# XIX.

## THE ABERRATIONS OF AN OPTICAL INSTRUMENT OF REVOLUTION

[1833 ?]

[Note Book 28, pp. 39–53.]

[After a discussion of spherical aberration for a pair of thin lenses, the incident rays being parallel to the axis or meeting it in a single point, in which discussion Herschel's results are verified, Hamilton continues as follows.]

Herschel recommends that after determining the powers $P$, $P_1$, so as to satisfy the condition of achromatism, and to make the whole power $P+P_1=$ some given quantity, the anterior curvatures $R$, $R_1$, should be determined so as to satisfy the equations $a=0$, $b=0$, and thereby to make the object glass aplanatic for nearly parallel as well as for exactly parallel incident rays.

But I think that for astronomical purposes it would be much better to confine ourselves to the condition $a=0$, that is, to make the achromatic object glass aplanatic only for exactly parallel incident rays, and then to remove the remaining indeterminateness of the question by destroying, if possible, the confusion which arises from an initial inclination to the axis.

To investigate this last confusion, we may proceed as follows. The function $T$ for the object glass* may be thus developed,

$$\begin{aligned} T = T^{(0)} &+ P(\alpha^2+\beta^2) + P_{\prime}(\alpha\alpha'+\beta\beta') + P'(\alpha'^2+\beta'^2) \\ &+ Q(\alpha^2+\beta^2)^2 + Q_{\prime}(\alpha\alpha'+\beta\beta')(\alpha^2+\beta^2) + Q'(\alpha'^2+\beta'^2)(\alpha^2+\beta^2) \\ &+ Q_{\prime\prime}(\alpha\alpha'+\beta\beta')^2 + Q_{\prime}'(\alpha\alpha'+\beta\beta')(\alpha'^2+\beta'^2) + Q''(\alpha'^2+\beta'^2)^2; \end{aligned}$$

$T^{(0)}$ $P$ $P_{\prime}$ $P'$ $Q$ $Q_{\prime}$ $Q'$ $Q_{\prime\prime}$ $Q_{\prime}'$ $Q''$ being functions of the two indices and the four curvatures, and also of the arbitrary distance of the infinitely thin object glass from the origin of coordinates, supposed placed on the axis of revolution which we take for the axis of $z$; $\alpha'$, $\beta'$ are initial and $\alpha$, $\beta$ final small direction cosines, that is, cosines of inclination of incident and emergent rays to rectangular semi-axes of $x$, $y$. Then $\alpha'$, $\beta'$ are considered as given, and $\gamma>0$, and the equations of emergent ray are

$$x - \frac{\alpha z}{\sqrt{1-\alpha^2-\beta^2}} = \frac{\delta T}{\delta \alpha}, \quad y - \frac{\beta z}{\sqrt{1-\alpha^2-\beta^2}} = \frac{\delta T}{\delta \beta}.$$

Supposing first that the incident rays have no inclination to the axis, and considering emergent rays in plane of $xz$, then

$$\alpha'=0, \quad \beta'=0, \quad \beta=0, \quad T=T^{(0)}+P\alpha^2+Q\alpha^4,$$

$$x = \alpha z + \tfrac{1}{2}\alpha^3 z + 2\alpha P + 4\alpha^3 Q,$$

and since $x$ must vanish independently of $\alpha$ at the aplanatic focus, we have for the ordinate of this focus

$$z=-2P,$$

and we have the relation

$$Q=\tfrac{1}{4}P:$$

* [The results of this note are applicable to any instrument of revolution.]

