XXI.

ON THE IMPROVEMENT OF THE DOUBLE ACHROMATIC OBJECT GLASS*

Feb. 9, 1844.

[Note Book 28, pp. 151–209.]

[1.] Single refractor: T for sphere, for any surface of revolution, for reflecting surface.
[2.] Alternative method: $\Delta T^{(2)}$, $\Delta T^{(4)}$ for any refracting surface of revolution.
[3.] Lens of revolution, preliminary.
[4.] $T^{(2)}$ for lens of revolution.
[5.] Alternative method. Foci.
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[7.] Combination of two lenses *in vacuo*: $T^{(2)}$, focal centres, focal length.
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* [This was the title of a paper read to the Royal Irish Academy on June 24, 1844, but never published. The manuscript which follows (obviously not prepared for publication) probably represents the work which led to that paper, and for that reason we prefix to the manuscript the title of the paper. There are no numbered sections in the manuscript; the sections as now printed correspond to pages in the note book. Slight verbal alterations to suit this mode of reference have been made in the text without comment. The formulæ underlined by Hamilton have been enclosed in rectangles. The synopsis of contents has been supplied by the Editors.]

[1.] *Single Refractor.*

Rigorous Equations.*

$$\begin{cases} \Delta T = x\Delta\sigma + y\Delta\tau + z\Delta v, \\ 0 \;\;= \delta x\Delta\sigma + \delta y\Delta\tau + \delta z\Delta v, \\ \delta z \;= p\delta x + q\delta y, \quad z - px - qy = f(-p, -q) \\ \Delta\sigma = -p\Delta v, \quad \Delta\tau = -q\Delta v, \\ \Delta T = (z - px - qy)\,\Delta v = \Delta v\, f\left(\dfrac{\Delta\sigma}{\Delta v}, \dfrac{\Delta\tau}{\Delta v}\right). \end{cases}$$

Ex. 1. Let

$$z = v + r^{-1}\{1 - \sqrt{1 - r^2(x^2 + y^2)}\} = c - r^{-1}\sqrt{1 - r^2(x^2+y^2)},$$

v being ordinate of vertex of hemispheric surface, c ordinate of centre, $r^{-1} = c - v =$ [radius of]

* [Cf. Third Supplement, (I^7), (K^7), p. 216. These equations are general, but the rest of the work deals with an instrument of revolution.]

curvature, positive when surface is concave upwards, so that rays proceeding upwards fall upon its convexity. The radical is supposed to be positive. Then

$$(z-c)^2+x^2+y^2=r^{-2},\quad p=-\frac{x}{z-c},\quad q=-\frac{y}{z-c},$$

$$z-px-qy=c+(z-c)\left\{1+\frac{x^2+y^2}{(z-c)^2}\right\}=c+\frac{r^{-2}}{z-c},$$

$$p^2+q^2=-1+\frac{r^{-2}}{(z-c)^2};$$

$z-c$ is negative if r be positive, and reciprocally,

$${}^*\because\ \sqrt{1+p^2+q^2}=\frac{-r^{-1}}{z-c},$$

$$\because\ f(-p,-q)=c-r^{-1}\sqrt{1+p^2+q^2},$$

$$\because\ \Delta T=c\Delta v-r^{-1}\Delta v\sqrt{1+\frac{\Delta\sigma^2+\Delta\tau^2}{\Delta v^2}},$$

rigorously, for a *refracting hemisphere* (as I have often found before,) and indeed for a reflecting hemisphere, and for all laws of refraction or reflexion, ordinary or extraordinary.

Ex. 2. Let

$$z=v+\tfrac{1}{2}r(x^2+y^2)+\tfrac{1}{4}s(x^2+y^2)^2;$$

then

$$\frac{p}{x}=\frac{q}{y}=r+s(x^2+y^2),\quad px+qy=r(x^2+y^2)+s(x^2+y^2)^2,$$

$$z-px-qy=v-\tfrac{1}{2}r(x^2+y^2)-\tfrac{3}{4}s(x^2+y^2)^2,$$

$$p^2+q^2=r^2(x^2+y^2)+2rs(x^2+y^2)^2+s^2(x^2+y^2)^3,$$

$$(p^2+q^2)^2=r^4(x^2+y^2)^2+\&c.;$$

therefore, neglecting $(x^2+y^2)^3$, we have

$$(x^2+y^2)^2=r^{-4}(p^2+q^2)^2,\quad x^2+y^2=r^{-2}(p^2+q^2)-2r^{-5}s(p^2+q^2)^2,$$

$$f(-p,-q)=v-\frac{p^2+q^2}{2r}+\frac{s(p^2+q^2)^2}{4r^4},$$

$$\because\ \Delta T=v\Delta v-\frac{\Delta\sigma^2+\Delta\tau^2}{2r\Delta v}+\frac{s(\Delta\sigma^2+\Delta\tau^2)^2}{4r^4\Delta v^3},$$

approximately, for a *surface of revolution*.

By making $s=\tfrac{1}{2}r^3$, the ellipticity vanishes, and we get

$$\Delta T=(c-r^{-1})\Delta v-r^{-1}\left(\frac{\Delta\sigma^2+\Delta\tau^2}{2\Delta v}-\frac{(\Delta\sigma^2+\Delta\tau^2)^2}{8\Delta v^3}\right),$$

as by developing the rigorous radical expression in the last example.

