383.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. I. to IV., 1863 to 1865.]

[Vol. I. (June 1863 to June 1864), pp. 18, 19.]

1373. (By T. T. WILKINSON, F.R.A.S.)—Given a circle (C) and any point A, either within or without the circle: through A draw BAD cutting the circle in B, D. Then it is required to find another point E, such that, if LEM be drawn cutting the circle in L, M, we may always have $AE^2 = LE \,.\, EM \pm BA \,.\, AD$.

Solution by PROFESSOR CAYLEY.

Consider a circle centre O and radius OA, and in relation thereto a point M either outside or inside the circle, and suppose that

$(OA)^2 - (OM)^2$, or the "squared inner potency" of M is denoted by $\square i \,.\, M$,

and

$(OM)^2 - (OA)^2$, or the "squared outer potency" of M is denoted by $\square o \,.\, M$,

so that, for an outside point, $\square o \,.\, M, = - \square i \,.\, M$, is the square of the tangential distance of M from the circle; and, for an inside point, $\square i \,.\, M, = - \square o \,.\, M$, is the square of the shortest semi-chord through M.

Suppose now that M is a given point; the proposed question is in effect to find the locus of a point P such that $\pm \square o \,.\, P \pm \square o \,.\, M = (MP)^2$; but we have thus in reality four different questions according as the signs are assumed to be $++$, $+-$, $-+$, or $--$; the case $++$, or when $\square o \,.\, P + \square o \,.\, M = (MP)^2$, is perhaps the most interesting.

Taking the radius as unity, (α, β) as the coordinates of M, and (x, y) as the coordinates of P, we have here

$$(x^2+y^2-1)+(\alpha^2+\beta^2-1)=(x-\alpha)^2+(y-\beta)^2, \text{ or } \alpha x+\beta y-1=0;$$

that is, the locus of P is a right line, the polar of M in regard to the circle.

It may be remarked, that, when M is an inside point, then throughout the locus P is an outside point; and, replacing the negative quantity $\square o\,.\,M$ by its value, $=-\square i\,.\,M$, we have $\square o\,.\,P-\square i\,.\,M=(MP)^2$. If, however, M is an outside point, then in part of the locus P is an outside point, and we have $\square o\,.\,P+\square o\,.\,M=(MP)^2$, while in the remainder of the locus P is an inside point, and, replacing the negative quantity $\square o\,.\,P$ by its value, $=-\square i\,.\,P$, we have $-\square i\,.\,P+\square o\,.\,M=(MP)^2$. For the case $+-$, the locus of P is a right line, but for each of the other two cases $-+$ and $--$ the locus is a circle; the discussion of the several cases presents no particular difficulty.

[Vol. I. pp. 43—45.]

1387. (By W. K. Clifford.)—1. Four common tangents are drawn to a circle and an ellipse which passes through the centre (O) of the circle; if A, B be opposite intersections of the tangents, prove that OA and OB are equally inclined to the tangent at O to the ellipse.

2. If a straight line A join the poles of B with respect to two conics, prove that the lines joining AB to a pair of opposite intersections of common tangents, form, with A, B, an harmonic pencil.

3. If a point A be the intersection of the polars of B with respect to two conics, and AB be cut by a pair of common chords in C, D, prove that $ACBD$ is an harmonic range.

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2. *Solution by* Professor Cayley.

This elegant theorem is included as a particular case in the known theorem, "Given three conics inscribed in the same quadrilateral, the tangents from any point to these conics form a pencil in involution."

Mr Clifford's theorem is in fact as follows: viz., Four common tangents are drawn to a circle and an ellipse which passes through the centre O of the circle; if A, B be opposite intersections of the tangents, then OA, OB are equally inclined to the tangent at O to the ellipse.

This comes to saying that the tangent at O to the ellipse, say OT, is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ drawn from O to the circular points at infinity; and if we replace the circle by an arbitrary conic S, and the line at infinity by an arbitrary line IJ, the theorem will be as follows:

Consider a conic S; a line meeting this conic in the points I, J; and the point O, the intersection of the tangents at I, J, or (what is the same thing) the pole of the line IJ in regard to the conic. If through the point O there be drawn any other conic Θ, and if A, B be opposite intersections of the common tangents of the conics S, Θ; then the tangent OT at the point O to the conic Θ is the double or sibi-conjugate line of the involution of the pencil formed by the lines OA, OB, and the lines OI, OJ; or, as we may also express it, the lines OT, OT, the lines OA, OB, and the lines OI, OJ form a pencil in involution.

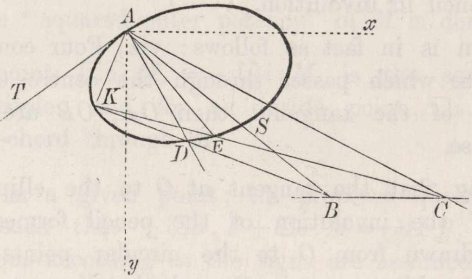
Now, considering the two points or point-pair (A, B) as a conic inscribed in the quadrilateral formed by the common tangents of the conics S and Θ, the conics S and Θ and the point-pair (A, B) are a system of three conics inscribed in the same quadrilateral; and hence, by the general theorem above referred to, if O' be any point whatever, the tangents from O to the conic S, the tangents from O' to the conic Θ, and the tangents from O' to the point-pair (that is, the two lines $O'A$, $O'B$) form a pencil in involution. But, if O' coincide with O, then the tangents to the conic S are the lines OI, OJ; and the tangents to the conic Θ are the coincident lines OT, OT; and we have thence the theorem in question; viz., that the lines OT, OT, the lines OI, OJ, and the lines OA, OB form a pencil in involution.

[Vol. I. pp. 77—79.]

1409. (By W. K. CLIFFORD.)—For every point A on a conic section there exists a straight line BC, not meeting the curve, such that, if through any other point K on the conic there be drawn any two straight lines meeting BC in B, C, and the curve in D, E, the angles BAC, DAE are either equal or supplementary.

Solution by PROFESSOR CAYLEY.

I find that this very elegant theorem depends on the lemma to be presently stated, and that it is intimately connected with Newton's theorem for the organic description of a conic, or, what is the same thing, with the theorem of the anharmonic relation of the points of a conic.



LEMMA. If AT be the tangent, and AS any other line through a point A of a conic, and if two lines equally inclined to AT and AS respectively meet the conic in the points K and D (viz., if $\angle TAK = SAD$, the two angles being measured in opposite directions from AT, AS respectively); then the line KD meets AS in a fixed point B, that is, a point the position of which is independent of the magnitude of the equal angles.

To prove this, take A for the origin, and the bisectors of the angle TAS for the axes of x and y: then the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2fy + 2gx = 0;$$

the equation of the tangent at the origin, that is, the line AT, is $gx + fy = 0$; and hence the equation of the line AS is $gx - fy = 0$. Taking $y = \alpha x$ for the equation of the line AK, we have, for the coordinates x_1, y_1 of the point K where this meets the conic,

$$(a + 2h\alpha + b\alpha^2)\, x_1 + 2\,(f\alpha + g) = 0, \quad y_1 = \alpha x_1;$$

and then the equation of the line AD will be $y = -\alpha x$, and we shall have, for the coordinates x_2, y_2 of the point D where this meets the conic,

$$(a - 2h\alpha + b\alpha^2)\, x_2 + 2\,(f\alpha + g) = 0, \quad y_2 = -\alpha x_2.$$

The equation of the line KD is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & \alpha x_1, & 1 \\ x_2, & -\alpha x_2, & 1 \end{vmatrix} = 0,$$

that is

$$\alpha x\,(x_1 + x_2) + y\,(x_2 - x_1) - 2\alpha x_1 x_2 = 0;$$

and for the coordinates of the point B where this meets the line AS, the equation whereof is $gx - fy = 0$, we have

$$x\,\{f\alpha\,(x_1 + x_2) + g\,(x_2 - x_1)\} - 2f\alpha x_1 x_2 = 0,$$

or, as this may be written,

$$x\left\{\frac{f\alpha + g}{x_1} - \frac{-f\alpha + g}{x_2}\right\} - 2f\alpha = 0.$$

But we have

$$\frac{f\alpha + g}{x_1} = -\tfrac{1}{2}\,(a + 2h\alpha + b\alpha^2), \quad \frac{-f\alpha + g}{x_2} = -\tfrac{1}{2}\,(a - 2h\alpha + b\alpha^2);$$

and hence the equation is

$$x\,(-2h\alpha) - 2f\alpha = 0,$$

giving $x = -\dfrac{f}{h}$, and thence $y = -\dfrac{g}{h}$, for the coordinates of the point B; and, these being independent of α, the lemma is seen to be true.

Consider now the points A, K as fixed points on the conic; then, revolving about A the constant angle DAB, and about K the constant (zero) angle DKB, the locus of B is (by the theorem of the anharmonic relation of the points of a conic) given in the first instance as a conic through the points A, K; but, observing that a position of the angle DAB is TAK, and that the corresponding position of DKB is AKA, the line AK is part of the locus; and the locus is made up of this line and a line BC. And, conversely, given the fixed points A, K, and the line BC, the original conic is, by Newton's theorem, described by means of the constant angles DAB, DKB revolving about these points in such a manner that the arms AB, KB generate by their intersections the line BC. This being so, the other two arms AD, KD generate by their intersections the conic.

And then, considering the two positions DAB, EAC of the angle DAB (so that D, B are in a line with K, and E, C are also in a line with K), we have $\angle DAB = \angle EAC$, that is, $\angle DAE = \angle BAC$, which is Mr Clifford's theorem.

It has been seen that, A being given, the same line BC is obtained whatever be the position of the point K; and, taking AK for the normal at A, it at once appears geometrically that (as remarked by Mr Clifford) the line BC is the polar of the point Θ of intersection of all the chords which subtend a right angle at A.

{Professor Cayley's lemma may be otherwise proved, as follows:

The trilinear equation of the conic, referred to two tangents (α at A, β at S) and their chord of contact (γ or AS), is $U = \lambda\alpha\beta - \gamma^2 = 0$; and the equation of two straight lines (AK, AD) equally inclined to α, γ is

$$(\alpha - \mu\gamma)(\mu\alpha - \gamma) = 0, \quad \text{or} \quad V = \alpha^2 + \gamma^2 - (\mu + \mu^{-1})\,\alpha\gamma = 0;$$

also $U + V = 0$ denotes a conic passing through the intersections of U and V; but $U + V$ is resolvable into $\alpha = 0$, or the tangent AT, and

$$\alpha + \lambda\beta - (\mu + \mu^{-1})\,\gamma = 0,$$

which is, therefore, the equation of the chord KD; whence we see that KD meets AS (or γ) in a point B (given by $\gamma = 0$, $\alpha + \lambda\beta = 0$) whose position is independent of μ, that is, of the equal angles SAD, TAK.}

[Vol. I. pp. 125—127.]

1478. (By J. McDowell, M.A.)—(α) Two sides of a given triangle always pass through two fixed points; prove that the third side always touches a fixed circle.

(β) Two sides of a given triangle touch two fixed circles; prove that the third side also touches a fixed circle.

(γ) Two sides of a given polygon touch fixed circles; prove that all the remaining sides also touch fixed circles.

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3. *Solution by* PROFESSOR CAYLEY.

Since the theorem (γ) follows at once from (β), and (α) is included in (β), it is only necessary to prove (β). Consider three given circles, and let it be proposed to construct a triangle the sides whereof touch the given circles, and which is similar to a given triangle; the direction of one side may be assumed at pleasure, and then the triangle is determined. Impose now on the triangle the condition that the area is equal to a given quantity; we obtain for the given area an expression involving the angle θ which fixes the direction of one of the sides, and we have thus an equation for the determination of the angle. θ. But, for a properly determined relation between the data of the problem, the expression for the area becomes independent of the angle θ, that is, every triangle, the sides whereof touch the three circles, and which is similar to a given triangle, is of the same area, or say, the area of every such triangle is equal to a given quantity Δ; and, this being so, it is clear that, if we construct a triangle similar to a given triangle and of the given area Δ (that is, a triangle equal to a given triangle), in such manner that two of the sides touch two of the given circles, then the envelope of the remaining side will be the remaining given circle; which is in fact the theorem (β).

It only remains therefore to show that the foregoing porismatic case of the problem exists.

For the first circle, let the coordinates of the centre be a, b, and the radius be c; and suppose in like manner that we have a', b', and c' for the second circle, and a'', b'', and c'' for the third circle. Let λ, λ', λ'' be the inclinations to the axis of x of the perpendiculars on the sides which touch these circles respectively; then the equations of the three sides respectively are

$$(x-a)\cos\lambda+(y-b)\sin\lambda-c=0,\ (x-a')\cos\lambda'+(y-b')\sin\lambda'-c'=0,$$
$$(x-a'')\cos\lambda''+(y-b'')\sin\lambda''-c''=0.$$

If the triangle be similar to a given triangle, then the differences of the angles λ, λ', λ'' will be given angles, or, what is the same thing, we may write

$$\lambda=\theta+\xi,\quad \lambda'=\theta'+\xi,\quad \lambda''=\theta''+\xi,$$

where θ, θ', θ'' are given angles, and ξ is a variable angle. Let Δ be the area of the triangle, then (disregarding a merely numerical factor) we have

$$\begin{aligned}\sqrt{\Delta}=&\sin(\lambda'-\lambda'')(a\cos\lambda+b\sin\lambda+c)\\&+\sin(\lambda''-\lambda)(a'\cos\lambda'+b'\sin\lambda'+c')+\sin(\lambda-\lambda')(a''\cos\lambda''+b''\sin\lambda''+c'');\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}\sqrt{\Delta}=&\ \sin(\theta'-\theta'')\{a\cos(\theta+\xi)+b\sin(\theta+\xi)+c\}\\&+\sin(\theta''-\theta)\{a'\cos(\theta'+\xi)+b'\sin(\theta'+\xi)+c'\}\\&+\sin(\theta-\theta')\{a''\cos(\theta''+\xi)+b''\sin(\theta''+\xi)+c''\}.\end{aligned}$$

It is now clear that the right-hand side will be independent of ξ, if only

$$\sin(\theta'-\theta'')(a\cos\theta+b\sin\theta)+\sin(\theta''-\theta)(a'\cos\theta'+b'\sin\theta') \\ +\sin(\theta-\theta')(a''\cos\theta''+b''\sin\theta'')=0,$$

$$\sin(\theta'-\theta'')(-a\sin\theta+b\cos\theta)+\sin(\theta''-\theta)(-a'\sin\theta'+b'\cos\theta') \\ +\sin(\theta-\theta')(-a''\sin\theta''+b''\cos\theta'')=0;$$

equations which show that, given the form of the triangle and the centres of two of the circles, the centre of the third circle (in the porismatic case) is a determinate unique point: and the theorem is thus proved.

[Vol. I. pp. 137—141.]

1273. (By the EDITOR [W. J. MILLER, B.A.].)—In a given triangle let three triangles be inscribed, by joining the points of contact of the inscribed circle, the points where the bisectors of the angles meet the sides, and the points where the perpendiculars meet the sides; then will the corresponding sides of these three triangles pass through the same point; also the triangle formed by the three points of intersection will be a circumscribed co-polar to the original triangle, and the pole will be on the straight line in which the sides of the given triangle meet the bisectors of its exterior angles.

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1. *Solution by* PROFESSOR CAYLEY.

The theorem is, in fact, included in the following more general

THEOREM. Let the points O, O', O'', ... lie on a conic circumscribed about a triangle ABC; then first the polars of the points O, O', O'', ... in regard to the triangle (see Note at the end of the Solution) pass through a fixed point Ω. And secondly, if by means of the point O, joining it with the vertices A, B, C, and taking the intersections of these lines with the sides BC, CA, AB, respectively, we form a triangle inscribed in the triangle ABC; and the like for the points O', O'', ...; the corresponding sides of the inscribed triangles meet in three points forming a triangle circumscribed about the original triangle ABC, and such that the lines joining the corresponding vertices of the last-mentioned two triangles meet in the point Ω.

But, in order to see that the proposed theorem 1273 is in fact included under the foregoing more general one, it is necessary to state the following

SUBSIDIARY THEOREM. Consider a conic inscribed in the triangle ABC, and passing through the points I, J.

Take O the pole of the line IJ in regard to the conic; O' the point of intersection of the lines joining the vertices of the triangle with the points of contact on the opposite sides respectively; O'' the point of intersection of the lines Al, Bm, Cn, where l is a point on BC such that the lines lA, lBC, lI, lJ form a harmonic pencil, and the like for the points m and n respectively.

Then the points O, O', O'' lie on a conic circumscribed about the triangle ABC.

In fact, if in the subsidiary theorem the inscribed conic be a circle, and the points I, J be the circular points at infinity, the point O will be the centre of the circle, that is, the point of intersection of the interior bisectors of the angles; O' will be the point of intersection of the lines to the points of contact of the inscribed circle; and O'' the point of intersection of the perpendiculars on the sides of the triangle; and, these three points being on a conic circumscribed about the triangle, the general theorem will apply to the three points in question.