if then we take the aplanatic focus on the axis of the object glass for the origin of coordinates, we shall have

$$P=0, \quad Q=0.$$

Suppose now that star is not in axis, but is in plane of $xz$; then

$$\beta'=0,$$

$$T=T^{(0)}+P_{,}\alpha\alpha'+P'\alpha'^2+Q''\alpha'^4$$
$$+Q_{,}\alpha\alpha'(\alpha^2+\beta^2)+Q'\alpha'^2(\alpha^2+\beta^2)+Q_{,,}\alpha'^2\alpha^2+Q_{,}'\alpha'^3\alpha;$$

and the equations of an emergent ray are, near the aplanatic focus,

$$\left.\begin{aligned} x&=\alpha z+P_{,}\alpha'+Q_{,}'\alpha'^3+Q_{,}\alpha'(3\alpha^2+\beta^2)+2(Q'+Q_{,,})\alpha\alpha'^2;\\ y&=\beta z+2\beta\alpha'(Q_{,}\alpha+Q'\alpha'). \end{aligned}\right\}$$

For the *principal* emergent rays and foci, we have therefore ($\alpha'$ not $=0$)

$$\left.\begin{aligned} 0&=z+2(Q'+Q_{,,})\alpha'^2+6\alpha\alpha'Q_{,};\\ 0&=\beta Q_{,};\\ 0&=z+2Q'\alpha'^2+2\alpha\alpha'Q_{,}; \end{aligned}\right\}$$

therefore, *unless* $Q_{,}=0$, we have one such ray and focus, for which

$$\beta=0, \quad \alpha=-\frac{Q_{,,}\alpha'}{2Q_{,}}, \quad Z=(Q_{,,}-2Q')\alpha'^2;$$

therefore $Y=0$, and

$$X=P_{,}\alpha'+Q_{,}'\alpha'^3+\alpha\alpha'^2\left(\frac{Z}{\alpha'^2}+2Q'+2Q_{,,}+3Q_{,}\frac{\alpha}{\alpha'}\right)$$

$$=P_{,}\alpha'+Q_{,}'\alpha'^3-\frac{Q_{,,}\alpha'^3}{2Q_{,}}(3Q_{,,}-\tfrac{3}{2}Q_{,,})$$

$$=P_{,}\alpha'+\left(Q_{,}'-\frac{3Q_{,,}^2}{4Q_{,}}\right)\alpha'^3;$$

consequently the equations of any near emergent ray may be put under the form

$$x-X-\alpha(z-Z)=\frac{3Q_{,,}^2\alpha'^3}{4Q_{,}}+3Q_{,}\alpha'\alpha^2+3Q_{,,}\alpha'^2\alpha+Q_{,}\alpha'\beta^2$$

$$=Q_{,}\alpha'\beta^2+3Q_{,}\alpha'\left(\alpha+\frac{Q_{,,}\alpha'}{2Q_{,}}\right)^2$$

$$=Q_{,}\alpha'\left\{3\left(\alpha+\frac{Q_{,,}\alpha'}{2Q_{,}}\right)^2+\beta^2\right\};$$

$$y-\beta(z-Z)=2Q_{,}\alpha'\alpha\beta+Q_{,,}\alpha'^2\beta$$

$$=2Q_{,}\alpha'\beta\left(\alpha+\frac{Q_{,,}\alpha'}{2Q_{,}}\right).$$

Thus *unless* $Q_{,}=0$, the equations of the emergent rays from the aplanatic object glass for oblique incident rays may, by removing the origin to the focus $X$, $Y$, $Z$, be thus written:

$$\left.\begin{aligned} x&=\alpha z+k(3\alpha'^2+\beta^2),\\ y&=\beta z+2k\alpha'\beta, \end{aligned}\right\}$$

in which

$$k = Q_{,}\alpha',$$

$$\alpha' = \alpha + \frac{Q_{,,}\alpha'}{2Q_{,}}.$$

*But if* $Q_{,} = 0$, then the equations of an emergent ray on last page become

$$x = \alpha z + P_{,}\alpha' + Q_{,}'\alpha'^3 + 2(Q' + Q_{,,})\alpha\alpha'^2,$$

$$y = \beta z + 2Q'\alpha'^2\beta;$$

and now the emergent rays may be considered as intersecting two distinct right lines, of which the equations are