* [See footnote to p. 370.]

Ex. 3. Let there be a single *reflecting* surface*; then v_0 and v_1 will have opposite signs, and we may suppose, considering the τ's as vanishing,†

$$v_0 = -\sqrt{1-\sigma_0^2} = -1 + \tfrac{1}{2}\sigma_0^2 + \tfrac{1}{8}\sigma_0^4,$$

$$v_1 = \sqrt{1-\sigma_1^2} = 1 - \tfrac{1}{2}\sigma_1^2 - \tfrac{1}{8}\sigma_1^4,$$

$$\Delta v = 2 - \tfrac{1}{2}(\sigma_1^2 + \sigma_0^2) - \tfrac{1}{8}(\sigma_1^4 + \sigma_0^4),$$

$$T = 2v - \frac{v}{2}(\sigma_1^2 + \sigma_0^2) - \frac{(\sigma_1 - \sigma_0)^2}{4r} - \frac{v}{8}(\sigma_1^4 + \sigma_0^4) - \frac{(\sigma_1-\sigma_0)^2(\sigma_1^2+\sigma_0^2)}{16r} + \frac{s(\sigma_1-\sigma_0)^4}{32r^4};$$

and for reflected ray,

$$x = \sigma_1 z + \tfrac{1}{2}\sigma_1^3 z + \frac{\delta T}{\delta \sigma_1}.$$

If incident rays be parallel to axis, then

$$\sigma_0 = 0,$$

$$T = 2v - \tfrac{1}{2}v\sigma_1^2 - \tfrac{1}{4}r^{-1}\sigma_1^2 - \tfrac{1}{8}v\sigma_1^4 - \frac{r^{-1}}{16}\sigma_1^4 + \frac{r^{-4}s}{32}\sigma_1^4;$$

$$x = \sigma_1(z - v - \tfrac{1}{2}r^{-1}) + \tfrac{1}{2}\sigma_1^3(z - v - \tfrac{1}{2}r^{-1} + \tfrac{1}{4}r^{-4}s);$$

and when

$$z = v + \tfrac{1}{2}r^{-1} = \text{ordinate of principal focus},$$

then

$$x = \text{lateral aberration} = \tfrac{1}{8}r^{-4}s\sigma_1^3;$$

or, reciprocally, when $x = 0$, then

$$z - v - \tfrac{1}{2}r^{-1} = \text{longitudinal aberration} = -\tfrac{1}{8}r^{-4}s\sigma_1^2.$$

For hemisphere, this last aberration $= -\frac{1}{16}r^{-1}\sigma_1^2$; for paraboloid, it vanishes. (Rigorously, for hemisphere, longitudinal aberration $= \frac{1}{2}r^{-1}(1 - \sec\frac{1}{2}\sin^{-1}\sigma_1)$.)

[2.] Another method of developing ΔT, as far as small quantities of the 4th. dimension inclusive, is to make

$$z = v + z^{(2)} + z^{(4)},$$

$$v = \mu + v^{(2)} + v^{(4)},$$

$$\Delta v = \Delta\mu + \Delta v^{(2)} + \Delta v^{(4)},$$

$$\Delta T = \Delta T^{(0)} + \Delta T^{(2)} + \Delta T^{(4)}, \quad \Delta T^{(0)} = v\Delta\mu,$$

$$\Delta T^{(2)} = x\Delta\sigma + y\Delta\tau + v\Delta v^{(2)} + z^{(2)}\Delta\mu, \quad z^{(2)} = \tfrac{1}{2}r(x^2+y^2),$$

$$\Delta T^{(4)} = v\Delta v^{(4)} + z^{(2)}\Delta v^{(2)} + z^{(4)}\Delta\mu, \quad z^{(4)} = \tfrac{1}{4}s(x^2+y^2)^2,$$

$$\Delta\sigma + rx\Delta\mu = 0, \quad \Delta\tau + ry\Delta\mu = 0,$$

these last two equations giving values of x, y, which are not indeed rigorous, but of which the want of rigour does not affect our present results;‡ hence,

$$x\Delta\sigma + y\Delta\tau = \frac{\Delta\sigma^2 + \Delta\tau^2}{-r\Delta\mu}, \quad z^{(2)}\Delta\mu = \frac{\Delta\sigma^2 + \Delta\tau^2}{2r\Delta\mu},$$

* [of revolution *in vacuo*].

† [That is, considering only rays in the diametral plane $y=0$. The medium lies on the positive side of the mirror.]