I first prove the subsidiary theorem. Taking $x=0$, $y=0$, $z=0$ for the equations of the sides of the triangle and (α, β, γ), $(\alpha', \beta', \gamma')$ for the coordinates of the points I, J respectively; the equation of the inscribed conic is

$$\begin{vmatrix} \sqrt{x}, & \sqrt{y}, & \sqrt{z} \\ \sqrt{\alpha}, & \sqrt{\beta}, & \sqrt{\gamma} \\ \sqrt{\alpha'}, & \sqrt{\beta'}, & \sqrt{\gamma'} \end{vmatrix} = 0,$$

or say

$$a\sqrt{x} + b\sqrt{y} + c\sqrt{z} = 0,$$

where

$$a = \sqrt{\beta\gamma'} - \sqrt{\beta'\gamma} = p - p_1, \quad b = \sqrt{\gamma\alpha'} - \sqrt{\gamma'\alpha} = q - q_1, \quad c = \sqrt{\alpha\beta'} - \sqrt{\alpha'\beta} = r - r_1, \text{ suppose.}$$

The coordinates of the point of intersection of the lines from the vertices to the points of contact on the opposite sides are

$$x : y : z = \frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2},$$

that is,

$$= \frac{1}{(p-p_1)^2} : \frac{1}{(q-q_1)^2} : \frac{1}{(r-r_1)^2}.$$

The equation of the line IJ is

$$(\beta\gamma' - \beta'\gamma)\,x + (\gamma\alpha' - \gamma'\alpha)\,y + (\alpha\beta' - \alpha'\beta)\,z = 0;$$

or, what is the same thing,

$$(p^2 - p_1^2)\,x + (q^2 - q_1^2)\,y + (r^2 - r_1^2)\,z = 0.$$

Representing this for a moment by $\lambda x + \mu y + \nu z = 0$, the coordinates of the pole of this line, in regard to the inscribed conic $a\sqrt{x} + b\sqrt{y} + c\sqrt{z} = 0$, are as

$$c^2\mu + b^2\nu : a^2\nu + c^2\lambda : b^2\lambda + a^2\mu.$$

Now

$$\begin{aligned} c^2\mu + b^2\nu &= (r-r_1)^2(q^2-q_1^2) + (q-q_1)^2(r^2-r_1^2), \\ &= (r-r_1)(q-q_1)\left[(r-r_1)(q+q_1) + (q-q_1)(r+r_1)\right], \\ &= 2(r-r_1)(q-q_1)(qr-q_1r_1), \end{aligned}$$

but, observing that $pqr = p_1q_1r_1$, we have

$$qr - q_1r_1 = \left(\frac{p_1}{p} - 1\right) q_1r_1 = -\frac{(p-p_1)\,q_1r_1}{p} = -\frac{(p-p_1)\,p_1q_1r_1}{pp_1};$$

hence

$$c^2\mu + b^2\nu = -\frac{2\,(p-p_1)\,(q-q_1)\,(r-r_1)\,p_1q_1r_1}{pp_1};$$

and we have the like values for $a^2\nu + c^2\lambda$ and $b^2\lambda + a^2\mu$ respectively; hence, omitting the symmetrical factor, we have, for the coordinates of the point in question,

$$x : y : z = \frac{1}{pp_1} : \frac{1}{qq_1} : \frac{1}{rr_1}.$$

Taking the equation of the line Al to be $Qy + Rz = 0$, those of the lines lI, lJ will be

$$x = \lambda\,(Qy + Rz),\ x = \lambda'\,(Qy + Rz),$$

where

$$\lambda = \frac{\alpha}{Q\beta + R\gamma},\quad \lambda' = \frac{\alpha'}{Q\beta' + R\gamma'};$$

and the harmonic condition gives $\lambda + \lambda' = 0$, that is,

$$Q\,(\alpha\beta' + \alpha'\beta) + R\,(\alpha\gamma' + \alpha'\gamma) = 0;$$

the equation of the line Al is thus found to be

$$(\gamma\alpha' + \gamma'\alpha)\,y = (\alpha\beta + \alpha'\beta)\,z;$$

and, since we have the like forms for the equations of the lines Bm and Cn, we have for the coordinates of the point of intersection of these three lines

$$x : y : z = \frac{1}{\beta\gamma' + \beta'\gamma} : \frac{1}{\gamma\alpha' + \gamma'\alpha} : \frac{1}{\alpha\beta' + \alpha'\beta},$$

that is

$$= \frac{1}{p^2 + p_1^2} : \frac{1}{q^2 + q_1^2} : \frac{1}{r^2 + r_1^2}.$$

The equation of a conic circumscribed about the triangle ABC is

$$\frac{\lambda}{x} + \frac{\mu}{y} + \frac{\nu}{z} = 0,$$

where λ, μ, ν are arbitrary coefficients; and the condition for the three points being in the conic is thus found to be

$$\begin{vmatrix} (p-p_1)^2, & (q-q_1)^2, & (r-r_1)^2 \\ pp_1\ \ , & qq_1\ \ , & rr_1 \\ p^2+p_1^2, & q^2+q_1^2, & r^2+r_1^2 \end{vmatrix} = 0,$$

but, in virtue of the relations

$$(p-p_1)^2=-2pp_1+(p^2+p_1^2), \text{ \&c.},$$

this equation is identically true, and the subsidiary theorem is thus proved.

Passing now to the general theorem, I prove the first part of it as follows:

The equation of a conic circumscribed about the triangle $x=0$, $y=0$, $z=0$ is

$$\frac{A}{x}+\frac{B}{y}+\frac{C}{z}=0;$$

hence, if (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ are the coordinates of any three points on the conic, we have

$$\frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}=0,\quad \frac{A}{\alpha'}+\frac{B}{\beta'}+\frac{C}{\gamma'}=0,\quad \frac{A}{\alpha''}+\frac{B}{\beta''}+\frac{C}{\gamma''}=0,$$

and thence

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix}=0,$$

which is the condition for the intersection in a point of the three lines

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0,$$

$$\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}=0,$$

$$\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}=0;$$

and the theorem in question is thus proved. I remark, in passing, that the theorem might also be stated as follows:—The locus of a point O, such that its polar in regard to the triangle ABC passes through a fixed point Ω, is a conic circumscribed about the triangle.

To prove the second part of the theorem, take for the coordinates of the points O, O', O'' respectively (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; then

$$\begin{vmatrix} \frac{1}{\alpha}, & \frac{1}{\beta}, & \frac{1}{\gamma} \\ \frac{1}{\alpha'}, & \frac{1}{\beta'}, & \frac{1}{\gamma'} \\ \frac{1}{\alpha''}, & \frac{1}{\beta''}, & \frac{1}{\gamma''} \end{vmatrix}=0,$$

and if X, Y, Z are the coordinates of the point Ω, then we have

$$\frac{X}{\alpha}+\frac{Y}{\beta}+\frac{Z}{\gamma}=0,$$

$$\frac{X}{\alpha'}+\frac{Y}{\beta'}+\frac{Z}{\gamma'}=0,$$

$$\frac{X}{\alpha''}+\frac{Y}{\beta''}+\frac{Z}{\gamma''}=0.$$

The equations of the sides of the inscribed triangle obtained by means of the point O are

$$-\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0, \qquad \frac{x}{\alpha}-\frac{y}{\beta}+\frac{z}{\gamma}=0, \qquad \frac{x}{\alpha}+\frac{y}{\beta}-\frac{z}{\gamma}=0,$$

and the like for the triangles obtained by means of the points O' and O'' respectively. Hence, for a set of corresponding sides of the three triangles, we have, e.g.,

$$-\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0, \quad -\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}=0, \quad -\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}=0,$$

and it is clear that these equations are simultaneously satisfied by the values

$$x : y : z = -X : Y : Z,$$

and we have the like expressions for the other sets of corresponding sides; that is, we have for the coordinates of the vertices of the resulting triangle

$$(-X : Y : Z), \quad (X : -Y : Z), \quad (X : Y : -Z);$$

and hence also the equations of the sides of the triangle in question are

$$\frac{y}{Y}+\frac{z}{Z}=0, \quad \frac{z}{Z}+\frac{x}{X}=0, \quad \frac{x}{X}+\frac{y}{Y}=0,$$

that is, it is a triangle circumscribed about the triangle ABC. The equations of the lines joining the corresponding vertices of the two triangles are

$$\frac{y}{Y}=\frac{z}{Z}, \quad \frac{z}{Z}=\frac{x}{X}, \quad \frac{x}{X}=\frac{y}{Y},$$

and these lines meet in the point $(X : Y : Z)$, which is the point Ω, the intersection of the polars of O, O', O''; the demonstration of the theorem is thus completed.

{The expression Polar of a point in regard to a triangle denotes a line constructed as follows:—viz., O being the point and ABC the triangle, then, taking on BC a point a, the harmonic in regard to the points B and C of the intersection of BC by AO; and in like manner on CA and AB the points b and c respectively, the three points a, b, c lie on a line which is the polar of the point O. If the equations of the sides are $x=0$, $y=0$, $z=0$, and the coordinates of the point are (α, β, γ), then the equation of the polar is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$; the equation may also be written $(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)^2 xyz=0$, and it thus appears that the line just defined as the polar is in fact the second or line polar of the point in regard to the three lines BC, CA, AB considered as forming a cubic curve.}

[Vol. II. July to December 1864, pp. 6—9.]

1505. (Proposed by Professor CAYLEY.)—If P, Q, 1, 2, 3, 4 be points on a conic, then the four points $P1$, $Q2$; $P2$, $Q1$; $P3$, $Q4$; $P4$, $Q3$ lie on a conic passing through the points P and Q.

Solution by the PROPOSER.

This is an immediate consequence of the theorem of the anharmonic property of the points of a conic. For if $(P1, P2, P3, P4)$ denote the anharmonic ratio of the lines $P1$, $P2$, $P3$, $P4$, and so in other cases; then

$$(P2,\ P1,\ P4,\ P3)=(P1,\ P2,\ P3,\ P4)=(Q1,\ Q2,\ Q3,\ Q4);$$

that is

$$(P2,\ P1,\ P4,\ P3)=(Q1,\ Q2,\ Q3,\ Q4),$$

which proves the theorem.

In particular, if P, Q are the circular points at infinity, then the conic is a circle. Moreover the points $P1$, $Q2$; $P2$, $Q1$ are the antifocal points of 1, 2; viz., calling these 1′, 2′, then 12 and 1′2′ are lines at right angles to each other, having a common centre O, but such that $1'2'=i\,.\,12$, $(i=\sqrt{-1}$, as usual); or, what is the same thing, $O1=O2=i\,.\,O1'=i\,.\,O2'$. And the theorem is as follows: viz., if 1, 2, 3, 4 are points on a circle, and

1′, 2′ are the antifocal points of 1, 2,

3′, 4′ „ „ „ 3, 4,

then 1′, 2′, 3′, 4′ are points on a circle.

As an *à posteriori* proof, take the centre of the given circle as origin, so that $(\alpha_1,\ \beta_1)$, $(\alpha_2,\ \beta_2)$, $(\alpha_3,\ \beta_3)$, $(\alpha_4,\ \beta_4)$ being the coordinates of 1, 2, 3, 4, and the radius being taken as unity, we have

$$\alpha_1^2+\beta_1^2=\alpha_2^2+\beta_2^2=\alpha_3^2+\beta_3^2=\alpha_4^2+\beta_4^2=1.$$

Suppose for a moment that x, y are the coordinates of the antifocal points of 1, 2; we have

$$x-\alpha_1\pm i\,(y-\beta_1)=0,\quad x-\alpha_2\mp i\,(y-\beta_2)=0,$$

that is

$$x+iy=\alpha_1+i\beta_1,\quad x-iy=\alpha_2-i\beta_2,$$

for the coordinates of the one point; and similarly

$$x - iy = \alpha_1 - i\beta_1, \quad x + iy = \alpha_2 + i\beta_2,$$

for the coordinates of the other point.

Hence, taking the new coordinates

$$X = x + iy, \quad Y = x - iy,$$

and similarly $A_1 = \alpha_1 + i\beta_1$, $B_1 = \alpha_1 - i\beta_1$, &c.; the coordinates of the antifocal points 1′, 2′ are (A_1, B_2) and (A_2, B_1) respectively; but we have $A_1B_1 = \alpha_1^2 + \beta_1^2 = 1$, $A_2B_2 = \alpha_2^2 + \beta_2^2 = 1$; so that $B_1 = \frac{1}{A_1}$, $B_2 = \frac{1}{A_2}$; and the coordinates are $\left(A_1, \frac{1}{A_2}\right)$, $\left(A_2, \frac{1}{A_1}\right)$ respectively. Similarly the coordinates of the antifocal points (3′, 4′) are $\left(A_3, \frac{1}{A_4}\right)$, $\left(A_4, \frac{1}{A_3}\right)$ respectively.

Take as the equation of the circle through the two pairs of antifocal points

$$x^2 + y^2 + 2\lambda x + 2\mu y + \nu = 0,$$

or, what is the same thing,

$$XY + \lambda(X + Y) - i\mu(X - Y) + \nu = 0,$$

that is

$$XY + LY + MX + N = 0,$$

if

$$L = \lambda + i\mu, \quad M = \lambda - i\mu, \quad N = \nu.$$

We ought then to have

$$\frac{A_1}{A_2} + L\frac{1}{A_2} + MA_1 + N = 0,$$

$$\frac{A_2}{A_1} + L\frac{1}{A_1} + MA_2 + N = 0,$$

$$\frac{A_3}{A_4} + L\frac{1}{A_4} + MA_3 + N = 0,$$

$$\frac{A_4}{A_3} + L\frac{1}{A_3} + MA_4 + N = 0;$$

and these will exist simultaneously, if

$$\begin{vmatrix} \frac{A_1}{A_2}, & \frac{1}{A_2}, & A_1, & 1 \\ \frac{A_2}{A_1}, & \frac{1}{A_1}, & A_2, & 1 \\ \frac{A_3}{A_4}, & \frac{1}{A_4}, & A_3, & 1 \\ \frac{A_4}{A_3}, & \frac{1}{A_3}, & A_4, & 1 \end{vmatrix} = 0,$$

an identical equation which is easily verified. It, in fact, gives

$$\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)-\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)+\left(\frac{A_1}{A_2}-\frac{A_2}{A_1}\right)\left(1-1\right)+$$

$$\left(1-1\right)\left(\frac{A_3}{A_4}-\frac{A_4}{A_3}\right)-\left(\frac{1}{A_2}-\frac{1}{A_1}\right)\left(A_3-A_4\right)+\left(A_1-A_2\right)\left(\frac{1}{A_4}-\frac{1}{A_3}\right)=0,$$

which is obviously true. The equation may also be written

$$\begin{vmatrix} 1, & A_1, & A_2, & A_1A_2 \\ 1, & A_2, & A_1, & A_1A_2 \\ 1, & A_3, & A_4, & A_3A_4 \\ 1, & A_4, & A_3, & A_3A_4 \end{vmatrix}=0,$$

and in this form it expresses the known theorem of the equality of the anharmonic ratios of (A_1, A_2, A_3, A_4) and (A_2, A_1, A_4, A_3).

But, in order to actually find the circle, we may write

$$\begin{aligned} XY+LY+MX \quad +N \quad &=0, \\ A_1 \quad +L \quad +MA_1A_2+NA_2&=0, \\ A_2 \quad +L \quad +MA_1A_2+NA_1&=0, \\ A_3 \quad +L \quad +MA_3A_4+NA_4&=0, \end{aligned}$$

and eliminating L, M, N, the equation of the circle is

$$\begin{vmatrix} XY, & Y, & X \;, & 1 \\ A_1\;, & 1, & A_1A_2, & A_2 \\ A_2\;, & 1, & A_1A_2, & A_1 \\ A_3\;, & 1, & A_3A_4, & A_4 \end{vmatrix}=0,$$

or, reducing, this is

$$(A_2-A_1)\,[XY(A_3A_4-A_1A_2)+Y\{A_1A_2(A_3+A_4)-A_3A_4(A_1+A_2)\}$$
$$+X(A_1+A_2-A_3-A_4)+(A_3A_4-A_1A_2)]=0,$$

or say

$$XY(A_1A_2-A_3A_4)+Y\{A_3A_4(A_1+A_2)-A_1A_2(A_3+A_4)\}$$
$$+X\{A_3+A_4-A_1-A_2\}+(A_1A_2-A_3A_4)=0:$$

that is

$$\begin{vmatrix} XY+1\;, & X & , & Y \\ A_1 \;+A_2, & A_1A_2, & 1 \\ A_3 \;+A_4, & A_3A_4, & 1 \end{vmatrix}=0,$$

which is the required equation; or, transforming to the original axes, we have $x+iy=X$, $x-iy=Y$, &c., and therefore $XY=x^2+y^2$; and the equation becomes

$$\begin{vmatrix} x^2+y^2+1, & x+iy, & x-iy \\ \alpha_1+\alpha_2+i(\beta_1+\beta_2), & (\alpha_1+i\beta_1)(\alpha_2+i\beta_2), & 1 \\ \alpha_3+\alpha_4+i(\beta_3+\beta_4), & (\alpha_3+i\beta_3)(\alpha_4+i\beta_4), & 1 \end{vmatrix}=0,$$

which is the equation of the circle through the two pairs of antifocal points.

{NOTE. The *second* form of the equation of the circle may be otherwise deduced from the *first*, without expanding the determinants, by the following method:

$$\begin{vmatrix} XY, & Y, & X, & 1 \\ A_1, & 1, & A_1A_2, & A_2 \\ A_2, & 1, & A_1A_2, & A_1 \\ A_3, & 1, & A_3A_4, & A_4 \end{vmatrix} = \begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ A_1+A_2, & 1, & A_1A_2, & A_1 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} =$$

$$\begin{vmatrix} XY+1, & Y, & X, & 1 \\ A_1+A_2, & 1, & A_1A_2, & A_2 \\ 0, & 0, & 0, & A_1-A_2 \\ A_3+A_4, & 1, & A_3A_4, & A_4 \end{vmatrix} = (A_1-A_2)\begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix};$$

therefore

$$\begin{vmatrix} XY+1, & X, & Y \\ A_1+A_2, & A_1A_2, & 1 \\ A_3+A_4, & A_3A_4, & 1 \end{vmatrix}=0.$$

ED. [W. J. M.]}

[Vol. II. pp. 22—24.]

1513. (Proposed by the Rev. J. BLISSARD, B.A.)—Prove the following formulæ:

(1) $$\frac{(x-1)(x-2)\,..\,(x-n)}{x(x+1)\,..\,(x+n-1)}=$$

$$1+(-)^n\left\{n\cdot\frac{1}{x}-\frac{n(n^2-1^2)}{1^2}\cdot\frac{1}{x+1}+\frac{n(n^2-1^2)(n^2-2^2)}{1^2.\,2^2}\cdot\frac{1}{x+2}-\text{\&c.}\right\}$$

(2) The above formula expressed as

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1\,.\,2\,.\,3}\cdot\frac{1}{x(x+1)(x+2)}+\&\text{c}.$$

and show that this equation is subject to the sole restriction that when n is not integral x must not be negative.