Ist. $\qquad x = P_{,}\alpha' + Q_{,}'\alpha'^3, \quad z = -2(Q' + Q_{,,})\alpha'^2;$

and

IInd. $\qquad y = 0, \quad z = -2Q'\alpha'^2.$

*The object glass may therefore be considered as perfect,* if besides the aplanaticity for incident rays of no obliquity, we have the two following conditions*

$$Q_{,} = 0, \quad Q_{,,} = 0:$$

*and the coordinates of the aplanatic focus for incident inclined rays are then*

$$X = P_{,}\alpha' + Q_{,}'\alpha'^3; \quad Y = 0; \quad Z = -2Q'\alpha'^2.$$

*But it is probably impossible to satisfy rigorously and simultaneously the four following equations by any thin achromatic spheric lens,*

$$P = 0, \quad Q = 0, \quad Q_{,} = 0, \quad Q_{,,} = 0.$$

*If so, what relation between* $Q_{,}$, $Q_{,,}$ *would be most favourable to distinct vision for a star not in the axis of the object glass?*

Let us remove the origin to the point† having for coordinates

$$P_{,}\alpha' + Q_{,}'\alpha'^3, \quad 0, \quad -(Q_{,,} + 2Q')\alpha'^2,$$

without supposing that the conditions of aplanaticity are satisfied; then the equations of a near emergent ray, referred to axes parallel to the old axes of coordinates, will become, by last page,

$$\left.\begin{aligned} x &= \alpha z + Q_{,}\alpha'(3\alpha^2 + \beta^2) + Q_{,,}\alpha'^2\alpha, \\ y &= \beta z + 2Q_{,}\alpha'\alpha\beta - Q_{,,}\alpha'^2\beta. \end{aligned}\right\}$$

That is, putting for abridgment $Q_{,}\alpha' = k$, $Q_{,,}\alpha'^2 = l$,

$$\left.\begin{aligned} x &= \alpha(z + l) + k(3\alpha^2 + \beta^2), \\ y &= \beta(z - l) + 2k\alpha\beta. \end{aligned}\right\}$$

When $k$ and $l$ both vanish, the rays thus determined all pass through the new origin: but otherwise they do not, and we may propose to determine $k$ and $l$, so that the area occupied on the new

* [The conditions, in fact, ensure that the emergent rays all cut the plane $z = -2Q'\alpha'^2$ at a distance of the order of $\alpha'^5$ from the point $X$, $Y$, $Z$; thus, to the order of approximation considered, a perfect point image of an infinitely distant point is formed. The instrument is thus corrected for spherical aberration, coma and astigmatism. See also p. 457. The remaining defects, curvature and distortion, only require consideration when several object points are to be taken into account simultaneously.]

† [This is the middle point of the common perpendicular to the two lines mentioned above.]

plane of $xy$ by the rays for which $\alpha^2+\beta^2 \not> \theta^2$, $\theta$ being some given small quantity, shall be a minimum, $k$ and $l$ being connected by some given relation. And the result of such determination must, I think, conduce to the improvement of the achromatic object glass.

[Hamilton then proceeds to investigate the ruled surface formed by the emergent rays for which

$$\alpha^2+\beta^2=\theta^2,$$

by elimination of $\beta$ between this equation and the two given above. This yields

$$x=\alpha(z+l)+2k\alpha^2+k\theta^2,$$

$$x^2+y^2-4\alpha lx=\theta^2\{k^2\theta^2+(z-l)^2\}+6\alpha k\theta^2(z-l)+4\alpha^2(2k^2\theta^2-l^2).$$

Considering the special case where $l=0$, $k^2>0$, he eliminates $\alpha$ and finds for the surface in question the following equation:]

$$(x^2+y^2-4xk\theta^2+3k^2\theta^4)^2=z^2\theta^2(x^2+y^2-2xk\theta^2+k^2\theta^4),$$

that is

$$(x-k\theta^2)(x-3k\theta^2)+y^2=\pm z\theta\sqrt{(x-k\theta^2)^2+y^2},$$

therefore

$$(x-k\theta^2)^2+y^2-2k\theta^2(x-k\theta^2)=\pm z\theta\sqrt{(x-k\theta^2)^2+y^2}.$$

Is this any known curve of the 4th. degree?