‡ [It is obvious that the substitution of the approximate values in $\Delta T^{(4)}$ introduces no error of the fourth order; but it is not so obvious in the case of $\Delta T^{(2)}$. If, however, we substitute more exact values for x, y from

and finally

$$\begin{cases} \Delta T^{(2)} = v\Delta v^{(2)} - \dfrac{\Delta\sigma^2 + \Delta\tau^2}{2r\Delta\mu}, \\ \Delta T^{(4)} = v\Delta v^{(4)} + \dfrac{(\Delta\sigma^2 + \Delta\tau^2)\,\Delta v^{(2)}}{2r\Delta\mu^2} + \dfrac{s\,(\Delta\sigma^2 + \Delta\tau^2)^2}{4r^4\Delta\mu^3}. \end{cases}$$

Also, for ordinary refraction or reflexion,

$$v = \mu\sqrt{1 - \frac{\sigma^2 + \tau^2}{\mu^2}};$$

$$\therefore\ v^{(2)} = -\frac{\sigma^2 + \tau^2}{2\mu};\quad v^{(4)} = -\frac{(\sigma^2 + \tau^2)^2}{8\mu^3}.$$

Thus, more explicitly, for *any ordinary refractor or reflector of revolution,*

$$\begin{cases} \Delta T^{(2)} = -\dfrac{v}{2}\Delta\dfrac{\sigma^2 + \tau^2}{\mu} - \dfrac{r^{-1}}{2}\dfrac{\Delta\sigma^2 + \Delta\tau^2}{\Delta\mu}; \\ \Delta T^{(4)} = -\dfrac{v}{8}\Delta\dfrac{(\sigma^2 + \tau^2)^2}{\mu^3} - \dfrac{r^{-1}}{4}\dfrac{\Delta\sigma^2 + \Delta\tau^2}{\Delta\mu^2}\Delta\dfrac{\sigma^2 + \tau^2}{\mu} + \dfrac{sr^{-4}}{4}\dfrac{(\Delta\sigma^2 + \Delta\tau^2)^2}{\Delta\mu^3}. \end{cases}$$

But we may also write, more concisely,*

$$\begin{cases} \Delta T^{(2)} = v\Delta v^{(2)} - z^{(2)}\Delta\mu; \\ \Delta T^{(4)} = v\Delta v^{(4)} + z^{(2)}\Delta v^{(2)} + z^{(4)}\Delta\mu; \end{cases}$$

or, more symmetrically,

$$\begin{cases} \Delta T^{(2)} = z^{(0)}\Delta v^{(2)} - z^{(2)}\Delta v^{(0)}; \\ \Delta T^{(4)} = z^{(0)}\Delta v^{(4)} + z^{(2)}\Delta v^{(2)} + z^{(4)}\Delta v^{(0)}; \end{cases}$$

in which $z^{(0)}$, $z^{(2)}$, $z^{(4)}$ are the three first terms of the development of the ordinate z of the surface, and $v^{(0)}$, $v^{(2)}$, $v^{(4)}$ are the three first terms of the development of the component v of normal slowness of the wave; while $z^{(2)}$ in $\Delta T^{(2)}$ is to receive its *first approximate value,* obtained by substituting for x, y, their own first approximate values, deduced from the two equations comprised in the formula

$$\Delta\sigma\,\delta x + \Delta\tau\,\delta y + \Delta v^{(0)}\,\delta z^{(2)} = 0.$$

[3.] *Lens of Revolution.*

Foci, Images, Focal Centres.

Now, let there be a lens. For it,

$$\boxed{\begin{aligned} T^{(2)} &= z_1^{(0)}(v_1^{(2)} - v_0^{(2)}) - z_1^{(2)}(v_1^{(0)} - v_0^{(0)}) \\ &\quad + z_2^{(0)}(v_2^{(2)} - v_1^{(2)}) - z_2^{(2)}(v_2^{(0)} - v_1^{(0)}); \end{aligned}}$$

$\Delta\sigma = -p\,\Delta v$, $\Delta\tau = -q\,\Delta v$, we find that the additional terms introduced cancel out, and we arrive at the expression for $\Delta T^{(2)}$ which follows. For a general justification of the method, see Appendix, Note 24, p. 507. Cf. also [15.] of the present paper.]

* [These equations are the same as those at the top of the page.]

in which $z_1^{(2)}$ is a function* of $\sigma_1-\sigma_0$, $\tau_1-\tau_0$; and $z_2^{(2)}$ is a function of $\sigma_2-\sigma_1$, $\tau_2-\tau_1$; also σ_1, τ_1 are to be eliminated by the condition† that

$$\delta_1 T^{(2)}=0,$$

or more fully that

$$0=(z_2^{(0)}-z_1^{(0)})\,\delta\upsilon_1^{(2)}+(\upsilon_1^{(0)}-\upsilon_0^{(0)})\,\delta_1 z_1^{(2)}+(\upsilon_2^{(0)}-\upsilon_1^{(0)})\,\delta_1 z_2^{(2)},$$

δ_1 referring only to the variations of σ_1, τ_1. More concisely, if t be thickness of lens,

$$0=t\delta\upsilon_1^{(2)}+\Delta\mu_0\delta_1 z_1^{(2)}+\Delta\mu_1\delta_1 z_2^{(2)}.$$