Solution by PROFESSOR CAYLEY; *and* X. U. J.

Let n be a positive integer, and suppose that $[x]^n$ denotes as usual the factorial $x(x-1)\ldots.(x-n+1)$; then we have

$$[x+k]^n=(1+\Delta)^k[x]^n=\left(1+k\Delta+\frac{k(k-1)}{1\,.\,2}\Delta^2+\&\text{c}.\right)[x]^n$$

$$=[x]^n+\frac{kn}{1}[x]^{n-1}+\frac{k(k-1)\,n\,(n-1)}{1\,.\,2}[x]^{n-2}+\&\text{c}.;$$

or putting $k=-n$ we have

$$[x-n]^n=[x]^n-\frac{n^2}{1}[x]^{n-1}+\frac{n^2(n^2-1^2)}{1\,.\,2}[x]^{n-2}-\&\text{c}.$$

Writing herein $(x+n-1)$ for x, and dividing by $[x+n-1]^n$, we have

$$\frac{[x-1]^n}{[x+n-1]^n}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\&\text{c}.;$$

or, what is the same thing,

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)}=1-\frac{n^2}{1}\cdot\frac{1}{x}+\frac{n^2(n^2-1^2)}{1\,.\,2}\cdot\frac{1}{x(x+1)}-\&\text{c}.,$$

which is the formula (2). The foregoing demonstration applies to the case of n a positive integer; but as the two sides are respectively unaltered when n is changed into $-n$, it is clear that the formula holds good also for n a negative integer. The right hand side is the hypergeometric series $F(n,\,-n,\,x,\,1)$ and the formula therefore is

$$\frac{(\Gamma x)^2}{\Gamma(x-n)\,\Gamma(x+n)}=F(n,\,-n,\,x,\,1),$$

a particular case of the known formula

$$\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}=F(\alpha,\,\beta,\,\gamma,\,1),$$

which when α or β is a positive integer is a mere identity, true therefore for all values of γ; but if neither α nor β is a positive integer, then the right hand side is an infinite series which is only convergent for $\gamma>\alpha+\beta$. In the particular case we have $\alpha=n$, $\beta=-n$, $\gamma=x$; hence if n be a positive or negative integer, the formula is an identity, but if n be fractional, the condition of convergency is $x>0$, that is, x must be positive.

To prove the formula (1) it is only necessary to remark, that (n being a positive integer) the quantity $\dfrac{[x-1]^n}{[x+n-1]^n}$ is a rational fraction, the numerator and denominator whereof are of the same degree n, and which becomes $=1$ for $x=\infty$. Hence, decomposing it into simple fractions, we may write

$$\frac{[x-1]^n}{[x+n-1]^n}=1+S_r\,.\,\frac{A_r}{x+r}$$

where the summation extends from $r=0$ to $r=n-1$ both inclusive. And we have

$$A_r=\left\{\frac{(x+r)\,[x-1]^n}{[x+n-1]^n}\right\}_{x=-r},$$

or, observing that $[x+n-1]^n=[x+n-1]^{n-r-1}\,(x+r)\,[x+r-1]^r$, we have

$$A_r=\left\{\frac{[x-1]^n}{[x+n-1]^{n-r-1}\,[x+r-1]^r}\right\}_{x=-r}=\frac{[-r-1]^n}{[n-r-1]^{n-r-1}\,[-1]^r}$$

$$=\frac{(-)^n\,[n+r]^n}{[n-r-1]^{n-r-1}\,(-)^r\,[r]^r}=(-)^{n+r}\,\frac{[n+r]^{n+r}}{[n-r-1]^{n-r-1}\,[r]^r\,[r]^r}=(-)^{n+r}\,\frac{[n+r]^{2r+1}}{[r]^r\,.\,[r]^r}.$$

Hence the formula is

$$\frac{[x-1]^n}{[x+n-1]^n}=1+(-)^n\,.\,S_r\,(-)^r\,.\,\frac{[n+r]^{2r+1}}{[r]^r\,[r]^r}\,\frac{1}{x+r},$$

or, as this may also be written,

$$\frac{(x-1)\,(x-2)\,..\,(x-n)}{x\,(x+1)\,..\,(x+n-1)}=1+(-)^n\left\{n\,.\,\frac{1}{x}-\frac{n\,(n^2-1^2)}{1^2}\,.\,\frac{1}{x+1}+\frac{n\,(n^2-1^2)\,(n^2-2^2)}{1^2\,.\,2^2}\,.\,\frac{1}{x+2}-\&\text{c.}\right\}$$

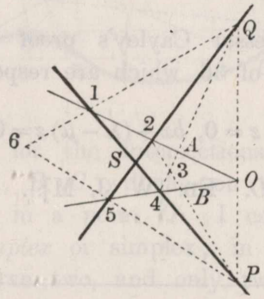
which is the formula in question.

[Vol. II. pp. 51, 52.]

1512. (Proposed by Professor CAYLEY.)—It is possible to construct a hexagon 123456, inscribed in a conic, and such that the diagonals 14, 25, 36 pass respectively through the Pascalian points (intersections of opposite sides) 23, 56; 34, 61; 45, 12. Given the points 1, 2; 4, 5; to construct the hexagon.

Solution by the PROPOSER.

Let 12, 45, meet in O, and through O draw at pleasure a line meeting 14 in P, and 25 in Q; let $P2$, $Q4$ meet in 3, and $P5$, $Q1$ in 6; then the line 36 will pass through O, and this being so, the hexagon 123456 satisfies the required conditions.



We have to show that 36 passes through O. Let $Q4$ meet $O12$ in A, and $P2$ meet $O45$ in B; then the points 6, 3, O, are the intersections of corresponding sides of the triangles $A1Q$, $B5P$; and in order that these points may lie in a line, the lines joining the corresponding vertices must meet in a point, that is, we have to show that the lines 15, AB, PQ meet in a point. The property is in fact as follows; viz., given the points 2, 4; and also the points Q, O, P lying in a line; then constructing the points 1, 5, A, B, which are the respective intersections of $P4$, $O2$; $Q2$, $O4$; $Q4$, $O2$; $P2$, $O4$; the lines 15, AB, PQ will meet in a point. Take $x=0$, $y=0$, $z=0$ for the respective equations of $P2$, $Q4$, PQ; then O is an arbitrary point in the line PQ, say that for the point O we have $z=0$, $ax+by=0$; also $O2$, $O4$ are arbitrary lines through O: say that their equations are $ax+by+\lambda z=0$; $ax+by+\mu z=0$; then we have for the points A and B, respectively, $ax+by+\mu z=0$, $y=0$; $ax+by+\mu z=0$, $x=0$; hence the equation of AB is $\mu ax+\lambda by+\lambda\mu z=0$. The equation of $P4$ is $ax+\mu z=0$, and that of $Q2$ is $by+\lambda z=0$; the point 1 is therefore given by $ax+\mu z=0$, $ax+by+\lambda z=0$; and 5 by $by+\lambda z=0$, $ax+by+\mu z=0$; hence the equation of 15 is $\mu ax+\lambda by+(\mu^2-\mu\lambda+\lambda^2)z=0$; and the equation of PQ being $z=0$, it is clear that the three lines AB, 15, PQ intersect in the point given by the equations $\mu ax+\lambda by=0$, $z=0$.

OBS. 1. By inspection of the figure we see that $3PQ$ is a triangle whereof the sides $3P$, $3Q$, PQ pass respectively through the fixed points 2, 4, O; while the vertices P and Q lie in the fixed lines 14, 25; the locus of the vertex 3 is consequently a conic; and the like as regards the triangle $6PQ$.

OBS. 2. The regular hexagon projects into a hexagon inscribed in a conic and circumscribed about another conic having double contact therewith; in the hexagon so obtained (as appears at once by the consideration of the regular hexagon) the above-mentioned property holds; but the in-and-circumscribed hexagon has the additional property that the three diagonals meet in a point, and it is therefore a less general figure than the hexagon of the foregoing theorem. It would, I think, be worth while to study further the hexagon of the theorem.

{NOTE. In the solution of Question 1548 it is shown that if two pairs of opposite sides of *any hexagon* intersect each on a diagonal produced, so likewise will the third pair.

A slight variation of Professor Cayley's proof may be obtained by finding the equations of $P5$, $Q1$, and thence of 36, which are respectively

$$ax-(\lambda-\mu)z=0,\ bx+(\lambda-\mu)z=0,\ ax+by=0,$$

showing that 36 passes through O. ED. [W. J. M].}

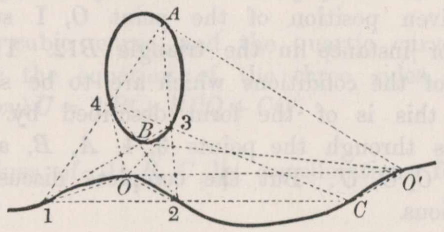
[Vol. II. pp. 70—72.]

1562. (Proposed by F. D. THOMSON, M.A.)—Find the locus of the points of contact of tangents drawn from a given point to a conic circumscribing a given quadrangle. The quadrangle being supposed convex, trace the changes of form of the locus for different positions of the given point.

Solution by PROFESSOR CAYLEY; *and the* PROPOSER.

Let O be the given point; 1, 2, 3, 4 the vertices of the given quadrangle; A, B, C the centres of the quadrangle, viz., A the intersection of the lines 14, 23; B of 24, 31; C of 34, 12. The polars of O in regard to the several circumscribed conics intersect in a point O'. This being so, the locus is a cubic passing through the nine points 1, 2, 3, 4, A, B, C, O, O', and which is moreover such that the tangents at the four points 1, 2, 3, 4 meet the cubic in the point O, and the tangents at the four points A, B, C, O meet the cubic in the point O'. It is to be remarked that the nine points are so related to each other that a cubic through any eight of these points passes through the remaining ninth point; say a cubic through 1, 2, 3, 4, A, B, C, O passes through O'; the nine points consequently do not determine the cubic; but the cubic will be determined, e.g., by the conditions that it passes through 1, 2, 3, 4, A, B, C, O, and has $O1$ for the tangent at 1. The series of cubics corresponding to different positions of the point O is identical with the series of cubics passing through the seven points 1, 2, 3, 4, A, B, C

Conversely any given cubic curve may be taken to be a cubic of the series; and the points 1, 2, 3, 4 will then be determined as follows, viz., 1, 2, 3, 4 are the points of contact of the tangents to the cubic from an arbitrary point O on the cubic; and



then taking as before A, B, C for the intersections of 14, 23, of 24, 31 and of 34, 12, respectively, the points A, B, C will lie on the cubic, and the tangents at A, B, C, O will meet the cubic in a point O'. I call to mind that a cubic curve without singularities is either *complex* or simplex; in the simplex kind there can be drawn from any point of the curve two, and only two, real tangents to the curve; in the complex kind, there can be drawn four real tangents or else no real tangent, viz. from any point on a certain branch of the curve there can be drawn four real tangents, from a point on the remaining portion of the curve no real tangent. Hence, in the foregoing construction, in order that the points 1, 2, 3, 4 may be real, the given cubic must be of the complex kind, and the point O must be taken on the branch which has through each of its points four real tangents.

The foregoing results may be established *geometrically* or *analytically;* but for brevity I merely indicate the analytical demonstration. Suppose first, that the points 1, 2, 3, 4 are given as the intersections of the conics $U=0$, $V=0$; let α, β, γ be the coordinates of the point O, and write $D=\alpha\delta_x+\beta\delta_y+\gamma\delta_z$, so that $DU=0$ and $DV=0$ are the equations of the polars of O in regard to the conics $U=0$, $V=0$ respectively. The equation of any conic through the four points is $U+kV=0$; and the equation of the polar of O in regard thereto is $DU+kDV=0$; eliminating k from these equations, we have $UDV-VDU=0$, which is the equation of the given locus. We see at once that it is a cubic curve passing through the points $(U=0,\ V=0)$, that is, the points 1, 2, 3, 4; and through the point $DU=0$, $DV=0$, that is, the point O'; it also follows without difficulty that the curve passes through the point O. But for the remaining results it is better to particularize the conics $U=0$, $V=0$. Let the equations of 12, 23, 34, 41 be $x=0$, $y=0$, $z=0$, $w=0$ respectively, (where $x+y+z+w=0$); and in the same system, let α, β, γ, δ be the coordinates of O $(\alpha+\beta+\gamma+\delta=0)$, then $xz=0$, $yw=0$ are each of them a conic (pair of lines) passing through the four points; and we may therefore write $U=yw$, $V=xz$; the equation $UDV-VDU=0$ thus becomes $yw(\alpha z+\gamma x)-xz(\beta w+\delta y)=0$, or, as this equation may also be written,

$$\frac{\alpha}{x}-\frac{\beta}{y}+\frac{\gamma}{z}-\frac{\delta}{w}=0,$$

which is the equation of the cubic curve; and from this form the several above-mentioned results may be obtained without difficulty.

To give an idea of the form of the curve corresponding to a given convex quadrangle 1234, and given position of the point O, I suppose that O is situate *within* the quadrangle, for instance in the triangle $B12$. The mere inspection of the figure, and consideration of the conditions which are to be satisfied by the cubic curve, is enough to show that this is of the form described by Newton as *anguinea cum ovali*, viz., the oval passes through the points 3, 4, A, B, and the serpentine branch through the points 1, 2, C, O, O'. But the complete discussion of the different cases would be somewhat laborious.

{A *geometrical* investigation of the locus is given on p. 124 of Cremona's *Teoria Geometrica delle Curve Piane*. ED. [W. J. M.].}

[Vol. II. pp. 89, 90.]

1533. (Proposed by Professor CAYLEY.)—If on the sides of a triangle there are taken three points, one on each side; and if through the three points and the three vertices of the triangle there are drawn a cubic curve and a quartic curve, intersecting in six other points; then there exists a quintic curve passing through each of the three points, and having each of the six points for a double point.

Solution by the PROPOSER.

Let $P=0$ be the equation of the quartic curve, $Q=0$ the equation of the cubic curve, $M=0$ the equation of the three sides of the triangle; then if we can find A, B, C functions of the orders 0, 1, 2 respectively, and U a function of the fifth order, such that we have identically $MU = AP^2 + BPQ + CQ^2$; we have $MU=0$, a curve of the eighth order, having a double point at each of the points $(P=0,\ Q=0)$, which points are the three vertices of the triangle, the three points, and the six points; but the curve $MU=0$ is made up of the curve $M=0$ (the three sides of the triangle, being a cubic curve having each of the vertices for a double point, and passing through each of the three points) and of a certain quintic curve $U=0$; hence the quintic curve must pass through each of the three points, and have a double point at each of the six points; or there exists a quintic curve satisfying the conditions of the theorem.

I take $x=0$, $y=0$, $z=0$ for the equations of the three sides of the triangle, and then (the constants being all of them arbitrary) writing for shortness

$$\begin{array}{lll} \xi = \ .\quad by + cz, & X = \ .\quad \beta y + \gamma z, & \Theta = \lambda x + \mu y + \nu z, \\ \eta = a'x \quad .\ + c'z, & Y = \alpha' x \quad .\ + \gamma' z, & \\ \zeta = a''x + b''y \quad .\ , & Z = \alpha'' x + \beta'' y \quad .\ , & \end{array}$$

I assume that the three points are given by the equations $(x=0, \xi=0)$, $(y=0, \eta=0)$, $(z=0, \zeta=0)$, respectively. This being so, we may write

$$Q = yz\xi\delta + zx\eta\delta' + xy\zeta\delta'' + xyz\epsilon = 0, \quad -P = yz\xi X + zx\eta Y + xy\zeta Z + xyz\Theta = 0,$$

for the equations of the cubic curve and the quartic curve respectively. We have of course $M = xyz = 0$ for the equation of the three sides of the triangle, and the identity to be satisfied is $xyzU = AP^2 + BPQ + CQ^2$.

I was led to the values of A, B, C by considerations founded on the theory of curves in space. We have

$$A = \delta\delta'\delta'', \quad B = (\delta'\alpha'' + \delta''\alpha)\,\delta x + (\delta''\beta + \delta\beta'')\,\delta' y + (\delta\gamma' + \delta'\gamma)\,\delta'' z,$$

$$C = \alpha'\alpha''\delta x^2 + \beta''\beta\delta' y^2 + \gamma\gamma'\delta'' z^2 + (\gamma\beta''\delta' + \gamma'\beta\delta'')\,yz + (\alpha'\gamma\delta'' + \alpha''\gamma'\delta)\,zx + (\beta''\alpha'\delta + \beta\alpha''\delta')\,xy;$$

and with these values it is easy to show that the function $AP^2 + BPQ + CQ^2$ contains the factor xyz; for substituting the values of P, Q, all the terms of $AP^2 + BPQ + CQ^2$ contain explicitly the factor xyz, except the terms

$$A\,(y^2z^2\xi^2X^2 + z^2x^2\eta^2Y^2 + x^2y^2\zeta^2Z^2) - B\,(y^2z^2\xi^2X\delta + z^2x^2\eta^2Y\delta' + x^2y^2\zeta^2Z\delta')$$
$$+ C\,(y^2z^2\xi^2\delta^2 + z^2x^2\eta^2\delta'^2 + x^2y^2\zeta^2\delta''^2);$$

and these terms will contain the factor xyz, if only the expressions $AX^2 - BX\delta + C\delta^2$, $AY^2 - BY\delta' + C\delta'^2$, $AZ^2 - BZ\delta'' + C\delta''^2$ contain respectively the factors x, y, z. But $AX^2 - BX\delta + C\delta^2$ will contain the factor x, if only the expression vanishes for $x=0$; and for $x=0$ we have

$$AX^2 - BX\delta + C\delta^2 = 0 =$$
$$\delta\delta'\delta''(\beta y + \gamma z)^2 - [\delta'\delta''(\beta y + \gamma z) + \delta(\beta''\delta' y + \gamma'\delta'' z)]\,\delta(\beta y + \gamma z) + (\beta y + \gamma z)(\beta''\delta' y + \gamma'\delta'' z)\,\delta^2;$$

that is, $AX^2 - BX\delta + C\delta^2$ contains the factor x; and by symmetry the other two expressions contain the factors y and z respectively. The excepted terms contain therefore the factor xyz; and there exists therefore a quintic function $U = (AP^2 + BPQ + CQ^2) \div xyz$; which proves the theorem.