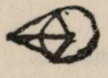
When $z=0$, it reduces itself simply to the following,

$$(x-k\theta^2)(x-3k\theta^2)+y^2=0,$$

that is

$$y=\pm\sqrt{(x-k\theta^2)(3k\theta^2-x)}:$$

it is therefore *a circle*, whose centre is at $2k\theta^2$, 0, 0, and whose radius $=k\theta^2$. The enveloppe of such circles, for different values of $\theta$, is composed of the two right lines*



$$y=\pm x\tan 30^\circ=\pm\frac{x}{\sqrt{3}};$$

and the whole space occupied on the plane of $xy$ (perpendicular to the central ray and passing through principal focus) by the near rays which make with central ray angles $\not> \theta$ is = two-thirds of the whole circular area + an equilateral triangle of side = diameter, that is,

$$=k^2\theta^4\left(\frac{2\pi}{3}+\sqrt{3}\right).$$

At the same time,

$$x-2k\theta^2=k(\alpha^2-\beta^2),\quad y=2k\alpha\beta,$$

that is

$$x-2k\theta^2=k\theta^2\cos 2\phi,$$

$$y=k\theta^2\sin 2\phi,$$

if

$$\alpha=\theta\cos\phi,\quad \beta=\theta\sin\phi;$$

therefore the upper point of contact with enveloppe, namely

$$x-2k\theta^2=-\tfrac{1}{2}k\theta^2,\quad y=k\theta^2\frac{\sqrt{3}}{2},$$

* [The phenomenon here described is now generally known as *coma*: see footnote to p. 48.]

corresponds to

$$\cos 2\phi = -\tfrac{1}{2}, \quad \sin 2\phi = \frac{\sqrt{3}}{2},$$

therefore

$$\phi = \frac{\pi}{3} \text{ or } = \frac{4\pi}{3}.$$

Therefore $\tan\phi = \frac{\beta}{\alpha} = \sqrt{3}$; and in like manner the lower point corresponds to $\frac{\beta}{\alpha} = -\sqrt{3}$; and in fact if we make $\beta = \pm\alpha\sqrt{3}$, in the equations

$$x = k(3\alpha^2 + \beta^2), \quad y = 2k\alpha\beta,$$

we find

$$x = 6k\alpha^2, \quad y = \pm 2\sqrt{3}\, k\alpha^2 = \pm\frac{x}{\sqrt{3}}.$$

If $z$ still $= 0$, then

$$\frac{\beta}{\alpha} = \tan\phi = \frac{\sin 2\phi}{1 + \cos 2\phi} = \frac{y}{x - k\theta^2}.$$

When $z^2 > 0$, $(k^2 > 0,\ l = 0)$ then putting $x - 2k\theta^2 = \xi$, we have

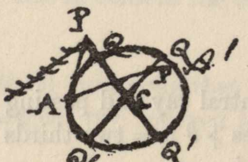
$$x - k\theta^2 = \xi + k\theta^2, \quad x - 3k\theta^2 = \xi - k\theta^2,$$

and the equation of the intersection of the plane $z =$ const. with the pencil $\theta =$ const. becomes

$$y^2 + \xi^2 - k^2\theta^4 = \pm z\theta\sqrt{y^2 + (\xi + k\theta^2)^2} = \pm\theta\sqrt{z^2}\sqrt{y^2 + (\xi + k\theta^2)^2};$$

*therefore the curves corresponding to* $z > 0$ *are exactly the same as those corresponding to* $z < 0$; and in fact it is evident from the equations of the rays that $x, y$ will not alter if we change at once the signs of $z, \alpha, \beta$.