Now

$$\delta z^{(2)}=p\,\delta x+q\,\delta y,\quad 2z^{(2)}=px+qy,\quad \because\ \delta z^{(2)}=x\,\delta p+y\,\delta q,$$

$$p=-\frac{\Delta\sigma}{\Delta\mu},\quad q=-\frac{\Delta\tau}{\Delta\mu},\quad \because ‡\ \Delta\mu\,\delta z^{(2)}=-(x\,\delta\Delta\sigma+y\,\delta\Delta\tau);$$

$$\because\ \Delta\mu_0\delta_1 z_1^{(2)}=-(x_1\delta\sigma_1+y_1\delta\tau_1);$$

$$\Delta\mu_1\delta_1 z_2^{(2)}=+(x_2\delta\sigma_1+y_2\delta\tau_1);$$

also

$$\upsilon_1^{(2)}=-\frac{\sigma_1^2+\tau_1^2}{2\mu_1},\quad \delta\upsilon_1^{(2)}=-\frac{\sigma_1\delta\sigma_1+\tau_1\delta\tau_1}{\mu_1};$$

hence

$$\mu_1^{-1}t\,(\sigma_1\delta\sigma_1+\tau_1\delta\tau_1)=(x_2-x_1)\,\delta\sigma_1+(y_2-y_1)\,\delta\tau_1;$$

that is

$$\boxed{0=\mu_1(x_2-x_1)-t\sigma_1,\quad 0=\mu_1(y_2-y_1)-t\tau_1.}$$

Another mode of considering the question is to observe that we have *rigorously*, if x_1, y_1, z_1, x_2, y_2, z_2, σ_1, τ_1, υ_1 be rigorous, the equation

$$0=(x_2-x_1)\,\delta\sigma_1+(y_2-y_1)\,\delta\tau_1+(z_2-z_1)\,\delta\upsilon_1;$$

therefore also *rigorously*

$$\frac{x_2-x_1}{z_2-z_1}=\frac{\sigma_1}{\upsilon_1};\quad \frac{y_2-y_1}{z_2-z_1}=\frac{\tau_1}{\upsilon_1};$$

that is, *rigorously*,§

$$\frac{x_2-x_1}{\sigma_1}=\frac{y_2-y_1}{\tau_1}=\frac{z_2-z_1}{\upsilon_1};$$

if then we change the last fraction to $\dfrac{t}{\mu_1}$, or simply to $\dfrac{t}{\mu}$, if μ be the index of the lens, supposed *in vacuo*, and substitute for x_1, y_1, x_2, y_2 their 1st. approximate values,|| we shall obtain corresponding approximate values for σ_1, τ_1, as linear functions of σ_0, τ_0, σ_2, τ_2, which will be sufficient to give $T^{(2)}$ and even $T^{(4)}$ to the required degree of accuracy.

* [$z_1^{(2)}=\dfrac{(\sigma_1-\sigma_0)^2+(\tau_1-\tau_0)^2}{2r_1(\upsilon_1^{(0)}-\upsilon_0^{(0)})^2}$, $\quad z_2^{(2)}=\dfrac{(\sigma_2-\sigma_1)^2+(\tau_2-\tau_1)^2}{2r_2(\upsilon_2^{(0)}-\upsilon_1^{(0)})^2}$.]

† [Cf. Third Supplement, 11, p. 217.]

‡ [The media being homogeneous, $\delta\Delta\mu=0$.]

§ [These equations are evident, since σ_1, τ_1, υ_1 are proportional to the direction cosines of the ray.]

|| [From the equations $\Delta\sigma+rx\Delta\mu=0$, $\Delta\tau+ry\Delta\mu=0$, of [2.].]

[4.] Thus for a lens of revolution *in vacuo*, $\mu_0 = \mu_2 = 1$, $\mu_1 = \mu$, we have

$$\left\{\begin{array}{l} T^{(2)} = v_2 v_2^{(2)} - v_1 v_0^{(2)} - t v_1^{(2)} + (\mu - 1)(z_2^{(2)} - z_1^{(2)}); \\ t\sigma_1 = \mu (x_2 - x_1), \quad t\tau_1 = \mu (y_2 - y_1); \\ r_1 (\mu - 1) x_1 = \sigma_0 - \sigma_1, \quad r_1 (\mu - 1) y_1 = \tau_0 - \tau_1; \\ r_2 (\mu - 1) x_2 = \sigma_2 - \sigma_1, \quad r_2 (\mu - 1) y_2 = \tau_2 - \tau_1; \\ z_1^{(2)} = \tfrac{1}{2} r_1 (x_1^2 + y_1^2), \quad z_2^{(2)} = \tfrac{1}{2} r_2 (x_2^2 + y_2^2); \\ v_0^{(2)} = -\tfrac{1}{2}(\sigma_0^2 + \tau_0^2), \quad v_1^{(2)} = -\dfrac{1}{2\mu}(\sigma_1^2 + \tau_1^2), \quad v_2^{(2)} = -\tfrac{1}{2}(\sigma_2^2 + \tau_2^2). \end{array}\right.$$