The values of A, B, C were obtained by considering the surface $w = \frac{P}{Q}$, which, as is at once seen, contains upon itself the three lines

$$\left(x=0,\ w=-\frac{X}{\delta}\right),\ \left(y=0,\ w=-\frac{Y}{\delta'}\right),\ \left(z=0,\ w=-\frac{Z}{\delta''}\right)$$

or as these equations may be written

$$\begin{array}{ll}(x=0, & \quad .\quad\ \ \beta y + \gamma z + \delta w\ = 0),\\ (y=0, & \alpha' x\quad .\quad + \gamma' z + \delta' w = 0),\\ (z=0, & \alpha'' x + \beta'' y\ \ .\ + \delta'' w = 0);\end{array}$$

and then seeking for the equation of the hyperboloid which passes through the three lines, this is found to be $Aw^2 + Bw + C = 0$, where A, B C have the before-mentioned values.

If in the foregoing theorem the cubic is considered as a given cubic curve, and the three points as three arbitrary points on the cubic, the question then arises to find the triangle; or we have the problem proposed as Question 1607.

[Vol. II. p. 91.]

1542. (Proposed by Professor CAYLEY.)—If a given line meet two given conics in the points (A, B) and (A', B') respectively; and if (A'', B'') be the sibi-conjugate points (or foci) of the pairs (A, A') and (B, B'), or of the pairs (A, B') and (A', B), then (A'', B'') lie on a conic passing through the four points of intersection of the two given conics.

[Vol. II. pp. 97—100.]

1606. (Proposed by the EDITOR, [W. J. M.]).—Solve the following problems:

(α) Through three given points to draw a conic whose foci shall lie in two given lines.

(β) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

(γ) Through two given points to draw a circle such that its chords of intersection with a given circle shall pass through a given point.

Solution by PROFESSOR CAYLEY.

(α) Through three given points to draw a conic whose foci shall lie in two given lines.

The focus of a conic is a point such that the lines joining it with the two circular points at infinity (say the points I, J) are tangents to the conic. Hence the question is, in a given line to find a point A, and in another given line to find a point B, such that there exists a conic touching the four lines AI, AJ, BI, BJ (where I, J are any given points) and besides passing through three given points. More generally, instead of the lines from A, B to the given points I, J, we may consider the tangents from A, B, to a given conic Θ; the question then is, in a given line to find a point A, and in another given line to find a point B, such that there exists a conic touching the tangents from A, B to a given conic Θ, and besides passing through three given points. It is rather more convenient to consider the reciprocal question—though it is to be borne in mind that for any two reciprocal questions a solution of the one question by means of coordinates (x, y, z) regarded as point-coordinates is in fact a solution of the other question by means of the same coordinates (x, y, z) regarded as line-coordinates. The reciprocal question is: through a given point to draw a line A, and through another given point to draw a line B, such that there exists a conic passing through the intersections of these lines with a given conic Θ, and besides touching three given lines. The given points may be taken to be $(x=0,\ z=0)$, $(y=0,\ z=0)$; this determines the line $z=0$, but not the lines $x=0$, $y=0$, so that the point $(x=0,\ y=0)$ may without loss of generality be supposed to lie on the conic Θ; the equation of this conic will therefore be

$$(a,\ b,\ 0,\ f,\ g,\ h\between x,\ y,\ z)^2 = 0.$$

I take $\alpha z + \gamma x = 0$ for the equation of the line A, $\mu y + \nu z = 0$ for the equation of the line B (so that the quantities to be determined are the ratios $\alpha : \gamma$ and $\mu : \nu$); this being so, the required conic passes through the intersections of these lines with the conic Θ; its equation will therefore be

$$(a,\ b,\ 0,\ f,\ g,\ h\between x,\ y,\ z)^2 + 2(\alpha x + \gamma z)(\mu y + \nu z) = 0\,;$$

or what is the same thing

$$(a,\ b,\ 2\nu\gamma,\ f+\mu\gamma,\ g+\nu\alpha,\ h+\mu\alpha\between x,\ y,\ z)^2 = 0\,;$$

where α, γ, μ, ν have to be determined in such manner that this conic may touch three given lines. It is to be observed that α, γ, μ, ν, enter into the equation through the combinations $\alpha\mu$, $\alpha : \gamma$, and $\mu : \nu$, so that there are really only three disposable quantities.

The condition in order that the conic may touch a line $\xi x + \eta y + \zeta z = 0$ is

$$\left(\begin{matrix} 2b\nu\gamma - (f+\mu\gamma)^2,\ 2a\nu\gamma - (g+\nu\alpha)^2,\ ab - (h+\mu\alpha)^2, \\ (g+\nu\alpha)(h+\mu\alpha) - \quad a(f+\mu\gamma), \\ (h-\mu\alpha)(f+\mu\gamma) - \quad b(g+\nu\alpha), \\ (f+\mu\gamma)(g+\nu\alpha) - 2\nu\gamma(h+\mu\alpha) \end{matrix}\right)\!\between \xi,\ \eta,\ \zeta)^2 = 0,$$

that is, putting for shortness $C = ab - h^2$, $F = gh - af$, $G = hf - bg$, and reversing the sign of the whole expression,

$$\begin{aligned} &\{\ f^2\xi^2 + g^2\eta^2 - C\zeta^2 - 2F\eta\zeta - 2G\zeta\xi - 2fg\xi\eta\} \\ +\ 2\mu\,&\{\ f\gamma\xi^2 + h\alpha\zeta^2 + (a\gamma - g\alpha)\,\eta\zeta - (h\gamma + f\alpha)\,\zeta\xi - g\gamma\xi\eta\} \\ +\ 2\nu\,&\{-b\gamma\xi^2 - (a\gamma - g\alpha)\,\eta^2 - h\alpha\eta\zeta + b\alpha\zeta\xi + (h\gamma - \alpha f)\,\xi\eta\} \\ +\ \ &\mu^2\,\{(\gamma\xi - \alpha\zeta)^2\} + 2\mu\nu\,\{\alpha\eta\,(\gamma\xi - \alpha\zeta)\} + \nu^2\,\{\alpha^2\eta^2\} = 0\,; \end{aligned}$$

or what is the same thing

$$\{\nu\alpha\eta+\mu(\gamma\xi-\alpha\zeta)\}^2+2\nu(p\alpha+q\gamma)+2\mu(r\alpha+s\gamma)+t=0;$$

where p, q, r, s, t are given functions of (ξ, η, ζ).

I write for greater convenience

$$\nu=\frac{1}{X},\quad \mu=\frac{1}{Y},\quad \alpha=W,\ \gamma=Z,$$

(so that the quantities to be determined will be the ratios $X : Y : Z : W$); the foregoing equation then becomes

$$\left\{\eta\frac{W}{X}+\frac{1}{Y}(\xi Z-\zeta W)\right\}^2+\frac{2}{X}(pW+qZ)+\frac{2}{Y}(rW+sZ)+t=0,$$

or what is the same thing

$$\{\eta YW+X(\xi Z-\zeta W)\}^2+2XY^2(pW+qZ)+2X^2Y(rW+sZ)+tX^2Y^2=0.$$

Hence, considering in place of the line $\xi x+\eta y+\zeta z=0$, the three given lines $\xi_1x+\eta_1y+\zeta_1z=0$, $\xi_2x+\eta_2y+\zeta_2z=0$, $\xi_3x+\eta_3y+\zeta_3z=0$ successively, we have the three equations

$$\begin{array}{ll}\{\eta_1YW+X(\xi_1Z-\zeta_1W)\}^2+2XY^2(p_1W+q_1Z)+2X^2Y(r_1W+s_1Z)+t_1X^2Y^2 & =0,\\ \{\eta_2YW+\&c.\qquad\qquad\}^2+\&c. & =0,\\ \{\eta_3YW+\&c.\qquad\qquad\}^2+\&c. & =0;\end{array}$$

which, treating X, Y, Z, W as the coordinates of a point in space, are each of them the equation of a quartic surface having the line $(X=0, Y=0)$ for a cuspidal line. The required values of X, Y, Z, W are the coordinates of a point of intersection of the three surfaces, and these being found the equation of the conic satisfying the conditions of the question is

$$(a, b, 0, f, g, h)(x, y, z)^2+2(Wx+Zz)\left(\frac{y}{Y}+\frac{z}{Z}\right)=0.$$

As to the intersection of surfaces having a common line, see Salmon's *Solid Geometry*, p. 257; but the case of a cuspidal line not having been hitherto discussed, I am not able to say now how many points of intersection there are of the three surfaces, nor consequently what is the number of the solutions of the question in hand. It of course appears that 64 is a superior limit.

(β) Through four given points to draw a conic such that one of its chords of intersection with a given conic shall pass through a given point.

Let the four points be given as the intersections of the conics $U=0$, $V=0$, and let $W=0$ be the equation of the given conic, (α, β, γ) the coordinates of the given point.

The equation of the required conic may be taken to be $\Theta = \lambda U + \mu V = 0$, and this being so, the equation of *any* conic passing through the points of intersection of the conic $\Theta = 0$ and the given conic $W = 0$, will be $\lambda U + \mu V + \nu W = 0$; and if ν be properly determined, viz. by the equation

$$\text{Disct. } (\lambda U + \mu V + \nu W) = 0,$$

which it will be observed is a cubic equation in (λ, μ, ν), then $\lambda U + \mu V + \nu W = 0$ will be the equation of a pair of the chords of intersection of the conics $\Theta = 0$, $W = 0$. The chord which passes through the given point (α, β, γ) may be taken to be one of the pair of chords; the pair of chords, regarded as a conic, then passes through the given point (α, β, γ); or if U_0, V_0, W_0 are what U, V, W become on substituting therein the values (α, β, γ) for the coordinates, we have

$$\lambda U_0 + \mu V_0 + \nu W_0 = 0,$$

which is a linear equation in (λ, μ, ν); and combining it with the before-mentioned cubic equation,

$$\text{Disct. } (\lambda U + \mu V + \nu W) = 0,$$

the two equations give the ratios $(\lambda : \mu : \nu)$, and the equation of the required conic is $\lambda U + \mu V = 0$. There are three systems of ratios $\lambda : \mu : \nu$, and consequently three conics satisfying the conditions of the Question.

Suppose that the conics $U = 0$, $V = 0$, $W = 0$, have a common chord, then the conics $\Theta = \lambda U + \mu V = 0$, $W = 0$, have this common chord, say the fixed chord; and they have besides another chord of intersection, say the proper chord, which is the line joining the two points of intersection not on the fixed chord. It follows that, in the equation $\lambda U + \mu V + \nu W = 0$, ν may be so determined that this equation shall represent the fixed and proper chords; the required value of ν is one of those given by the before-mentioned cubic equation, which will then have a single rational factor of the form $a\lambda + b\mu + c\nu$, and the value of ν is that obtained by means of this factor, that is, by the equation $a\lambda + b\mu + c\nu = 0$.

{The value in question may, however, be found independently of the cubic equation; thus, if the three conics have the common chord $P = 0$, then their equations may be taken to be $U = 0$, $U + PQ = 0$, $U + PR = 0$; we have then $\Theta = \lambda U + \mu (U + PQ)$, and the value of ν is at once seen to be $\nu = -(\lambda + \mu)$, for then

$$\lambda U + \mu V + \nu W = \lambda U + \mu (U + PQ) - (\lambda + \mu)(U + PR) = 0,$$

that is, $P[\mu Q - (\lambda + \mu) R] = 0$, which is the conic made up of the fixed chord $P = 0$ and the proper chord $\mu Q - (\lambda + \mu) R = 0$.}

But returning to the equations $U = 0$, $V = 0$, $W = 0$, the value of ν is given by the equation $a\lambda + b\mu + c\nu = 0$, obtained by equating to zero the rational factor of the cubic equation. Suppose now that the *proper chord* passes through the given point (α, β, γ), then, as before, the equation $\lambda U + \mu V + \nu W = 0$ is satisfied by these values of the coordinates, or we have $\lambda U_0 + \mu V_0 + \nu W_0 = 0$; which, with the before-mentioned linear equation $a\lambda + b\mu + c\nu = 0$, determines the ratios $\lambda : \mu : \nu$; and the required conic is $\lambda U + \mu V = 0$; there is, then, in the present case only one conic satisfying the conditions of the Question.

(γ) Through two given points to draw a circle such that its chord of intersection with a given circle shall pass through a given point.

The foregoing discussion of the case of three conics having a common chord is of course directly applicable to the present question, the common chord being the line infinity; it is therefore sufficient to give the final analytical result; viz., if the given points are $y = 0$, $x = \pm 1$, and if the given circle is $x^2 + y^2 + c + 2fy + 2gx = 0$, and the point through which passes the chord is $x = \alpha$, $y = \beta$, then the equation of the required circle is

$$x^2 + y^2 - 1 + \frac{1}{\beta}(2g\alpha + 2f\beta + 1 + c)\, y = 0.$$

The equation of the chord of intersection is, in fact,

$$1 + c - \frac{1}{\beta}(2g\alpha + 2f\beta + 1 + c)\, y + 2gx + 2fy = 0,$$

which is satisfied, as it should be, by $x = \alpha$, $y = \beta$.

But the geometrical solution is even more simple. Let A, B, be the given points, C the point through which passes the chord of intersection; then, joining C, A, and taking on this line a point H such that $CA \, . \, CH$ is equal to the square on the tangential distance of C from the given circle, it is at once seen that *any* circle through A, H is such that its chord of intersection with the given circle passes through C; hence the required circle is that drawn through the three points A, H, B.

[Vol. III. January to July, 1865, p. 29.]

1607. (Proposed by Professor CAYLEY.)—In a given cubic curve to inscribe a triangle such that the three sides shall pass respectively through three given points on the curve.

[Vol. III. pp. 60—63.]

1647. (Proposed by Professor CAYLEY.)—Find the locus of the foci of an ellipse of given major axis, passing through three given points.

{In connexion with the problem the Proposer remarks as follows:

Let A, B, C be the given points; take P an arbitrary point (not in general in the plane of the three given points), then we may find a point Q (not in general in the plane of the three given points) such that $QA+AP=QB+BP=QC+CP=$ given major axis. And this being so, if the locus of P be a given surface, then we shall have a certain surface, the locus of Q; and so if the locus of P be a given curve in space, then we shall have a given curve in space, the locus of Q. In particular, if the locus of P be the plane of the three given points, then the locus of Q will be a certain surface, cutting the plane in a curve which is the locus in the foregoing problem; and when Q is situate on this curve, then also P will be situate on the same curve. Or if the locus of P be the curve in question, then the locus of Q will be the same curve. Say, in general, that the loci of P and Q are reciprocal loci, then the curve in the problem *is its own reciprocal.* And we may propose the following question:

Find the curve or surface, the locus of P, which is its own reciprocal.

We have also analogous to the original problem the following question in Solid Geometry:

Given the four points A, B, C, D in space, to find the locus of the points P, Q such that

$$PA+AQ=PB+BQ=PC+CQ=PD+DQ=\text{ a given line.}\}$$

Solution by the PROPOSER.

In general if a, b, c be the sides of a triangle, and f, g, h the distances of any point from the angles of the triangle (or, what is the same thing, if (a, b, c, f, g, h) be the distances of any four points in a plane from each other), then we have a certain relation

$$\phi(a, b, c, f, g, h)=0.$$

Hence if r, s, t be the distances of the one focus from the angles of the triangle, and the major axis is $=2\lambda$, then the distances for the other focus are $2\lambda-r$, $2\lambda-s$, $2\lambda-t$; and considering the three angles and the other focus as a system of four points, we have

$$\phi(a, b, c, 2\lambda-r, 2\lambda-s, 2\lambda-t)=0,$$

which is a relation between the distances r, s, t of the first focus from the angles of the triangle, and which, treating these distances as coordinates (of course in a generalised sense of the term "Coordinate"), may be regarded as the equation of the required locus. It is to be observed, that we have identically

$$\phi(a, b, c, r, s, t)=0,$$

and the equation may be expressed in the simplified form

$$\phi(a, b, c, 2\lambda-r, 2\lambda-s, 2\lambda-t)-\phi(a, b, c, r, s, t)=0.$$

To develope the solution, I notice that the expression for the equation $\phi(a, b, c, f, g, h)=0$ is

$$\begin{aligned}
b^2c^2(g^2+h^2)+c^2a^2(h^2+f^2)+a^2b^2(f^2+g^2)&\\
+g^2h^2(b^2+c^2)+h^2f^2(c^2+a^2)+f^2g^2(a^2+b^2)&\\
-a^2f^2(a^2+f^2)-b^2g^2(b^2+g^2)-c^2h^2(c^2+h^2)&\\
-a^2g^2h^2-b^2h^2f^2-c^2f^2g^2-a^2b^2c^2=0\,;&
\end{aligned}$$

see my paper, "Note on the value of certain determinants &c.," *Quart. Math. Journ.* vol. III. (1860), pp. 275—277, [286]. Or, as this may also be written

$$\Sigma\{(b^2+c^2-a^2)(g^2h^2+a^2f^2)-a^2f^4\}-a^2b^2c^2=0,$$

where Σ refers to the simultaneous cyclical permutation of (a, b, c) and of (f, g, h). Hence we have only in this equation to write $2\lambda-r$, $2\lambda-s$, $2\lambda-t$ in place of (f, g, h), and to omit the terms independent of λ, being in fact those which are equal to $\phi(a, b, c, r, s, t)$. Observing that we have

$$\begin{aligned}
g^2h^2+a^2f^2&=\{4\lambda^2-2\lambda(s+t)+st\}^2+a^2(2\lambda-r)^2\\
&=16\lambda^4-16\lambda^3(s+t)+4\lambda^2(s^2+t^2+4st+a^2)-4\lambda[st(s+t)+a^2r]+s^2t^2+a^2r^2\,;\\
f^4&=(2\lambda-r)^4=16\lambda^4-32\lambda^3r+24\lambda^2r^2-8\lambda r^3+r^4,
\end{aligned}$$

the equation becomes

$$\begin{aligned}
&16\lambda^4\{\Sigma(b^2+c^2-a^2)-\Sigma a^2\}\\
&-16\lambda^3\{\Sigma(b^2+c^2-a^2)(s+t)-2\Sigma a^2r\}\\
&+\ 4\lambda^2\{\Sigma(b^2+c^2-a^2)(s^2+t^2+4st+a^2)-6\Sigma a^2r^2\}\\
&-\ 4\lambda\ \{\Sigma(b^2+c^2-a^2)[st(s+t)+a^2r]-2\Sigma a^2r^3\}=0,
\end{aligned}$$

where the Σ's refer to the simultaneous cyclical permutation of the (a, b, c) and the (r, s, t). The coefficients of λ^4 and λ^3 are, it is easy to see, each $=0$; and in the coefficient of λ^2 the terms $\Sigma(b^2+c^2-a^2)(s^2+t^2)-6\Sigma a^2r^2$ are $=-4\Sigma a^2r^2$; hence dividing the whole equation by 4λ, we find

$$\lambda\{\Sigma(b^2+c^2-a^2)(4st+a^2)-4\Sigma a^2r^2\}-\{\Sigma(b^2+c^2-a^2)[st(s+t)+a^2r]-2\Sigma a^2r^3\}=0,$$

which is the required relation between (r, s, t).

