma_3)}{\gamma_1 + \gamma_3 - \gamma_2 - \gamma_4},$$

and those of the line through $\tfrac{1}{2}AD$, $\tfrac{1}{2}BC$ are

$$\frac{x - \tfrac{1}{2}(\alpha_1 + \alpha_4)}{\alpha_2 + \alpha_3 - \alpha_1 - \alpha_4} = \frac{y - \tfrac{1}{2}(\beta_1 + \beta_4)}{\beta_2 + \beta_3 - \beta_1 - \beta_4} = \frac{z - \tfrac{1}{2}(\gamma_1 + \gamma_4)}{\gamma_2 + \gamma_3 - \gamma_1 - \gamma_4};$$

and the condition that these may cut at right angles is

$$\Sigma\,(\alpha_1 + \alpha_3 - \alpha_2 - \alpha_4)\,(\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3) = 0,$$

that is

$$\Sigma\,\{(\alpha_1 - \alpha_2)^2 - (\alpha_3 - \alpha_4)^2\} = 0,$$

which is in fact the condition $AB = CD$.

Combining the two theorems we see that if in a tetrahedron the pairs of opposite sides are respectively equal, then the line joining the centres of opposite sides cuts these sides at right angles, and moreover the three joining lines cut each other at right angles.

A tetrahedron of the form in question may be constructed as follows: viz. taking a parallelogram $ABCD$, whereof the diagonals AC, BD are unequal, then bending the parallelogram about its shorter diagonal AC in such manner that in the solid figure BD becomes equal to AC, we have a tetrahedron the opposite sides whereof are respectively equal.

Or it may be constructed even more simply as follows: viz. if $AB'CD'$ and $A'BC'D$ be parallel faces of any rectangular parallelopiped (the angles A and A', B and B', C and C', D and D' being respectively opposite to each other), then $ABCD$ or $A'B'C'D'$ is a tetrahedron of the form in question. The consideration of the rectangular parallelopiped puts in evidence the foregoing geometrical property.

In such a tetrahedron the line joining the centres of a pair of opposite sides is in the language of Bravais, see his "Mémoire sur les polyèdres de forme symétrique," *Liouville*, t. XIV. (1849), pp. 141—180, a binary axis of symmetry: viz. the figure is not altered by turning it round such axis through an angle $= \frac{1}{2} 360°$. There are thus three such axes at right angles to each other, but the figure has not any centre of symmetry, nor (assuming that it is not further particularised) any plane of symmetry: each of the three axes is a principal axis, and the figure belongs to the sixth of Bravais' twenty-three classes of polyhedra, see the table p. 179. It was in fact by seeking to construct a figure of this class that I was led to the investigation.