Eliminating, we find

$$-\mu^{-1}(\mu - 1) r_1 r_2 t \sigma_1 = r_2(\sigma_0 - \sigma_1) - r_1(\sigma_2 - \sigma_1), \text{ \&c.},$$

$$\because \quad R\sigma_1 = r_1\sigma_2 - r_2\sigma_0, \quad R\tau_1 = r_1\tau_2 - r_2\tau_0,$$

if we make

$$R = r_1 - r_2 + (1 - \mu^{-1}) r_1 r_2 t.$$

Hence

$$r_1^{-1} R (\sigma_0 - \sigma_1) = (R + r_2) r_1^{-1} \sigma_0 - \sigma_2 = \sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_2 t \sigma_0,$$

$$r_2^{-1} R (\sigma_2 - \sigma_1) = (R - r_1) r_2^{-1} \sigma_2 + \sigma_0 = \sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_1 t \sigma_2;$$

$$\because \quad (\mu - 1) R x_1 = \sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_2 t \sigma_0; \quad (\mu - 1) R y_1 = \tau_0 - \tau_2 + (1 - \mu^{-1}) r_2 t \tau_0;$$

$$(\mu - 1) R x_2 = \sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_1 t \sigma_2; \quad (\mu - 1) R y_2 = \tau_0 - \tau_2 + (1 - \mu^{-1}) r_1 t \tau_2.$$

As a verification, these give

$$\mu (x_2 - x_1) = t R^{-1} (r_1 \sigma_2 - r_2 \sigma_0) = t \sigma_1.$$

We have now the system of expressions

$$\left\{\begin{array}{ll} \sigma_1 = R^{-1}(r_1\sigma_2 - r_2\sigma_0); & \tau_1 = R^{-1}(r_1\tau_2 - r_2\tau_0); \\ x_1 = \dfrac{\sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_2 t \sigma_0}{(\mu - 1) R}; & y_1 = \dfrac{\tau_0 - \tau_2 + (1 - \mu^{-1}) r_2 t \tau_0}{(\mu - 1) R}; \\ x_2 = \dfrac{\sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_1 t \sigma_2}{(\mu - 1) R}; & y_2 = \dfrac{\tau_0 - \tau_2 + (1 - \mu^{-1}) r_1 t \tau_2}{(\mu - 1) R}; \end{array}\right.$$

and therefore

$$\begin{aligned} 2\mu (\mu - 1) R^2 (T^{(2)} - v_2 v_2^{(2)} + v_1 v_0^{(2)}) &= (\mu - 1) t \{(r_1\sigma_2 - r_2\sigma_0)^2 + (r_1\tau_2 - r_2\tau_0)^2\} \\ &\quad + \mu r_2 \{(\sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_1 t \sigma_2)^2 + (\tau_0 - \tau_2 + (1 - \mu^{-1}) r_1 t \tau_2)^2\} \\ &\quad - \mu r_1 \{(\sigma_0 - \sigma_2 + (1 - \mu^{-1}) r_2 t \sigma_0)^2 + (\tau_0 - \tau_2 + (1 - \mu^{-1}) r_2 t \tau_0)^2\} \\ &= -\mu (r_1 - r_2)\{(\sigma_2 - \sigma_0)^2 + (\tau_2 - \tau_0)^2\} + \mu (1 - \mu^{-1})^2 r_1 r_2 t^2 \{r_1(\sigma_2^2 + \tau_2^2) - r_2(\sigma_0^2 + \tau_0^2)\} \\ &\quad + (\mu - 1) t \times (\&c.); \end{aligned}$$

and this will be

$$= -\mu R \{(\sigma_2 - \sigma_0)^2 + (\tau_2 - \tau_0)^2 - (1 - \mu^{-1}) t (r_1(\sigma_2^2 + \tau_2^2) - r_2(\sigma_0^2 + \tau_0^2))\},$$

if

$$(r_1\sigma_2 - r_2\sigma_0)^2 + (r_1\tau_2 - r_2\tau_0)^2 - 2 r_1 r_2 \{(\sigma_0 - \sigma_2)^2 + (\tau_0 - \tau_2)^2\},$$

which is the (&c.), shall be found to be

$$= (r_1 - r_2)(r_1(\sigma_2^2 + \tau_2^2) - r_2(\sigma_0^2 + \tau_0^2)) - r_1 r_2 ((\sigma_2 - \sigma_0)^2 + (\tau_2 - \tau_0)^2);$$

and, in fact, each

$$= r_1^2(\sigma_2^2 + \tau_2^2) + r_2^2(\sigma_0^2 + \tau_0^2) - 2 r_1 r_2 (\sigma_2^2 + \tau_2^2 - \sigma_0\sigma_2 - \tau_0\tau_2 + \sigma_0^2 + \tau_0^2).$$

Hence, finally, *for any lens of revolution in vacuo*, changing $\sigma_0, \tau_0, \sigma_2, \tau_2$ to $\alpha_0, \beta_0, \alpha_2, \beta_2$, we have

$$T^{(2)} = -\tfrac{1}{2}v_2(\alpha_2^2+\beta_2^2) + \tfrac{1}{2}v_1(\alpha_0^2+\beta_0^2) - \frac{(\alpha_2-\alpha_0)^2+(\beta_2-\beta_0)^2}{2(\mu-1)R} + \frac{t\{r_1(\alpha_2^2+\beta_2^2) - r_2(\alpha_0^2+\beta_0^2)\}}{2\mu R};$$

$$R = r_1 - r_2 + (1-\mu^{-1})r_1 r_2 t; \quad t = v_2 - v_1.$$

[5.] Another mode of eliminating σ_1, τ_1 is to form first the explicit expression (σ, τ being written instead of σ_1, τ_1),