The locus may be thus defined*




$$\frac{QPQ'}{AP} = \frac{\text{rectangle under segments}}{\text{distance from point } A \text{ of circumference}} = \theta\sqrt{z^2} = \text{constant}.$$

But $QPQ' = APA'$; therefore $PA' = \text{const.} = \theta\sqrt{z^2}$. Now, putting $AP = \rho$ $PAC = \psi$, and considering $AA'$, $PA'$, and $\rho$ as positive, we have $PA' = AA' \mp \rho$, when $\cos\psi \gtrless 0$. Therefore considering $\cos\psi$ as $> 0$, but $\rho$ as sometimes positive and sometimes negative, we may put

$$PA' = AA' \sim \rho,$$

$$\because\ \rho = AA' \pm PA',$$

$$\because\ \rho = 2k\theta^2\cos\psi \pm \theta\sqrt{z^2} = 2k\theta^2\cos\psi \pm \theta z,$$

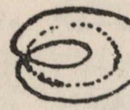
or, supposing for simplicity, $k > 0$, $z > 0$, as well as $\theta > 0$, and $\rho$, $\cos\psi$, as both capable of becoming negative,

$$\rho = 2k\theta^2\cos\psi + z\theta.$$

*The locus may therefore be constructed by drawing the chords* $AA'$ *from* $A$, *and then measuring forward and backward from* $A'$ *a constant length* $A'P = z\theta$.

* [The circle is $(x - 2k\theta^2)^2 + y^2 = k^2\theta^4$, and the point $A$ is $x = k\theta^2$, $y = 0$. The curve described by $P$ is a limaçon. For the early history of this curve, see F. G. Teixeira, *Courbes spéciales*, t. 1, pp. 199, 200.]

If this constant length be less than diameter, $z < 2k\theta$, the curve will have some such shape as this:



the dotted curve representing the generating circle.

But if $z > 2k\theta$, the curve will have no node at $A$, and will lie entirely outside the generating circle.

In the case $z < 2k\theta$, $\rho$ is positive for the outer part of the curve, but negative for the inner part or loop: in the case $z > 2k\theta$, $\rho$ is positive throughout.

To find the point or points of contact of the curve with its enveloppe, we are to observe that when $\theta$ becomes $\theta + \delta\theta$, the pole of $\rho$, $\theta$* is pushed on through the quantity $\delta x = 2k\theta\,\delta\theta$; so that, for an unchanged point $P$, $\rho$ becomes

$$\rho - \cos\psi \,.\, \delta x = \rho - \cos\psi \,.\, 2k\theta \,.\, \delta\theta,$$

and $\psi$ becomes

$$\psi + \frac{\sin\psi \,.\, \delta x}{\rho} = \psi + \frac{\sin\psi}{\rho} \,.\, 2k\theta \,.\, \delta\theta,$$

at least when $\rho > 0$; and even if $\rho < 0$ (inner loop) similar formulæ hold. Thus the equation

$$\rho = 2k\theta^2 \cos\psi + z\theta$$

gives, by differentiation,

$$6k\theta\cos\psi + z = \frac{4k^2\theta^3\sin\psi^2}{\rho} = \frac{4k^2\theta^2\sin\psi^2}{2k\theta\cos\psi + z};$$

$$\therefore\ 0 = (z + 2k\theta\cos\psi)(z + 6k\theta\cos\psi) + 4k^2\theta^2\cos\psi^2 - 4k^2\theta^2$$