It may be noticed that, expressing the distances r, s, t in terms of Cartesian or trilinear coordinates (x, y) or (x, y, z), then r^2, s^2, t^2 are rational and integral functions of the coordinates, and the form of the equation therefore is

$$A_2+B_2r+C_2s+D_2t+E_0st+F_0tr+G_0rs=0,$$

where the subscript numbers denote the degrees in regard to the coordinates. Multiplying this equation successively by 1, r, s, t, st, tr, rs, rst, we have eight equations linear in the last-mentioned eight quantities, the coefficients being of known degrees respectively;

and eliminating the eight quantities, we have the rationalised equation expressed in the form, determinant (of order 8) = 0; viz. this is

$$\begin{vmatrix} A_2, & B_2, & C_2, & D_2, & E_0, & F_0, & G_0, & 0 \\ B_2r^2, & A_2, & G_0r^2, & F_0r^2, & 0, & D_2, & C_2, & E_0 \\ C_2s^2, & G_0s^2, & A_2, & E_0s^2, & D_2, & 0, & B_2, & F_0 \\ D_2t^2, & F_0t^2, & E_0t^2, & A_2, & C_2, & B_2, & 0, & G_0 \\ E_0s^2t^2, & 0, & D_2t^2, & C_2s^2, & A_2, & G_0s^2, & F_0t^2, & B_2 \\ F_0t^2r^2, & D_2t^2, & 0, & B_2r^2, & G_0r^2, & A_2, & E_0t^2, & C_2 \\ G_0r^2s^2, & C_2s^2, & B_2r^2, & 0, & F_0r^2, & E_0s^2, & A_2, & D_2 \\ 0, & E_0s^2t^2, & F_0t^2r^2, & G_0r^2s^2, & B_2r^2, & C_2s^2, & D_2t^2, & A_2 \end{vmatrix} = 0.$$

This seems to be of the degree 18 in the coordinates, but it is probable that the real degree is lower.

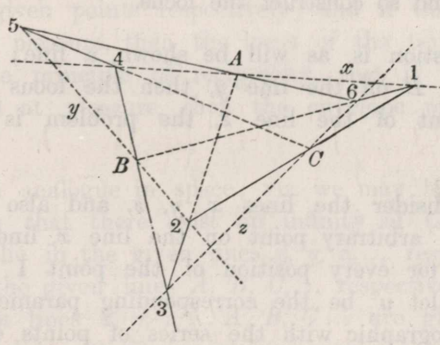
[Vol. III. pp. 63—65.]

1652. (Proposed by W. K. CLIFFORD.)—Through the angles A, B, C of a plane triangle three straight lines Aa, Bb, Cc are drawn. A straight line AR meets Cc in R; RB meets Aa in P; PC meets Bb in Q; QA meets Cc in r; and so on. Prove that, after going twice round the triangle in this way, we always come back to the same point.

Show that the theorem is its own reciprocal. Find the analogous properties of a skew quadrilateral in space, and of a polygon of n sides in a plane.

Solution by PROFESSOR CAYLEY.

1. The theorem may be thus stated: Given three lines x, y, z, and in these lines respectively the points A, B, C; then there exist an infinity of hexagons, such that



the pairs of opposite angles lie in the lines x, y, z, respectively, and that the pairs of opposite sides pass through the points A, B, C, respectively.

2. The demonstration is as follows: We have to show that, starting from an arbitrary point 1 in the line x, and constructing in the prescribed manner (as shown successively in the figure) the points 2, 3, 4, 5, 6, the last side 61 of the hexagon 123456 will pass through B. By the construction, we have A, 2, 3 in a line, and likewise C, 4, 5; hence, by Pascal's theorem, applied to the six points in a pair of lines, the points of intersection of the lines (25, 34), ($3C$, $A5$), ($A4$, $C2$), that is, the points B, 6, 1, lie in a line; which is the required theorem.

3. More generally suppose that the points A, B, C are not on the lines x, y, z, respectively. I remark that it is not in general possible to describe a hexagon such that the opposite angles lie in the lines x, y, z, respectively, and the opposite sides pass through the points A, B, C, respectively; but if there exists one hexagon (viz., a proper hexagon, not a triangle twice repeated), then there exists an infinity of such hexagons.

4. In fact, if it be required to find a polygon, the angles whereof lie in given lines respectively, and the sides whereof pass through given points respectively; the problem is either indeterminate or admits of only *two* solutions. If therefore in any particular case there are three or more solutions, the problem is indeterminate, and has an infinity of solutions. Now, in the above-mentioned case of the three lines and the three points, there exist *two* triangles, the angles whereof lie in the given lines, and the sides pass through the given points; and each triangle, taking the angles twice over in the same order 123123, is a hexagon satisfying the conditions of the problem; hence, if we have besides a proper hexagon satisfying the conditions of the problem, there are really *three* solutions, and the problem is therefore indeterminate.

5. Suppose that the three lines x, y, z, and also two of the three points, say the points A and B, are given; we may construct geometrically a locus, such that, taking for C any point of this locus, the problem shall be indeterminate: in fact, starting with the point 4, and constructing successively the points 3, 2; taking an arbitrary direction for the line 21, and constructing successively the points 1, 6, 5; then the intersection of the lines 21 and 54 is a position of the point C: and by taking any number of directions for the line 21, we obtain for each of them a different position of the point C; and so construct the locus.

6. The locus in question is, as will be shown, a line; and if the point A is on the line x, and the point B on the line y, then the locus of C will be the line z; that is, C being any point of the line z, the problem is indeterminate; which is Mr Clifford's theorem.

7. To prove this, consider the lines x, y, z, and also the points A, B, C, as given; the point 1 is an arbitrary point on the line x, linearly determined by means of a parameter u; and for every position of the point 1 we have a corresponding position of the point 4; let u' be the corresponding parameter for the point 4; the series of points 1 is homographic with the series of points 4; that is, the parameters u, u' are connected by an equation of the form $auu' + bu + cu + d = 0$, (where of course a, b, c, d are functions of the parameters which determine the given lines x, y, z and

points A, B, C). But if the problem be indeterminate, then starting from the point 1 and constructing the point 4, and again starting from the point 4 and making the very same construction, we arrive at the original point 1, that is, u must be the same function of u' that u' is of u; and this will be the case if $b=c$; hence $b=c$ is the condition in order that the problem may be indeterminate.

8. To effect the calculation, take $x=0$, $y=0$, $z=0$ for the equations of the lines x, y, z respectively; and let (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ be the coordinates of the points A, B, C respectively. Let 1 and 4 be given as the intersections of the line $x=0$ with the lines $y-uz=0$, $y-u'z=0$, respectively; and assume that for the point 2 we have $y=0$, $z-vx=0$, and for the point 3, $z=0$, $x-wy=0$. Then 1, C, 2 are in a line; as are also 2, A, 3; 3, B, 4; hence we obtain

$$v=\frac{\gamma''u-\beta''}{\alpha''u},\quad w=\frac{\alpha v-\gamma}{\beta v},\quad u'=\frac{\beta' w-\alpha}{\gamma' w};$$

therefore, eliminating v and w, we have

$$(\alpha\gamma''-\alpha''\gamma)\,\gamma' uu'-\alpha\beta''\gamma' u'-(\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma)\,u-\beta''(\alpha'\beta-\alpha\beta')=0.$$

The required condition, therefore, is

$$\alpha\beta''\gamma'=\alpha\beta'\gamma''-\alpha'\beta\gamma''-\alpha''\beta'\gamma,\quad\text{or}\quad \alpha\beta'\gamma''-\alpha\beta''\gamma'-\alpha'\beta\gamma''-\alpha''\beta'\gamma=0;$$

which is linear in regard to each of the three sets (α, β, γ), $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, separately; that is, two of the points A, B, C being given, the locus of the remaining point is a line. In particular, if $\alpha=0$, $\beta'=0$; then the equation becomes $\alpha'\beta\gamma''=0$, and assuming that neither $\alpha'=0$, or $\beta=0$, then the equation becomes $\gamma''=0$, that is, A, B being arbitrary points on the lines $x=0$, $y=0$ respectively, the locus of C is the line $z=0$.

9. Mr Clifford's theorem is clearly its own reciprocal. I do not know the *precise* analogues of his special form of the theorem; but the analogue of the more general theorem stated in (6) is as follows: viz., we may have in the plane n lines $x, y, z, \ldots$ and n points $A, B, C, \ldots$, such that there exist an infinity of $2n$-gons whereof the pairs of opposite angles lie in the given lines respectively; and the pairs of opposite sides pass through the given points respectively; and if the n lines and $n-1$ of the n points be assumed at pleasure, then the locus of the remaining point is a line. It is moreover clear by the principle of reciprocity, that if the n points and $n-1$ of the n lines be assumed at pleasure, then the envelope of the remaining line is a point.

There exists also an analogue in space; viz. we may have n lines $x, y, z, \ldots$ and n lines $A, B, C, \ldots$ such that there exist an infinity of (skew) $2n$-gons whereof the pairs of opposite angles lie in the given lines $x, y, z, \ldots$ respectively; and the pairs of opposite sides meet in the given lines $A, B, C, \ldots$ respectively. It may be added, that if all but one of the $2n$ lines $x, y, z, \ldots A, B, C \ldots$ are given, then the 'six coordinates' of the remaining line satisfy a certain linear equation, but I do not stop to explain the geometrical interpretation of this theorem.

10. Referring to the foregoing figure, if instead of the point 1 we take on the line x, a point $1'$, and construct therewith the hexagon $1'2'3'4'5'6'$; then if α, α' be the (foci or) sibi-conjugate points of the range 1, 4, $1'$, $4'$ on the line x; β, β' the sibi-conjugate points of the range 2, 5, $2'$, $5'$ on the line y; and γ, γ' the sibi-conjugate points of the range 3, 6, $3'$, $6'$ on the line z; the points in question form two triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, such that for each triangle the angles lie in the given lines and the sides pass through the given points. This is an elegant geometrical construction for the problem of the in-and-circumscribed triangle, in the particular case where the given points A, B, C lie in the given lines x, y, z, respectively.

11. The points 1, 2, 3, 4, 5, 6, A, B, C constitute a system of 9 points which lie in 9 lines of 3 each. The points α, β, γ, α', β', γ', A, B, C constitute a radically distinct system of 9 points lying in 9 lines of 3 each; viz., in the former system there are 3 sets of 3 lines which contain all the 9 points; in the latter system there is only the set of lines $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$ which contains all the nine points. The last-mentioned system may be constructed as follows: The points β, β' and γ, γ' are arbitrary: A is the intersection of the lines $\beta\gamma$ and $\beta'\gamma'$; and then joining A with the point of intersection of the lines $\beta\gamma'$ and $\beta'\gamma$ we have α an arbitrary point on the joining line; the lines $\alpha\gamma$ and $\beta\beta'$ meet in the point B, the lines $\alpha\beta$ and $\gamma\gamma'$ in the point C; the lines $C\beta'$ and $B\gamma'$ will then meet in a point α' on the line $A\alpha$; and we have thus the figure of the nine points α, β, γ, α', β', γ', A, B, C.

[Vol. III. pp. 78, 79.]

1667. (Proposed by Professor SYLVESTER.)

Show that the discriminant of the form

$$ax^5 + b\lambda x^4y + c\lambda^2x^3y^2 + c\mu^2x^2y^3 + b\mu xy^4 + ay^5$$

will be a rational integral function of the quantities a, b, c, $\lambda\mu$, $\lambda^5 + \mu^5$, and of the second degree only in respect to the last of them.

Solution by PROFESSOR CAYLEY.

In general

$$\text{Disct.}\ (a, b, c, d, e, f \between \lambda x + \mu y,\ \lambda' x + \mu' y)^5 = (\lambda\mu' - \lambda'\mu)^{20}\ \text{Disct.}\ (a, b, c, d, e, f \between x, y)^5.$$

Hence first, if $(\lambda, \mu, \lambda', \mu') = (0, 1, 1, 0)$, then

$$\text{Disct.}\ (a, b, c, d, e, f \between y, x)^5 = \text{Disct.}\ (a, b, c, d, e, f \between x, y)^5;$$

and secondly, if ω be an imaginary fifth root of unity and $(\lambda, \mu, \lambda', \mu') = (\omega, 0, 0, 1)$, then

$$\text{Disct.}\ (a, b, c, d, e, f \between \omega x, y)^5 = \text{Disct.}\ (a, b, c, d, e, f \between x, y)^5.$$

These two results may also be written,

$$\text{Disct.}\,(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5 = \text{Disct.}\,(f,\ e,\ d,\ c,\ b,\ a\between x,\ y)^5,$$

$$\text{Disct.}\,(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5 = \text{Disct.}\,(a,\ b\omega^4,\ c\omega^3,\ d\omega^2,\ e\omega,\ f\between x,\ y)^5;$$

that is, the discriminant of $(a, b, c, d, e, f \between x, y)^5$ is not altered by taking the coefficients in a reverse order, or by multiplying the several coefficients by the powers $\omega^5, \omega^4, \omega^3, \omega^2, \omega$, of an imaginary fifth root of unity. Applying these theorems to the form $(a, b\lambda, c\lambda^2, c\mu^2, b\mu, a \between x, y)^5$, the discriminant is not altered by changing the coefficients into $(a, b\mu, c\mu^2, c\lambda^2, b\lambda, a)$; that is, by the interchange of λ and μ; nor by changing the coefficients into

$$(a,\ b\omega^4\lambda,\ c\omega^3\lambda^2,\ c\omega^2\mu^2,\ b\omega\mu,\ a), \quad \text{or} \quad \{a,\ b\,(\lambda\omega^4),\ c\,(\lambda\omega^4)^2,\ c\,(\mu\omega)^2,\ b\,(\mu\omega),\ a\};$$

that is, the discriminant is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. The discriminant is therefore a rational and integral function, symmetrical in regard to λ, μ, and which is not altered by the change of λ, μ into $\lambda\omega^4$, $\mu\omega$ respectively. In virtue of the second property the discriminant is a rational integral function of $(\lambda\mu, \lambda^5, \mu^5)$, and then in virtue of the first property it is a rational integral function of $(\lambda\mu, \lambda^5\mu^5, \lambda^5+\mu^5)$, that is, of $\lambda\mu$, $\lambda^5+\mu^5$. For the general form $(a, b, c, d, e, f \between x, y)^5$, if a term of the discriminant be $a^\alpha b^\beta c^\gamma d^\delta e^\epsilon f^\phi$, then we have $\alpha+\beta+\gamma+\delta+\epsilon+\phi=8$, $5\alpha+4\beta+3\gamma+2\delta+\epsilon=20$; hence attending only to the indices α, β, γ we have $5\alpha+4\beta+3\gamma>20$, and therefore *à fortiori* $3\beta+3\gamma>20$, so that $\beta+\gamma$ is $=6$ at most. It follows that for the form $(a, b\lambda, c\lambda^2, c\mu^2, b\mu, a \between x, y)^5$, the sum of the indices of $b\lambda$, $c\lambda^2$ is $=6$ at most, and therefore, even if the index of $c\lambda^2$ is $=6$, the index of λ will be only $=12$, that is, the discriminant contains no power of λ higher than λ^{12}: hence considered as a function of $\lambda\mu$, $\lambda^5+\mu^5$, the highest power of $\lambda^5+\mu^5$ is $(\lambda^5+\mu^5)^2$; which completes the theorem.

[Vol. III. p. 90.]

1687. (Proposed by Professor CAYLEY.)—To describe a spherical triangle such that the angles thereof and of the polar triangle lie on a spherical conic.

On the sphere, the locus of a point such that the perpendiculars from it upon the sides of a given spherical triangle have their feet on a line (great circle), is in general a spherical cubic; if however the triangle be such as is mentioned in the above Problem, then the locus breaks up into a line (great circle) and into the conic through the angles of the given and polar triangles.

[Vol. III. pp. 92—96.]

1690. (Proposed by W. A. WHITWORTH, M.A.)—If ABC be the triangle formed by the three diagonals aa', bb', cc' of a complete quadrilateral $aa'bb'cc'$, then a conic can be found having double contact in the chord aa' with the critical conic of the quadrilateral $bb'cc'$, double contact in the chord bb' with the critical conic of the quadrilateral $cc'aa'$, and double contact in the chord cc' with the critical conic of the quadrilateral $aa'bb'$.

The same conic will also intersect in the chord $a'b'c'$, the three conics which pass through the intersection of Aa, Bb, Cc and touch any two sides of the triangle abc at the extremities of the third side.