$$T^{(2)} = \tfrac{1}{2}v_1(\alpha_0^2+\beta_0^2) - \tfrac{1}{2}v_2(\alpha_2^2+\beta_2^2) + \frac{t}{2\mu}(\sigma^2+\tau^2) + \frac{(\sigma-\alpha_2)^2+(\tau-\beta_2)^2}{2(\mu-1)r_2} - \frac{(\sigma-\alpha_0)^2+(\tau-\beta_0)^2}{2(\mu-1)r_1}$$

$$= \frac{R(\sigma^2+\tau^2) - 2r_1(\alpha_2\sigma+\beta_2\tau) + 2r_2(\alpha_0\sigma+\beta_0\tau)}{2(\mu-1)r_1r_2} + \left(v_1 - \frac{r_1^{-1}}{\mu-1}\right)\frac{\alpha_0^2+\beta_0^2}{2} - \left(v_2 - \frac{r_2^{-1}}{\mu-1}\right)\frac{\alpha_2^2+\beta_2^2}{2};$$

but

$$R\sigma = r_1\alpha_2 - r_2\alpha_0,$$

$$\therefore \; R\sigma^2 - 2(r_1\alpha_2 - r_2\alpha_0)\sigma = -\frac{(r_1\alpha_2-r_2\alpha_0)^2}{R},$$

and

$$-2(\mu-1)r_1r_2R\left(T^{(2)} - \frac{v_1(\alpha_0^2+\beta_0^2)}{2} + \frac{v_2(\alpha_2^2+\beta_2^2)}{2}\right)$$
$$= (r_1\alpha_2 - r_2\alpha_0)^2 + (r_1\beta_2 - r_2\beta_0)^2 + R\{r_2(\alpha_0^2+\beta_0^2) - r_1(\alpha_2^2+\beta_2^2)\};$$

which is already under a tolerably convenient form. But substituting for R its value

$$r_1 - r_2 + (1-\mu^{-1})r_1r_2t,$$

we are conducted to the reduction

$$(r_1\alpha_2 - r_2\alpha_0)^2 + (r_1 - r_2)(r_2\alpha_0^2 - r_1\alpha_2^2) = r_1r_2(\alpha_0^2 - 2\alpha_0\alpha_2 + \alpha_2^2) = r_1r_2(\alpha_2-\alpha_0)^2;$$

so that the second member of the recent expression becomes

$$r_1r_2\{(\alpha_2-\alpha_0)^2 + (\beta_2-\beta_0)^2\} + (1-\mu^{-1})r_1r_2t\{r_2(\alpha_0^2+\beta_0^2) - r_1(\alpha_2^2+\beta_2^2)\};$$

consequently

$$2T^{(2)} = v_1(\alpha_0^2+\beta_0^2) - v_2(\alpha_2^2+\beta_2^2) - \frac{(\alpha_2-\alpha_0)^2+(\beta_2-\beta_0)^2}{(\mu-1)R} - \frac{t\{r_2(\alpha_0^2+\beta_0^2) - r_1(\alpha_2^2+\beta_2^2)\}}{\mu R};$$

which may also be written thus:

$$T^{(2)} = \left(v_1 - \frac{r_2t}{\mu R}\right)\frac{\alpha_0^2+\beta_0^2}{2} - \left(v_2 - \frac{r_1t}{\mu R}\right)\frac{\alpha_2^2+\beta_2^2}{2} - \frac{(\alpha_2-\alpha_0)^2+(\beta_2-\beta_0)^2}{2(\mu-1)R}.$$

*Such is the function $T^{(2)}$ for a lens of revolution in vacuo; index μ; curvatures r_1, r_2, positive when convex to incident light; ordinates of vertices v_1, v_2; thickness $t = v_2 - v_1$; $R = r_1 - r_2 + (1 - \mu^{-1}) r_1 r_2 t$; α_0, β_0, direction cosines for incident ray, and α_2, β_2 for emergent; approximate equations of incident ray**

$$x_0 - \alpha_0 z_0 = -\frac{\delta T^{(2)}}{\delta \alpha_0}, \quad y_0 - \beta_0 z_0 = -\frac{\delta T^{(2)}}{\delta \beta_0};$$

and approximate equations of emergent ray

$$x_3 - \alpha_2 z_3 = +\frac{\delta T^{(2)}}{\delta \alpha_2}, \quad y_3 - \beta_2 z_3 = +\frac{\delta T^{(2)}}{\delta \beta_2}.$$

Parallel incident rays converge to (or diverge from) the focus

$$X_3 = \frac{\alpha_0}{(\mu - 1) R}, \quad Y_3 = \frac{\beta_0}{(\mu - 1) R}, \quad Z_3 = v_2 - \frac{r_1 t}{\mu R} + \frac{1}{(\mu - 1) R};$$

and the emergent rays are parallel, if incident diverge from (or converge to)

$$X_0 = \frac{-\alpha_2}{(\mu - 1) R}, \quad Y_0 = \frac{-\beta_2}{(\mu - 1) R}, \quad Z_0 = v_1 - \frac{r_2 t}{\mu R} - \frac{1}{(\mu - 1) R}.$$

[6.] $v_1 - \frac{r_2 t}{\mu R}$, and $v_2 - \frac{r_1 t}{\mu R}$, in the expression for $T^{(2)}$, are the ordinates of two points on the axis, which are sometimes called the *focal centres* of the lens. *They are the points in which the axis is intersected by the directions of the incident and emergent rays, respectively, when those two directions are parallel to each other;*† and it is not difficult to deduce their ordinates by geometrical considerations. And $\frac{1}{(\mu - 1) R}$ may not improperly be called the *focal length*, or $(\mu - 1) R$ the *power*, of the lens. This focal length × the sine of the semi-diameter of a planet, will give the radius of its image formed by the lens.‡ This image will remain unaltered in magnitude when the lens is reversed.