$$= z^2 + 8zk\theta\cos\psi + 16k^2\theta^2\cos\psi^2 - 4k^2\theta^2,$$

$$\therefore\ (z + 4k\theta\cos\psi)^2 = 4k^2\theta^2,$$

$$\therefore\ \cos\psi = \pm\tfrac{1}{2} - \frac{z}{4k\theta},$$

$$\therefore\ \rho = \pm k\theta^2 + \tfrac{1}{2}z\theta;$$

$$\therefore\ \rho\left(\tfrac{1}{2} \mp \cos\psi\right) = \left(k\theta^2 \pm \tfrac{1}{2}z\theta\right)\frac{z}{4k\theta} = \tfrac{1}{4}z\theta \pm \frac{z^2}{8k},$$

$$\therefore\ 4k\rho\,(1 \mp 2\cos\psi) \mp z^2 = 2zk\theta = \frac{2k\theta}{z} \times z^2 = z^2\left(\frac{z}{2k\theta}\right)^{-1}$$

$$= \frac{z^2}{\pm 1 - 2\cos\psi} = \frac{\pm z^2}{1 \mp 2\cos\psi},$$

$$\therefore\ 2k\rho\,(1 \mp 2\cos\psi) = \frac{\pm z^2(1 \mp \cos\psi)}{1 \mp 2\cos\psi}.$$

Therefore finally the equation obtained by eliminating $\theta$ between the expressions for $\rho$, $\psi$ of the curve may be thus written

$$2k\rho\,(1 \mp 2\cos\psi)^2 = \pm z^2(1 \mp \cos\psi).$$

* [Read $\psi$.]

But this equation still involves $\theta$, because the pole of $\rho$, $\psi$ depends on it. We are therefore to eliminate $\rho$, $\psi$, $\theta$ between the four following equations,

$$\rho\cos\psi = x - k\theta^2, \quad \rho\sin\psi = y,$$

$$\cos\psi = \pm\tfrac{1}{2} - \frac{z}{4k\theta}, \quad \rho = \pm k\theta^2 + \tfrac{1}{2}z\theta.$$

The two last give

$$\rho\cos\psi = \tfrac{1}{2}k\theta^2 - \frac{z^2}{8k} = x - k\theta^2;$$

$$\therefore \tfrac{3}{2}k\theta^2 = x + \frac{z^2}{8k},$$

$$\therefore \theta^2 = \frac{2x}{3k} + \frac{z^2}{12k^2},$$

$$\therefore x - k\theta^2 = \frac{x}{3} - \frac{z^2}{12k},$$

$$\therefore \rho^2 = y^2 + \left(\frac{x}{3} - \frac{z^2}{12k}\right)^2;$$

also

$$\rho = \pm\left(\frac{2x}{3} + \frac{z^2}{12k}\right) + \tfrac{1}{2}z\sqrt{\frac{2x}{3k} + \frac{z^2}{12k^2}}.$$

[Hence he deduces the equation of the envelope to be:]

$$0 = 9y^4 - 6y^2\left(x^2 + \frac{xz^2}{k} + \frac{z^4}{16k^2}\right) + x^4 - \tfrac{2}{3}\frac{x^3z^2}{k} + \tfrac{1}{8}\frac{x^2z^4}{k^2} - \frac{z^8}{768k^4}.$$

*The equation*

$$x = \tfrac{3}{2}k\theta^2 - \frac{z^2}{8k},$$

*shews that the four points, when real, in which the locus touches its enveloppe, are on one common parallel to the axis of y.* And we might obtain the equation of the enveloppe by eliminating $\theta^2$ between the two equations

$$x = \tfrac{3}{2}k\theta^2 - \frac{z^2}{8k},$$

$$(x - k\theta^2)(x - 3k\theta^2) + y^2 = \pm z\theta\sqrt{(x - k\theta^2)^2 + y^2}.$$

[This he proceeds to do, obtaining the equation given above, which he also throws into the form:]

$$y^2 = \tfrac{1}{3}x^2 + \frac{xz^2}{3k} + \frac{z^4}{48k^2} \pm zk\left(\frac{2x}{3k} + \frac{z^2}{12k^2}\right)^{\frac{3}{2}}.$$