It will intersect in the chord $a'bc$ the three conics which pass through the intersection of Aa, Bb', Cc' and touch any two sides of the triangle $ab'c'$ at the extremities of the third side.

It will intersect in the chord $ab'c$ the three conics which pass through the intersection of Aa', Bb, Cc' and touch any two sides of the triangle $a'bc'$ at the extremities of the third side.

It will intersect in the chord abc' the three conics which pass through the intersection of Aa', Bb', Cc and touch any two sides of the triangle $a'b'c$ at the extremities of the third side.

DEF. *The critical conic of any quadrilateral is a circumscribed conic such that the tangent at any angular point forms a harmonic pencil with the sides and diagonal meeting at that point.*

It is obvious that if the quadrilateral be projected into a square, the critical conic will become the circumscribed circle.

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3. *Solution by* PROFESSOR CAYLEY.

1. The equations of the sides of the quadrilateral may be taken to be respectively $x=0$, $y=0$, $z=0$, $w=0$, where the implicit constants are so determined that we have *identically*

$$x+y+z+w=0;$$

this being so, the equations of the three diagonals are respectively

$$\begin{array}{lllll} x+y=0, & \text{or} & z+w=0, & \text{or} & x+y-z-w=0 \text{ (three equivalent forms)} \\ x+z=0, & \text{or} & y+w=0, & \text{or} & x-y+z-w=0 \;(\;\;,, \quad ,, \quad ,, \;) \\ x+w=0, & \text{or} & y+z=0, & \text{or} & x-y-z+w=0 \;(\;\;,, \quad ,, \quad ,, \;) \end{array}$$

and the equations of the critical conics are respectively

$$xy+zw=0, \quad xz+yw=0, \quad xw+yz=0.$$

Hence we see that the equation of the required conic is

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0.$$

In fact this equation may be written

$$\Delta = (x+y-z-w)^2 - 4(xy+zw) = 0,$$
$$\Delta = (x-y+z-w)^2 - 4(xz+yw) = 0,$$
$$\Delta = (x-y-z+w)^2 - 4(xw+yz) = 0,$$

equations which put in evidence the double contact with the three critical conics respectively. We have also, identically,

$$\Delta = (x+y+z+w)(x+y-3z-w) - 2w(x+y-z-w) + 4(z^2-xy),$$

and the equation $\Delta = 0$ may therefore be written

$$\Delta = -2w(x+y-z-w) + 4(z^2-xy) = 0,$$

a form which shows that the conic $z^2-xy=0$ meets the line $w=0$ in the same two points in which it is met by the conic $\Delta=0$. And it hence appears by symmetry that the conics

$\Delta=0,\ \ x^2-yz=0,\ \ y^2-zx=0,\ \ z^2-xy=0$ meet the line $w=0$ in the same two points,
$\Delta=0,\ \ w^2-yz=0,\ \ y^2-zw=0,\ \ z^2-wy=0$ meet the line $x=0$ in the same two points,
$\Delta=0,\ \ w^2-xz=0,\ \ x^2-zw=0,\ \ z^2-wx=0$ meet the line $y=0$ in the same two points,
$\Delta=0,\ \ w^2-xy=0,\ \ x^2-yw=0,\ \ y^2-wx=0$ meet the line $z=0$ in the same two points,

which are the relations constituting the latter part of the proposed theorem.

2. The analogous theorems in space may be briefly referred to. Taking $x=0$, $y=0$, $z=0$, $w=0$ as the equations of the faces of a tetrahedron $ABCD$, then the implicit constants may be so determined that the coordinates of a given arbitrary point O shall be (1, 1, 1, 1). We may by lines drawn from the vertices of the tetrahedron project the point O on the faces, so as to obtain a point in each of the four faces; and then in each face, by lines drawn from the vertices of the face, project the point in that face upon the edges of the face; the two points thus obtained on each edge of the tetrahedron are (it is easy to see) one and the same point; that is, we have on each edge of the tetrahedron a point; and there exists a quadric surface

$$\Delta = x^2+y^2+z^2+w^2-2yz-2zx-2xy-2xw-2yw-2zw=0$$

touching the edges of the tetrahedron in these six points respectively.

The surface in question has plane contact with

the hyperboloid $xy+zw=0$ along the intersection with $x+y-z-w=0$,
" " $xz+yw=0$ " " " $x-y+z-w=0$,
" " $xw+yz=0$ " " " $x-y-z+w=0$,

and moreover the surfaces

$\Delta=0,\ \ x^2-yz=0,\ \ y^2-zx=0,\ \ z^2-xy=0$ meet the line $w=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ \ w^2-yz=0,\ \ y^2-zw=0,\ \ z^2-wy=0$ meet the line $x=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ \ w^2-xz=0,\ \ x^2-zw=0,\ \ z^2-wx=0$ meet the line $y=0,\ x+y+z+w=0$ in the same two points;
$\Delta=0,\ \ w^2-xy=0,\ \ x^2-yw=0,\ \ y^2-wx=0$ meet the line $z=0,\ x+y+z+w=0$ in the same two points.

With respect to the construction of the four planes,

$$x+y-z-w=0, \quad x-y+z-w=0, \quad x-y-z+w=0, \quad x+y+z+w=0,$$

it is to be observed that if through any edge of the tetrahedron, for instance the edge $x=0$, $y=0$, we draw the plane $x-y=0$ through the point O, then the harmonic of this in regard to the planes $x=0$, $y=0$ is the plane $x+y=0$; we have thus six planes, one through each edge of the tetrahedron, viz., these are $y+z=0$, $z+x=0$, $x+y=0$, $x+w=0$, $y+w=0$, $z+w=0$; the six planes being the faces of a hexahedron, which is such that the vertices of the tetrahedron are four of the eight vertices of the hexahedron. The pairs of opposite faces of the hexahedron meet in three lines lying in the plane $x+y+z+w=0$, and consequently forming a triangle such that through each side of the triangle there pass two opposite faces of the hexahedron; the planes $x+y-z-w=0$, $x-y+z-w=0$, $x-y-z+w=0$ are the harmonics of the plane $x+y+z+w=0$ in respect of the pairs of opposite faces of the hexahedron; viz., the plane $x+y-z-w=0$ is the harmonic of the plane $x+y+z+w=0$ in respect to the planes $x+y=0$, $z+w=0$; and the like for the other two planes $x-y+z-w=0$ and $x-y-z+w=0$ respectively.

[Vol. IV. July to December, 1865, pp. 17, 18.]

1710. (Proposed by Professor CAYLEY.)—Trace the curve $y^4-2y^2zx-z^4=0$, where the coordinates are such that $x+y+z=0$ is the line *infinity*.

Solution by the PROPOSER.

We have $x=\dfrac{y^4-z^4}{2y^2z}$; or writing $y=\theta z$, then $x=\dfrac{\theta^4-1}{2\theta^2}z$, that is

$$x:y:z=\theta^4-1:2\theta^3:2\theta^2.$$

Hence, we see that y, z are indefinitely small in comparison of x,

if $\theta=\infty$, and then $x:y:z=\theta^4:2\theta^3:2\theta^2$, that is $y^2=2zx$;

or, if $\theta=0$, and then $x:y:z=-1:2\theta^3:2\theta^2$, that is $z^3=-2y^2x$;

so that in the neighbourhood of the point $(y=0, z=0)$ there are two branches coinciding with the parabola $y^2=2zx$ and with the semicubical parabola $z^3=-2y^2xz$, respectively.

To find the points at infinity we have $x+y+z=0$, that is $\theta^4+2\theta^3+2\theta^2-1=(\theta+1)(\theta^3+\theta^2+\theta-1)=0$; and observing that the equation $\theta^3+\theta^2+\theta-1=0$ has one real root, say $\theta=k$, if k be the real root of the equation $k^3+k^2+k-1=0$ ($k=\cdot505$ nearly),—there are two real points at infinity, viz., corresponding to $\theta=-1$, we have the point $(0, -1, 1)$, and corresponding to $\theta=k$ the point $(-1-k, k, 1)$.

The equation of the tangent at a point (α, β, γ) is

$$x(-\beta^2\gamma)+y(2\beta^3-2\alpha\beta\gamma)+z(-\alpha\beta^2-2\gamma^3)=0,$$

and hence writing $(\alpha, \beta, \gamma) = (0, -1, 1)$ we have the asymptote $x + 2y + 2z = 0$: to find where this meets the curve, we have $\theta^4 + 4\theta^3 + 4\theta^2 - 1 = 0$, that is $(\theta + 1)^2 (\theta^2 + 2\theta - 1) = 0$, or at the points of intersection $\theta^2 + 2\theta - 1 = 0$, that is $\theta = -1 \pm \sqrt{2}$, or there are two real points of intersection.

Again writing $(\alpha, \beta, \gamma) = (-1 - k, k, 1)$ we find the asymptote $k^2 x - 2y + (k+1) z = 0$: to find where this meets the curve, we have $k^2 (\theta^4 - 1) - 4k\theta^3 + (2k + 2)\theta^2 = 0$, that is $k^2\theta^4 - 4k\theta^3 + (2k+2)\theta^2 - k^2 = (\theta - k)^2 \{k^2\theta^2 - 2(k^2 + k + 1)\theta - 1\} = 0$; or for the points of intersection $k^2\theta^2 - 2(k^2 + k + 1)\theta - 1 = 0$, an equation in θ with two real roots, hence the points of intersection are real.

It is now easy to lay down the curve; viz., if, to fix the ideas, the fundamental triangle is taken to be equilateral, and the coordinates x, y, z are considered to be positive for points *within* the triangle, then the curve is as shown in the annexed figure 1.

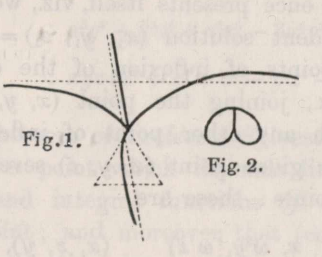


Fig. 1.

Fig. 2.

It may be remarked that the curve is met by every real line in two real points at least, and consequently that it is not the projection of any finite curve whatever. By a modification of the constants of the equation, we might obtain curves which are finite, such as the curve in figure 2; or curves with two or four infinite branches, which are the projections of such a finite curve.

[Vol. IV. pp. 32—37.]

1744. (Proposed by W. S. BURNSIDE, B.A.)—It is required to find (x_1, y_1, z_1), functions of (x, y, z), such that we may have identically

$$\frac{x_1^3 + y_1^3 + z_1^3}{x_1 y_1 z_1} = \frac{x^3 + y^3 + z^3}{xyz}.$$

Solution by PROFESSOR CAYLEY.

The Solution is in fact given in my "Memoir on Curves of the Third Order," *Philosophical Transactions*, vol. CXLVII. (1857), pp. 415—446, [146].

Write $\dfrac{x^3 + y^3 + z^3}{xyz} = -6l$; then, taking (X, Y, Z) as current coordinates, (x, y, z) are, it is clear, the coordinates of a point on the cubic curve $X^3 + Y^3 + Z^3 + 6lXYZ = 0$;

and if (x_1, y_1, z_1) are the coordinates of any other point on the same cubic curve, then we shall have

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1} = -6l = \frac{x^3+y^3+z^3}{xyz},$$

so that (x_1, y_1, z_1) will satisfy the condition in question. Hence, if from a given point (x, y, z) on the cubic curve we obtain by any geometrical construction another point on the curve, the coordinates of this new point will be functions (and, if the construction is such as to lead to a single point only, they will be *rational* functions) of (x, y, z), satisfying the condition in question.

For instance, if the point (x, y, z) be joined with any point (α, β, γ) on the curve, the joining line will again meet the curve in a single point, which may be taken to be the point (x_1, y_1, z_1). This assumes that we know on the cubic curve a point (α, β, γ); but such a point at once presents itself, viz., we may write $(\alpha, \beta, \gamma) = (1, -1, 0)$; which gives only the self-evident solution $(x_1, y_1, z_1) = (y, x, z)$. The point $(1, -1, 0)$ is clearly one of the nine points of inflexion of the cubic curve, and by using these in any manner whatever, viz., joining the point (x, y, z) with any point of inflexion, and then the new point with any other point of inflexion, and so on indefinitely, we obtain in connexion with the given point (x, y, z) seventeen other points on the curve, in all a system of eighteen points: these are

$$\begin{array}{lll|lll}
(x, y, z), & (x, \omega y, \omega^2 z), & (x, \omega^2 y, \omega z) & (x, z, y), & (x, \omega z, \omega^2 y), & (x, \omega^2 z, \omega y) \\
(y, z, x), & (\omega y, \omega^2 z, x), & (\omega^2 y, \omega z, x) & (z, y, x), & (\omega z, \omega^2 y, x), & (\omega^2 z, \omega y, x) \\
(z, x, y), & (\omega^2 z, x, \omega y), & (\omega z, x, \omega^2 y) & (y, x, z), & (\omega^2 y, x, \omega z), & (\omega y, x, \omega^2 z)
\end{array}$$

possessing remarkable geometrical properties; and of course each of the seventeen new points furnishes a (self-evident) solution of the given identity.

But we may take $(\alpha, \beta, \gamma) = (x, y, z)$; the point (x_1, y_1, z_1) is here the point of intersection of the cubic by the tangent at the point (x, y, z); or say it is the "tangential" of the point (x, y, z). The values thus obtained for (x_1, y_1, z_1) are

$$(x_1, y_1, z_1) = \{x(y^3-z^3), \quad y(z^3-x^3), \quad z(x^3-y^3)\},$$

which (excluding the above-mentioned self-evident solutions) is in fact the most simple solution of the proposed identity. In order to verify that the last-mentioned values of (x_1, y_1, z_1) are in fact the coordinates of the tangential of (x, y, z), I observe that this will be the case if only we have

$$(x^2+2lyz)\,x_1 + (y^2+2lzx)\,y_1 + (z^2+2lxy)\,z_1 = 0, \quad x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = 0,$$

the first of which is obviously satisfied by the values in question; and for the verification of the second equation,

$$\begin{aligned}
x_1^3+y_1^3+z_1^3 &= x^3(y^3-z^3)^3 + y^3(z^3-x^3)^3 + z^3(x^3-y^3)^3, \\
&= -x^9(y^3-z^3) - y^9(z^3-x^3) - z^9(x^3-y^3), \\
&= (x^3+y^3+z^3)(y^3-z^3)(z^3-x^3)(x^3-y^3), \\
x_1y_1z_1 &= xyz(y^3-z^3)(z^3-x^3)(x^3-y^3),
\end{aligned}$$

therefore

$$x_1^3+y_1^3+z_1^3+6lx_1y_1z_1=(x^3+y^3+z^3+6lxyz)(y^3-z^3)(z^3-x^3)(x^3-y^3),\ \ =0$$

if $x^3+y^3+z^3+6lxyz=0$; the same equations verify at once the identity

$$\frac{x_1^3+y_1^3+z_1^3}{x_1y_1z_1}=\frac{x^3+y^3+z^3}{xyz}.$$

Another solution is as follows: viz., if we take the third intersection with the cubic of the line joining the points (y, x, z) and $\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\}$, the coordinates of the line in question are

$$\begin{aligned} x_1 : y_1 : z_1 = \ & x^6y^3+y^6z^3+z^6x^3-3x^3y^3z^3 \\ : \ & x^3y^6+y^3z^6+z^3x^6-3x^3y^3z^3 \\ : \ & xyz(x^6+y^6+z^6-y^3z^3-z^3x^3-x^3y^3). \end{aligned}$$

According to a very beautiful theorem of Professor Sylvester's in relation to the theory of cubic curves, the coordinates of a point which depends linearly on a given point of the curve are necessarily rational and integral functions *of a square degree* of the coordinates (x, y, z) of the given point; and moreover that (considering as *one* solution those which can be derived from each other by a mere permutation of the coordinates, or change of x into ωx, &c.), there is only one solution of a given square degree m^2; the solutions of the degrees 4 and 9 are given above. The tangential of the tangential, or second tangential of the point (x, y, z), gives the solution of the degree 16; joining this second tangential with the original point (x, y, z), we have the solution of the degree 25; and the same solution is also given as the sixth point of intersection with the cubic, of the conic of 5-pointic intersection at the point (x, y, z). See my memoir "On the conic of 5-pointic contact at any point of a plane curve," *Phil. Trans.* vol. CXLIX. (1859), pp. 371—400, [261].

Addition to the foregoing Solution. On a system of Eighteen Points on a Cubic Curve.

Considering the cubic curve $x^3+y^3+z^3+6lxyz=0$, we have the nine points of inflexion, which I represent as follows:

$$\begin{array}{lll} a=(0,\ 1,\ -1), & d=(-1,\ 0,\ 1), & g=(1,\ -1,\ 0), \\ b=(0,\ 1,\ -\omega), & e=(-\omega,\ 0,\ 1), & h=(1,\ -\omega,\ 0), \\ c=(0,\ 1,\ -\omega^2), & f=(-\omega^2,\ 0,\ 1), & i=(1,\ -\omega^2,\ 0), \end{array}$$

viz., ω being an imaginary cube root of unity, the coordinates of a are $(0, 1, -1)$, those of b, $(0, 1, -\omega)$, &c.

The points of inflexion lie (as is known) by threes on twelve lines; viz., the lines are

$$\begin{array}{llll} abc, & afh, & bfg, & cfi, \\ adg, & bdi, & cdh, & def, \\ aei, & beh, & ceg, & ghi. \end{array}$$

Consider now a point on the curve, the coordinates whereof are (x, y, z), where of course $x^3+y^3+z^3+6lxyz=0$; this is one of a system of eighteen points on the curve, which may be represented as follows:

$$\begin{array}{lll} A=(x,\ y,\ z), & D=(\ x,\ \omega y,\ \omega^2 z), & G=(\ x,\ \omega^2 y,\ \omega z), \\ B=(y,\ z,\ x), & E=(\omega y,\ \omega^2 z,\ \ x), & H=(\omega^2 y,\ \omega z,\ \ x), \\ C=(z,\ x,\ y), & F=(\omega^2 z,\ \ x,\ \omega y), & I=(\omega z,\ \ x,\ \omega^2 y), \\ & & \\ J=(x,\ z,\ y), & M=(\ x,\ \omega z,\ \omega^2 y), & P=(\ x,\ \omega^2 z,\ \omega y), \\ K=(z,\ y,\ x), & N=(\omega z,\ \omega^2 y,\ \ x), & Q=(\omega^2 z,\ \omega y,\ \ x), \\ L=(y,\ x,\ z), & O=(\omega^2 y,\ \ x,\ \omega z), & R=(\omega y,\ \ x,\ \omega^2 z), \end{array}$$

viz., the coordinates of A are (x, y, z); those of B are (y, z, x), &c.