For the case of a *sphere*, the two focal centres ought to coincide in the centre of the sphere. Accordingly we have, for a sphere,

$$r_2 = -r_1, \quad t = 2r_1^{-1}, \quad \mu R = 2r_1,$$

and the ordinates become $v + r_1^{-1}, v_2 - r_1^{-1}$, which are equal each to the central ordinate. This ordinate of the centre being c, we have then, *for a sphere*,

$$\boxed{T^{(2)} = \frac{c}{2}(\alpha_0^2 + \beta_0^2 - \alpha_2^2 - \beta_2^2) - \frac{(\alpha_2 - \alpha_0)^2 + (\beta_2 - \beta_0)^2}{4(1 - \mu^{-1}) r_1}.}$$

The focal length (from centre) is $\frac{1}{2(1 - \mu^{-1}) r_1}$, and the power is $2(1 - \mu^{-1}) r_1$. This focal

* [The subscripts 1, 2 refer to the first and second faces of the lens; 0, 3 refer to the incident and emergent regions respectively.

† [These are the "nodal points," coincident in the present case with the "principal points." See Appendix, Note 25, p. 508.]

‡ [This agrees with the general definition of focal length given by C. F. Gauss, "Dioptrische Untersuchungen," *Abhand. Kgl. Ges. Wiss. Göttingen*, 1 (1838–1841), *Math. Cl.*, p. 14. Cf. von Rohr, *The Formation of Images in Optical Instruments* (English translation), London (1920), p. 103, or J. P. C. Southall, *Geometrical Optics*, p. 233. Hamilton does not appear to have been acquainted with the optical work of Gauss.]

length = curvature, or image is on second surface, when $2(1-\mu^{-1})=1$, that is, when $\mu=2$. Accordingly, for this index, the focal length of 1st. surface* is = diameter;

$$\frac{\mu}{(\mu-1)r}=\frac{2}{r}.$$

The focal length of a sphere, from its 2nd. surface, is

$$\frac{(2-\mu)r_1^{-1}}{2(\mu-1)}.$$

If $\mu=\frac{3}{2}$, this last length $=\frac{1}{2}r_1^{-1}$ = half the radius.

For any lens of revolution in vacuo, if we denote the ordinates of the two focal centres by F', F'', and the focal length by F, we have the expression†

$$\boxed{T^{(2)}=\tfrac{1}{2}F'(\alpha_0^2+\beta_0^2)-\tfrac{1}{2}F''(\alpha_2^2+\beta_2^2)-\tfrac{1}{2}F\{(\alpha_2-\alpha_0)^2+(\beta_2-\beta_0)^2\}.}$$

And the properties of the lens, independent of its position (and of aberrations), depend only on $F''-F'$ and F; in which

$$F''-F'=t\left(1-\frac{r_1-r_2}{\mu R}\right)=\frac{(\mu-1)t}{\mu R}(r_1-r_2+r_1r_2t)=\frac{(\mu-1)ti}{\mu i+t},$$

i being interval of centres of curvatures,

$$=c_1-c_2=r_1^{-1}-r_2^{-1}-t.$$

The focal centres close up into one, 1st. for $t=0$, infinitely thin lens; 2nd. for $i=0$, concentric surfaces.

[7.] For a combination of two coaxal lenses of revolution *in vacuo*, we have (the order being α', α, α''):

$$\begin{aligned}T^{(2)}=\;&\tfrac{1}{2}F_1'(\alpha'^2+\beta'^2)-\tfrac{1}{2}F_1''(\alpha^2+\beta^2)-\tfrac{1}{2}F_1\{(\alpha-\alpha')^2+(\beta-\beta')^2\}\\&-\tfrac{1}{2}F_2''(\alpha''^2+\beta''^2)+\tfrac{1}{2}F_2'(\alpha^2+\beta^2)-\tfrac{1}{2}F_2\{(\alpha-\alpha'')^2+(\beta-\beta'')^2\}\\=\;&\tfrac{1}{2}F'(\alpha'^2+\beta'^2)-\tfrac{1}{2}F''(\alpha''^2+\beta''^2)-\tfrac{1}{2}F\{(\alpha''-\alpha')^2+(\beta''-\beta')^2\};\end{aligned}$$

$$(F_2'-F_1''-F_2-F_1)\alpha+F_2\alpha''+F_1\alpha'=0;$$

$$(F_2'-F_1''-F_2-F_1)\beta+F_2\beta''+F_1\beta'=0;$$

$$\tfrac{1}{2}(F_2'-F_1''-F_2-F_1)\alpha^2+(F_2\alpha''+F_1\alpha')\alpha=-\frac{(F_2\alpha''+F_1\alpha')^2}{2(F_2'-F_1''-F_2-F_1)};\ \&\text{c.};$$