The tangent at A meets the curve in a point, "the tangential of A," the coordinates whereof are $x(y^3-z^3)$, $y(z^3-y^3)$, $z(x^3-y^3)$; which point may be called A'. And we have thus the eighteen tangentials

$$A',\ B',\ C',\ D',\ E',\ F',\ G',\ H',\ I',\ J',\ K',\ L',\ M',\ N',\ O',\ P',\ Q',\ R'.$$

The eighteen points A, B, &c., have the following property; viz., the line joining any two of them meets the cubic in a third point, which is either one of the nine points of inflexion, or one of the eighteen tangentials; there are through each point of inflexion 9 such lines, and through each tangential 4 such lines; $(9\times 9)+(18\times 4)=153=\frac{1}{2}(18.17)$, the number of pairs of points AB, AC, &c. The lines through the inflexions are the 81 lines obtained by joining any one of the points $(A, B, C, D, E, F, G, H, I)$ with any one of the points $(J, K, L, M, N, O, P, Q, R)$, as shown in the following Table:

	A	B	C	D	E	F	G	H	I
J	a	d	g	c	f	i	b	e	h
K	d	g	a	f	i	c	e	h	b
L	g	a	d	i	c	f	h	b	e
M	c	f	i	b	e	h	a	d	g
N	f	i	c	e	h	b	d	g	a
O	i	c	f	h	b	e	g	a	d
P	b	e	h	a	d	g	c	f	i
Q	e	h	b	d	g	a	f	i	c
R	h	b	e	g	a	d	i	c	f

viz., the line AJ passes through a, the line AK through d, &c.; the proof that AJ passes through a depends on the identical equation

$$\begin{vmatrix} x, & y, & z \\ x, & z, & y \\ 0, & 1, & -1 \end{vmatrix} = 0;$$

and the like for the other lines AK, AL, &c.

The lines through the tangentials are the 36 lines obtained by joining any two of the points $(A, B, C, D, E, F, G, H, I)$ and the 36 lines obtained by joining any two of the points $(J, K, L, M, N, O, P, Q, R)$; and these 72 lines pass through the tangentials, as shown by the table

ABC,	BDI,	CEG,	JKL,	KMR,	LNP,
ADG,	BEH,	CFI,	JMP,	KNQ,	LOR,
AEI,	BFG,	DEF,	JNR,	JOP,	MNO,
AFH,	CDH,	GHI,	JOQ,	LMQ,	PQR,

viz., in the triad ABC, BC passes through A', CA through B', AB through C'; and the like for the other triads. The proof that BC passes through A depends on the identical equation

$$\begin{vmatrix} y & , & z & , & x \\ z & , & x & , & y \\ x(x^3-z^3), & & y(z^3-x^3), & & z(x^3-y^3) \end{vmatrix} = 0;$$

and the like for the other combinations of points.

If we attend only to the points A, B, C and their tangentials A', B', C'; then we have on the cubic three points A, B, C, such that the line joining any two of them passes through the tangential of the third point. And the figure may be constructed by means of the three real points of inflexion a, d, g, as follows, viz., joining these with any point J on the cubic, the lines so obtained respectively meet the cubic in three new points which may be taken for the points A, B, C. Or if one of these points, say A, be given, then joining it with one of the three real inflexions, this line again meets the cubic in the point J, and from it by means of the other two real inflexions we obtain the remaining points B and C; it is clear that, A being given, the construction gives three points, say J, K, L, each of them leading to the same two points B and C.

We may consider the question from a different point of view. Let A, B, C be given points, and let there be given also three lines passing through these three points respectively; through the given points, touching at these points the given lines respectively, describe a cubic; and let the given lines again meet the cubic in the points A', B', C' respectively. The equation of the cubic contains three arbitrary parameters; but when two of these are properly determined, the points A, B, C and their tangentials A', B', C' will be related as in the theorem; viz., the line through any two of the points will pass through the tangential of the third point. The analytical investigation is as follows:

Let the equations of the three tangents be $x=0$, $y=0$, $z=0$, and suppose that, for the points A, B, C respectively, we have

$$(x=0,\ y=\lambda z),\quad (y=0,\ z=\mu x),\quad (z=0,\ x=\nu y),$$

then the equation of a cubic touching the three lines at the three points respectively will be

$$(y-\lambda z)^2(\nu^2By+Cz)+(z-\mu x)^2(\lambda^2Cz+Ax)+(x-\nu y)^2(\mu^2Ax+By)$$
$$-\mu^2Ax^3-\nu^2By^3-\lambda^2Cz^3+Kxyz=0,$$

where A, B, C, K are arbitrary coefficients; but if $A:B:C=\lambda:\mu:\nu$, then the equation is

$$(y-\lambda z)^2\,\nu\,(\mu\nu y+z)+(z-\mu x)^2\,\lambda\,(\nu\lambda z+x)+(x-\nu y)^2\,\mu\,(\lambda\mu x+y)$$
$$-\lambda\mu^2x^3-\mu\nu^2y^3-\nu\lambda^2z^3+Kxyz=0,$$

where

A, A' are the intersections of $x=0$, by $y-\lambda z=0$, $\mu\nu y+z=0$ respectively,

B, B' „ „ $y=0$, „ $z-\mu x=0$, $\nu\lambda z+x=0$ „ ,

C, C' „ „ $z=0$, „ $x-\nu y=0$, $\lambda\mu x+y=0$ „ ;

the equations of BC, CA, AB thus are

$$-\mu x+\mu\nu y+z=0,\quad x-\nu y+\nu\lambda z=0,\quad \lambda\mu x+y-\lambda z=0,$$

which pass through A', B' and C' respectively.

If we consider along with the points A, B, C the points J, K, L, and their respective tangentials, then we have inscribed in the cubic a hexagon $ALBJCK$ which has the following properties, viz., the pairs of opposite sides and the three diagonals pass through the three real inflexions *in linea*, viz.,

AL, JC, BK, through g

LB, CK, JA, „ a

BJ, KA, CL, „ d.

This shows that the six points A, B, C, J, K, L are the intersections of the cubic by a conic; and moreover, considering the triangles ABC, JKL formed by the alternate vertices, then in each triangle the sides pass through the tangentials of the opposite vertices respectively.

In what precedes we have in effect found the coordinates (z, x, y) of the third point of intersection with the cubic, of the line joining the points (y, z, x) and

$\{x(y^3-z^3),\ y(z^3-x^3),\ z(x^3-y^3)\}$. The coordinates of the same point may be otherwise found by a direct investigation, as follows: Write

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3); \quad x_1 : y_1 : z_1 = y : z : x.$$

If in the equation of the curve we substitute for x, y, z, the values ux_1+vx_2, uy_1+vy_2, uz_1+vz_2, we find

$$\begin{aligned} &u\{x_1^2x_2 + y_1^2y_2 + z_1^2z_2 + 2l(x_2y_1z_1 + y_2z_1x_1 + z_2x_1y_1)\} \\ &+v\{x_1x_2^2 + y_1y_2^2 + z_1z_2^2 + 2l(x_1y_2z_2 + y_1z_2x_2 + z_1x_2y_2)\} = 0, \end{aligned}$$

say $uP+vQ=0$; we may therefore write $u=Q$, $v=-P$, and the coordinates of the third point are Qx_1-Px_2, Qy_1-Py_2, Qz_1-Pz_2. Now

$$\begin{aligned} Qx_1 - Px_2 &= y_1y_2(x_1y_2 - x_2y_1) + z_1z_2(x_1z_2 - x_2z_1) + 2l(x_1^2y_2z_2 - x_2^2y_1z_1) \\ &= yz(z^3-x^3)\{y^2(z^3-x^3) - zx(y^3-z^3)\} \\ &\quad + zx(x^3-y^3)\{yz(x^3-y^3) - x^2(y^3-z^3)\} \\ &\quad + 2l\{y^2 . yz . (z^3-x^3)(x^3-y^3) - x^2(y^3-z^3)^2 zx\}. \\ &= (x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3)z \\ &\quad + xyz(x^6+y^6+z^6 - y^3z^3 - z^3x^3 - x^3y^3)z \\ &\quad - 2l(x^6y^3 + y^6z^3 + z^6x^3 - 3x^3y^3z^3)z; \end{aligned}$$

so that we have $Qx_1-Px_2 = \Pi z$; and in like manner $Qy_1-Py_2=\Pi x$, $Qz_1-Pz_2=\Pi y$; and therefore $Qx_1-Px_2 : Qy_1-Qy_2 : Qz_1-Pz_2 = z : x : y$, which proves the theorem.

I consider in like manner the following question; viz., if (y, x, z) be joined with the tangential of (x, y, z); to find the third point of intersection. We have here

$$x_2 : y_2 : z_2 = x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3); \quad x_1 : y_1 : z_1 = y : x : z;$$

and P, Q as before; and the coordinates of the third point are

$$Qx_1-Px_2 : Qy_1-Py_2 : Qz_1-Pz_2;$$

also

$$\begin{aligned} Qx_1 - Px_2 = &\ xy(z^3-x^3)\{y^2(z^3-x^3) - x^2(y^3-z^3)\} \\ &+ z^2(x^3-y^3)\{yz(x^3-y^3) - zx(y^3-z^3)\} \\ &+ 2l\{y^3z(z^3-x^3)(x^3-y^3) - x^2(y^3-z^3)^2 zx\}, \\ = &\ x\{y^3(z^3-x^3)^2 - z^3(x^3-y^3)(y^3-z^3)\} \\ &+ y\{z^3(x^3-y^3)^2 - x^3(y^3-z^3)(z^3-x^3)\} \\ &+ 2lz\{y^3(z^3-x^3)(x^3-y^3) - x^3(y^3-z^3)^2\}, \end{aligned}$$

that is $\quad Qx_1-Px_2 = (x+y-2lz)(x^3z^6 + y^3x^6 + z^3y^6 - 3x^3y^3z^3)$;

similarly $\quad Qy_1-Py_2 = (x+y-2lz)(y^3z^3 + y^6z^3 + z^3x^6 - 3x^3y^3z^3)$;

also $\quad Qz_1-Pz_2 = (x+y-2lz)(x^6+y^6+z^6 - y^3z^3 - x^3y^3)xyz$;

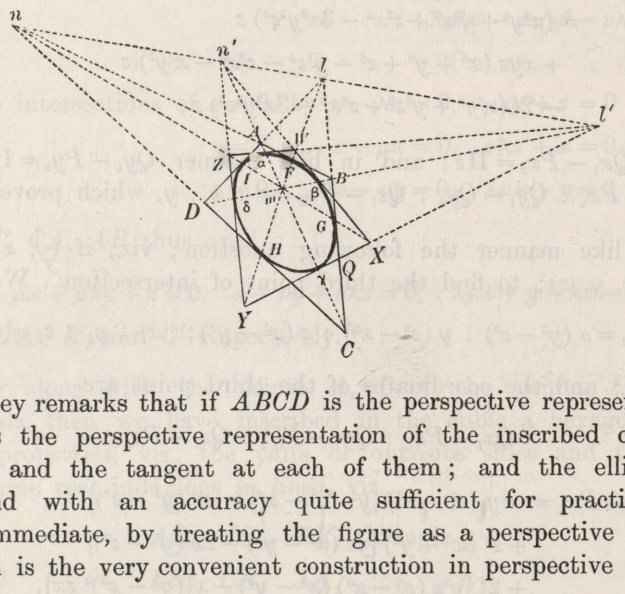
and we hence have the values

$$X : Y : Z = x^6y^3 + y^6z^3 + z^6x^3 - 3x^3y^3z^3 : x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3$$
$$: xyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)$$

for the coordinates of the point in question.

[Vol. IV. pp. 38, 39.]

1751. (Proposed by Professor CAYLEY.)—Let $ABCD$ be any quadrilateral. Construct, as shown in the figure, the points F, G, H, I: in BC find a point Q such that $\dfrac{BG}{BC}\cdot\dfrac{CQ}{GQ}=\dfrac{1}{\sqrt{2}}$; and complete the construction as shown in the figure. Show that an ellipse may be drawn passing through the eight points F, G, H, I, α, β, γ, δ, and having at these points respectively the tangents shown in the figure.



{Professor Cayley remarks that if $ABCD$ is the perspective representation of a square, then the ellipse is the perspective representation of the inscribed circle; the theorem gives eight points and the tangent at each of them; and the ellipse may therefore be drawn by hand with an accuracy quite sufficient for practical purposes. The demonstration is immediate, by treating the figure as a perspective representation: the gist of the theorem is the very convenient construction in perspective which it furnishes.}

[Vol. IV. pp. 65—67.]

1775. (Proposed by W. K. CLIFFORD.)—If a straight line meet the faces of the tetrahedron $ABCD$ in the points a, b, c, d, respectively; the spheres whose diameters are Aa, Bb, Cc, Dd have a common radical axis.

Solution by PROFESSOR CAYLEY.

Let the given line be taken for the axis of z; the axes of x, y being any rectangular axes in the plane perpendicular thereto; the equations of the given line are therefore $(x=0,\ y=0)$. Take $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ for the coordinates of the points A, B, C, D respectively; and $(0, 0, c_1)$, $(0, 0, c_2)$, $(0, 0, c_3)$, $(0, 0, c_4)$ for the coordinates of the points a, b, c, d respectively. Then, to determine c_1, the equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

and cutting this by the line $x=0$, $y=0$, we have

$$\begin{vmatrix} 0, & 0, & c_1, & 1 \\ \alpha_2, & \beta_2, & \gamma_2, & 1 \\ \alpha_3, & \beta_3, & \gamma_3, & 1 \\ \alpha_4, & \beta_4, & \gamma_4, & 1 \end{vmatrix} = 0,$$

with similar equations for c_2, c_3, c_4 respectively. The four equations may be united into the single equation

$$\begin{vmatrix} c_1p_1, & 1, & \alpha_1, & \beta_1 \\ c_2p_2, & 1, & \alpha_2, & \beta_2 \\ c_3p_3, & 1, & \alpha_3, & \beta_3 \\ c_4p_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} p_1, & \alpha_1, & \beta_1, & \gamma_1 \\ p_2, & \alpha_2, & \beta_2, & \gamma_2 \\ p_3, & \alpha_3, & \beta_3, & \gamma_3 \\ p_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

where p_1, p_2, p_3, p_4 are arbitrary multipliers. Hence, writing successively $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ and $(p_1, p_2, p_3, p_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, we have *first*

$$\begin{vmatrix} c_1, & 1, & \alpha_1, & \beta_1 \\ c_2, & 1, & \alpha_2, & \beta_2 \\ c_3, & 1, & \alpha_3, & \beta_3 \\ c_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} 1, & \alpha_1, & \beta_1, & \gamma_1 \\ 1, & \alpha_2, & \beta_2, & \gamma_2 \\ 1, & \alpha_3, & \beta_3, & \gamma_3 \\ 1, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4 \end{vmatrix} = 0;$$

and *secondly*,

$$\begin{vmatrix} c_1\gamma_1, & 1, & \alpha_1, & \beta_1 \\ c_2\gamma_2, & 1, & \alpha_2, & \beta_2 \\ c_3\gamma_3, & 1, & \alpha_3, & \beta_3 \\ c_4\gamma_4, & 1, & \alpha_4, & \beta_4 \end{vmatrix} = \begin{vmatrix} \gamma_1, & \alpha_1, & \beta_1, & \gamma_1 \\ \gamma_2, & \alpha_2, & \beta_2, & \gamma_2 \\ \gamma_3, & \alpha_3, & \beta_3, & \gamma_3 \\ \gamma_4, & \alpha_4, & \beta_4, & \gamma_4 \end{vmatrix},$$

that is,

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4\gamma_4 \end{vmatrix} = 0;$$

and these two results may be united into the single formula

$$\begin{vmatrix} 1, & \alpha_1, & \beta_1, & c_1+\gamma_1, & c_1\gamma_1 \\ 1, & \alpha_2, & \beta_2, & c_2+\gamma_2, & c_2\gamma_2 \\ 1, & \alpha_3, & \beta_3, & c_3+\gamma_3, & c_3\gamma_3 \\ 1, & \alpha_4, & \beta_4, & c_4+\gamma_4, & c_4\gamma_4 \end{vmatrix} = 0.$$

Now the equation of a sphere having for the extremities of a diameter the points (α, β, γ) and (a, b, c) is

$$[x-\tfrac{1}{2}(a+\alpha)]^2+[y-\tfrac{1}{2}(b+\beta)]^2+[z-\tfrac{1}{2}(c+\gamma)]^2=\tfrac{1}{4}[(a-\alpha)^2+(b-\beta)^2+(c-\gamma)^2],$$

or

$$(x-a)(x-\alpha)+(y-b)(y-\beta)+(z-c)(z-\gamma)=0,$$

or

$$x^2+y^2+z^2-(a+\alpha)x-(b+\beta)y-(c+\gamma)z+a\alpha+b\beta+c\gamma=0;$$

therefore, when the two points are (α, β, γ) and $(0, 0, c)$, the equation is

$$x^2+y^2+z^2-\alpha x-\beta y-(c+\gamma)z+c\gamma=0.$$

Hence, putting for shortness $P=-\alpha x-\beta y-(c+\gamma)z+c\gamma$, viz., $P_1=-\alpha_1 x-\beta_1 y-(c_1+\gamma_1)z+c_1\gamma_1$, &c., the equations of the four spheres are

$$x^2+y^2+z^2+P_1=0,\quad x^2+y^2+z^2+P_2=0,\quad x^2+y^2+z^2+P_3=0,\quad x^2+y^2+z^2+P_4=0,$$

and the four spheres will have a common radical axis, if for proper values of the multipliers μ, ν, ρ we have

$$\mu(P_1-P_2)+\nu(P_1-P_3)+\rho(P_1-P_4)=0,$$

or what is the same thing, if for proper values of λ, μ, ν, ρ we have

$$\lambda P_1+\mu P_2+\nu P_3+\rho P_4=0,\quad \lambda+\mu+\nu+\rho=0;$$

that is, if

$$\begin{aligned}
&\lambda + \mu + \nu + \rho = 0,\\
&\lambda\alpha_1 + \mu\alpha_2 + \nu\alpha_3 + \rho\alpha_4 = 0,\\
&\lambda\beta_1 + \mu\beta_2 + \nu\beta_3 + \rho\beta_4 = 0,\\
&\lambda(c_1+\gamma_1) + \mu(c_2+\gamma_2) + \nu(c_3+\gamma_3) + \rho(c_4+\gamma_4) = 0,\\
&\lambda c_1\gamma_1 + \mu c_2\gamma_2 + \nu c_3\gamma_3 + \rho c_4\gamma_4 = 0;
\end{aligned}$$

and eliminating from these equations $(\lambda, \mu, \nu, \rho)$, we find the above-mentioned relation between α_1, β_1, γ_1, c_1, &c.; which proves the theorem.

[Vol. IV. pp. 70, 71.]

1771. (Proposed by Professor CAYLEY.)—Given a circle and a line, it is required to find a parabola, having the line for its directrix, and the circle for a circle of curvature.

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2. *Solution by the* PROPOSER.

Let $x^2 + y^2 - 1 = 0$ be the equation of the given circle, $x = m$ that of the given line. Taking on the circle an arbitrary point $(\cos\theta, \sin\theta)$, we may find a parabola having the given line for its directrix, and touching the circle at the last-mentioned point; viz., the equation of the parabola is found to be

$$y^2 - 2y\sin\theta\,(1 + 2\cos^2\theta - 2m\cos\theta) - 4x\cos^2\theta\,(\cos\theta - m) + 1 + 3\cos^2\theta - 4m\cos\theta = 0.$$

{There is no difficulty in *verifying* that this parabola has for its directrix the line $x - m = 0$, that the equation is satisfied by the values $x = \cos\theta$, $y = \sin\theta$, and that the derived equation is satisfied by the values $x = \cos\theta$, $y = \sin\theta$, $\dfrac{dy}{dx} = -\cot\theta$.}

Representing for a moment the left-hand side of the equation by U, we have identically

$$\begin{aligned}
&U - \cos^2\theta\,(x^2 + y^2 - 1)\\
&\quad = y^2\sin^2\theta - x^2\cos^2\theta - 2y\sin\theta\,(1 + 2\cos^2\theta - 2m\cos\theta) - 4x\cos^2\theta\,(\cos\theta - m)\\
&\qquad\qquad + 1 + 4\cos^2\theta - 4m\cos\theta,\\
&\quad = (y\sin\theta + x\cos\theta - 1)(y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta).
\end{aligned}$$

Hence to find the intersections of the parabola with the circle, we have first

$$x^2 + y^2 - 1 = 0, \quad y\sin\theta + x\cos\theta - 1 = 0,$$

giving the point $(\cos\theta, \sin\theta)$ *twice*, since $y\sin\theta + x\cos\theta - 1 = 0$ is the equation of the tangent to the circle at the point in question; and secondly

$$x^2 + y^2 - 1 = 0, \quad y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta = 0,$$

giving the remaining two points of intersection. If the circle be a circle of curvature, one of these must coincide with the point $(\cos\theta, \sin\theta)$, that is the equation $y\sin\theta - x\cos\theta - 1 - 4\cos^2\theta + 4m\cos\theta = 0$, must be satisfied by the values $x = \cos\theta$, $y = \sin\theta$; this will be the case if $-6\cos^2\theta + 4m\cos\theta = 0$, that is $\cos\theta = 0$, giving for the parabola $y^2 \pm 2y + 1 = 0$, which is not a proper Solution, or else $\cos\theta = \frac{2}{3}$, giving $\sin\theta = \pm(1 - \frac{4}{9}m^2)^{\frac{1}{2}}$, so that there are *two* parabolas satisfying the conditions of the problem; if to fix the ideas we take the upper sign, the equation of the corresponding parabola is

$$y^2 - 2(1 - \tfrac{4}{9}m^2)^{\frac{3}{2}}y + \tfrac{16}{27}m^3x + 1 - \tfrac{4}{3}m^2 = 0;$$

and it may be added that the coordinates of the focus are

$$x = m - \tfrac{8}{27}m^3, \quad y = (1 - \tfrac{4}{9}m^2)^{\frac{3}{2}}.$$

The equation of the other parabola and the coordinates of the focus are of course found by merely changing the sign of the radical. The parabolas are real if $m < \frac{3}{2}$; if $m = \frac{3}{2}$ we have a single parabola, the point of contact being in this case the vertex of the parabola; and if $m > \frac{3}{2}$ the parabolas are imaginary.

{Professor Cayley states that he was led to the foregoing problem by the consideration of the curve (proposed for investigation in Quest. 1812) which is the envelope of a variable circle having its centre in the given circle and touching the given line. The required curve (which is of the sixth order) has two cusps which, it is easy to see geometrically, are the foci of the parabolas in the problem. Taking $(\cos\theta, \sin\theta)$ for the coordinates of the centre of the variable circle, we shall have

$$x = \tfrac{3}{2}\cos\theta - m\cos 2\theta + \tfrac{1}{2}\cos 3\theta, \quad y = \tfrac{3}{2}\sin\theta - m\sin 2\theta + \tfrac{1}{2}\sin 3\theta,$$

for the coordinates of a point on the envelope.}

[Vol. IV. pp. 81—83.]

1790. (Proposed by Professor SYLVESTER.)—(1) If a set of six points be respectively represented by the six permutations of $a : b : c$, show that they lie in a conic, and write down its equation.

(2) *Hence* prove that if AB, BC, CA be three real lines containing the nine points of inflexion of a cubic curve having an oval, the pairs of tangents drawn to the oval from A, B, C will meet it in six points lying in a conic.

Solution by PROFESSOR CAYLEY.

1. That the six points,

$$1 = (a, b, c), \quad 2 = (b, c, a), \quad 3 = (c, a, b),$$
$$4 = (a, c, b), \quad 5 = (b, a, c), \quad 6 = (c, b, a),$$

are situate on a conic, appears at once by writing down its equation: viz.,

$$(bc + ca + ab)(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)(yz + zx + xy) = 0,$$

which is satisfied by the coordinates of each of the six points.

2. It is interesting to remark that the six points on the conic form, not a general inscribed hexagon, but a hexagon such as is mentioned in Prob. 1512 (vol. II. p. 51), viz., one in which the three diagonals pass respectively through the Pascalian points (intersections of opposite sides): in fact, in the hexagon 143526, forming the equations of the sides and diagonals, these are

$$\begin{array}{llll}
14. & (b+c)\,x - a\,y - a\,z = 0, & 25. & (c+a)\,x - b\,y - b\,z = 0, \\
15. & -c\,x - c\,y + (a+b)\,z = 0, & 26. & -a\,x - a\,y + (b+c)\,z = 0, \\
16. & -b\,x + (c+a)\,y - b\,z = 0, & 24. & -c\,x + (a+b)\,y - c\,z = 0,
\end{array}$$

$$\begin{array}{ll}
36. & (a+b)\,x - c\,y - c\,z = 0, \\
34. & -b\,x - b\,y + (c+a)\,z = 0, \\
35. & -a\,x + (b+c)\,y - a\,z = 0;
\end{array}$$

so that the lines 14, 25, 36 meet in the point $x = 0,\ y + z = 0,$

„ 16, 24, 35 „ $y = 0,\ z + x = 0,$

„ 15, 26, 34 „ $z = 0,\ x + y = 0.$

3. It is further to be remarked that the six points lie on the cubic curve

$$\frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3} - \frac{xyz}{abc} = 0,$$

and are consequently the six points of intersection of this cubic by the above mentioned conic.

4. The points $(x = 0,\ y + z = 0)$, $(y = 0,\ z + x = 0)$, $(z = 0,\ x + y = 0)$ are the three real inflexions of the cubic; hence, attending only to the cubic, and starting from the arbitrary point (a, b, c) on this curve, it appears by what precedes, that we may, by means of the three real inflexions of the cubic, construct the system of six points, (the construction is, in fact, identical with that given in my Solution of Problem 1744, vol. IV. pp. 32—37, [*ante* p. 597] the six points being six out of the therein mentioned eighteen points); and it further appears, that these six points lie on a conic.

5. As regards the second part of the proposed Problem, consider the cubic curve $x^3 + y^3 + z^3 + 6lxyz = 0$; the three real lines containing the nine points of inflexion are the lines $x = 0$, $y = 0$, $z = 0$; and the points A, B, C are therefore $(y = 0,\ z = 0)$, $(z = 0,\ x = 0)$, $(x = 0,\ y = 0)$ respectively. From each of these points we may draw to the curve six tangents, and we have thus on the curve eighteen points, which are a particular case of the system in the Solution of Prob. 1744. Or if from each of the points we draw two properly selected tangents, (when the cubic has an oval these

may be the two tangents to the oval,) then we obtain a system of six points, (part of the system of eighteen points); viz., the coordinates of the six points are of the form (a, b, c), (b, c, a), (c, a, b), (a, c, b), (b, a, c), (c, b, a) and therefore the six points are in a conic.

6. To verify this, if we take $y=\theta x$ for the equation of a tangent from the point $(x=0,\ y=0)$, the equation $(1+\theta^3)x^3+6l\theta x^2z+z^3=0$ must have a pair of equal roots, giving for θ the equation $(1+\theta^3)^2+32l^3\theta^3=0$; and we then find $z=-\dfrac{1+\theta^3}{4l\theta}x$, that is, θ being determined by the foregoing equation, the coordinates of the point of contact are $x:y:z=1:\theta:-\dfrac{1+\theta^3}{4l\theta}$. The roots of the equation in θ are of the form $\theta_1,\ \theta_2,\ \theta_3,\ \dfrac{1}{\theta_1},\ \dfrac{1}{\theta_2},\ \dfrac{1}{\theta_3}$; and assuming that the curve has an oval, there are two real roots $\theta_1, \dfrac{1}{\theta_1}$. Hence, writing $x:y:z=1:\theta_1:-\dfrac{1+\theta_1^3}{4l\theta_1}=a:b:c$, the substitution $\dfrac{1}{\theta_1}$ for θ, gives $x:y:z=b:a:c$, that is, the coordinates of the points of contact of the tangents to the oval, from the point $(x=0,\ y=0)$ are $(a,\ b,\ c)$ and $(b,\ a,\ c)$ respectively; and writing successively (y, z, x) and (z, x, y) in place of (x, y, z), the coordinates for the tangents from $(y=0,\ z=0)$ are (b, c, a), (c, b, a); and those for the tangents from $(z=0,\ x=0)$ are (c, a, b) and (a, c, b); so that the coordinates of the six points of contact are a system of the form in question.

[Vol. IV. p. 107.]

1812. (Proposed by Professor CAYLEY.)—Find the envelope of a series of circles which touch a given straight line and have their centres in the circumference of a given circle. {See Quest. 1771.}

[Vol. IV. pp. 108, 109.]

1816. (Proposed by R. BALL, M.A.)—Express the roots of the equation

$$(ae-4bd+3c^2)(ax^4+4bx^3+6cx^2+4dx+e)^2-3\{(ac-b^2)x^4+2(ad-bc)x^3$$
$$+(ae+2bd-3c^2)x^2+2(be-cd)x+(ce-d^2)\}^2=0,$$

in terms of the roots $\alpha, \beta, \gamma, \delta$ of $x^4+4bx^3+6cx^2+4dx+e=0$.

Solution by PROFESSOR CAYLEY.

Writing

$$U=(a,\ b,\ c,\ d,\ e \between x,\ y)^4=a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y),$$

$$H=\left(ac-b^2,\ \tfrac{1}{2}(ad-bc),\ \tfrac{1}{6}(ae+2bd-3c^2),\ \tfrac{1}{2}(be-cd),\ ce-d^2 \between x,\ y\right)^4,$$

$$I=ae-4bd+3c^2;$$

then considering $x : y$ as the unknown quantity, it is required to find the roots of the equation $IU^2 - 3H^2 = 0$ in terms of the roots $(\alpha, \beta, \gamma, \delta)$ of the equation $U = 0$; or, what is the same thing, it is required to find the linear factors of the function $IU^2 - 3H^2$. The function in question is the product of four quadratic factors, rational functions of $(\alpha, \beta, \gamma, \delta)$; and these being known, the four pairs of linear factors can be determined each of them by the solution of a quadratic equation. In fact, writing

$$\begin{aligned}
\Theta_\alpha &= a\{(\beta-\alpha)(x-\gamma y)(x-\delta y)+(\gamma-\alpha)(x-\delta y)(x-\beta y)+(\delta-\alpha)(x-\beta y)(x-\gamma y)\},\\
\Theta_\beta &= a\{(\gamma-\beta)(x-\delta y)(x-\alpha y)+(\delta-\beta)(x-\alpha y)(x-\gamma y)+(\alpha-\beta)(x-\gamma y)(x-\delta y)\},\\
\Theta_\gamma &= a\{(\delta-\gamma)(x-\alpha y)(x-\beta y)+(\alpha-\gamma)(x-\beta y)(x-\delta y)+(\beta-\gamma)(x-\delta y)(x-\alpha y)\},\\
\Theta_\delta &= a\{(\alpha-\delta)(x-\beta y)(x-\gamma y)+(\beta-\delta)(x-\gamma y)(x-\alpha y)+(\gamma-\delta)(x-\alpha y)(x-\beta y)\},
\end{aligned}$$

we have identically $256(IU^2 - 3H^2) = \Theta_\alpha\Theta_\gamma\Theta_\beta\Theta_\delta$; so that the quadratic factors of $IU^2 - 3H^2$ are $\Theta_\alpha, \Theta_\beta, \Theta_\gamma, \Theta_\delta$. To show that this is so, it is to be remarked that the product $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is a symmetrical function of the roots $\alpha, \beta, \gamma, \delta$, and consequently a rational and integral function of the coefficients (a, b, c, d, e) of U; moreover $\Theta_\alpha, \Theta_\beta, \Theta_\gamma, \Theta_\delta$ being each of them a covariant (an irrational one) of U, the product in question must be a covariant. But a covariant is completely determined when the leading coefficient is given; hence it will be sufficient to show that the leading coefficients, or coefficients of x^8, in the functions $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ and $256(IU^2 - 3H^2)$ are equal to each other. Writing for a moment $\Sigma\alpha = p$, $\Sigma\alpha\beta = q$, $\Sigma\alpha\beta\gamma = r$, $\alpha\beta\gamma\delta = s$, the coefficient of x^2 in $a^{-1}\Theta_\alpha$ is $\beta + \gamma + \delta - 3\alpha$, which $= p - 4\alpha$; we have thence the product $(p-4\alpha)(p-4\beta)(p-4\gamma)(p-4\delta)$, which is $= p^4 - 4p^3 . p + 16p^2 . q - 64p . r + 256s, = 256s - 64pr + 16p^2q - 3p^4$.

Hence, restoring the omitted factor a^4, and observing that we have p, q, r, s equal to $-4b, 6c, -4d, e$, each divided by a, the coefficient of x^8 in $\Theta_\alpha\Theta_\beta\Theta_\gamma\Theta_\delta$ is

$$256(a^3e - 4a^2bd + 6ab^2c - 3b^4), \text{ or } 256\{(ae - 4bd + 3c^2)a^2 - 3(ac - b^2)^2\},$$

and is consequently equal to the coefficient of x^8 in $256(IU^2 - 3H^2)$; which proves the theorem.

It may be remarked that the leading coefficient of $IU^2 - 3H^2$ is $= a^{-1}(a, b, c, d, e \between b, -a)^4$; and that for a quantic $U = (a, b, ..)(x, y)^n$ of the order n we have a corresponding covariant of the order $n(n-2)$, the leading coefficient of which is $= a^{-1}(a, b, ... \between b, -a)^{2n}$. For $n = 2$, this is the invariant (discriminant) $ac - b^2$; for $n = 3$ it is the cubicovariant $(a^2d - 3abc + 2b^3, ... \between x, y)^3$; for $n = 4$ it is, as we have seen, the covariant $IU^2 - 3H^2$. For $n = 5$, the leading coefficient $a^4f - 5a^3be + 10a^2b^2d - 10ab^3c + 4b^5$ is $= a^2(a^2f - 5abe + 2acd + 8b^2d - 6bc^2) - 2(ac - b^2)(a^2d - 3abc + 2b^3)$, which shows that the covariant in question (of the order 15) is $= U^2$(No. 17) $- 2$(No. 15)(No. 18), where the Nos. refer to the Tables of my Second Memoir on Quantics, *Phil. Trans.*, vol. CXLVI. (1856), pp. 101—126, [141; in the notation there explained, the expression for the covariant is $A^2E - 2CF$].

{The roots of $\Theta_\alpha = 0$ are readily found to be

$$\frac{\alpha(\beta+\gamma+\delta) - (\gamma\delta + \delta\beta + \beta\gamma) \pm \{\tfrac{1}{2}[(\alpha-\beta)^2(\gamma-\delta)^2 + (\alpha-\gamma)^2(\delta-\beta)^2 + (\alpha-\delta)^2(\beta-\gamma)^2]\}^{\frac{1}{2}}}{3\alpha - (\beta+\gamma+\delta)};$$

these then, with three similar pairs, express the eight roots as required.}

In the following Contents, the Problems are referred to, each by its No. and the Proposer's name, and the subject is briefly indicated: an Asterisk shows that no solution was given.