$$\left\{\begin{aligned}F\;&=\frac{F_1F_2}{F_1+F_2+F_1''-F_2'};\quad (F_1F_2(\alpha''-\alpha')^2+(F_2\alpha''+F_1\alpha')^2=(F_1+F_2)(F_2\alpha''^2+F_1\alpha'^2);)\\F'\;&=F_1'-F_1+\frac{F_1(F_1+F_2)}{F_1+F_2+F_1''-F_2'}=F_1'+\frac{F_1(F_2'-F_1'')}{F_1+F_2+F_1''-F_2'};\\F''&=F_2''+F_2-\frac{F_2(F_1+F_2)}{F_1+F_2+F_1''-F_2'}=F_2''-\frac{F_2(F_2'-F_1'')}{F_1+F_2+F_1''-F_2'}.\end{aligned}\right.$$

* [Marginal note by Hamilton.] For a single refraction out of *vacuo*, at origin, of direct parallel indiametral rays,

$$T^{(2)}=-\frac{\sigma^2}{2r(\mu-1)};$$

for focus,

$$-\frac{\sigma z}{\mu}=\frac{\delta T^{(2)}}{\delta\sigma},\quad\therefore z=\frac{\mu r^{-1}}{\mu-1}.$$

† [This expression is valid for *any* optical instrument of revolution *in vacuo*.]

Thus, if $F_1'' = F_2'$, that is if 1*st. focal centre of* 2*nd. lens coincide with* 2*nd. focal centre of* 1*st. lens, we have*

$$F = \frac{F_1 F_2}{F_1 + F_2}; \quad F' = F_1'; \quad F'' = F_2'';$$

that is, the 1*st. focal centre of* 1*st. lens will be the* 1*st. of the combination; the* 2*nd. focal centre of the* 2*nd. lens will be the* 2*nd. centre of the combination; and the sum of the powers of the two component lenses will be the power of the combination.*

For example let there be two hemispheres, vertex to vertex, as in the figure, not necessarily

of equal radii, nor of equal indices; the last emergent ray will be parallel to the first incident, if the 1st. refracted ray pass through the common vertex of the two hemispheres; and then the focal centres F_1, F_2, of the combination, will evidently coincide with F_1' and F_2'', of the two hemispheres. As to the power of the combination, let μ_1, μ_2 be the two indices; ρ_1, ρ_2 the radii; the common vertex origin; then for parallel direct incident rays, the ordinate of the point of convergence after passing through the 1st. lens is $\frac{\rho_1}{\mu_1 - 1}$; the power of that lens is therefore $\frac{\mu_1 - 1}{\rho_1}$ (because vertex is 2nd. focal centre and is at origin); the convergence, immediately after entering the 2nd. lens, is

$$(1 - \mu_2^{-1})\, \rho_2^{-1} + \frac{\mu_1 - 1}{\mu_2 \rho_1} = \frac{1}{\mu_2} \left(\frac{\mu_1 - 1}{\rho_1} + \frac{\mu_2 - 1}{\rho_2} \right);$$

corresponding focal distance

$$= \frac{\mu_2 \rho_1 \rho_2}{(\mu_1 - 1)\rho_2 + (\mu_2 - 1)\rho_1};$$

subtract ρ_2, and there remains

$$\frac{\rho_2 \{\rho_1 - (\mu_1 - 1)\rho_2\}}{(\mu_1 - 1)\rho_2 + (\mu_2 - 1)\rho_1};$$

add $\frac{\text{this}}{\mu_2}$ to $\frac{\rho_2}{\mu_2}$, and we get focal length of combination (measured from F_2'')

$$= \frac{\rho_1 \rho_2}{(\mu_1 - 1)\rho_2 + (\mu_2 - 1)\rho_1};$$

$\therefore$ power of combination

$$= \frac{\mu_1 - 1}{\rho_1} + \frac{\mu_2 - 1}{\rho_2}$$

= sum of powers of the two component hemispheric lenses, as it ought to be.

The same theorems hold good for any two plano-spheric lenses, with vertices placed in contact.

[8.] Using the expressions of [6.] and [7.] for the function $T^{(2)}$ of a lens of revolution *in vacuo**

$$2T^{(2)} = F'(\alpha'^2 + \beta'^2) - F''(\alpha''^2 + \beta''^2) - F\{(\alpha'' - \alpha')^2 + (\beta'' - \beta')^2\},$$

in which F', F'' are the ordinates of the two focal centres, and F is focal length; the equations

* [The single accent refers to the incident system, the double accent to the emergent. The expression is valid for any instrument of revolution *in vacuo*.]

of an incident and of the corresponding emergent ray are, respectively, in the present order of approximation,

$$x' = \alpha'(z' - F') - F(\alpha'' - \alpha'),$$
$$y' = \beta'(z' - F') - F(\beta'' - \beta');$$

and

$$x'' = \alpha''(z'' - F'') - F(\alpha'' - \alpha'),$$
$$y'' = \beta''(z'' - F'') - F(\beta'' - \beta').$$

Hence

$$x' = -F\alpha'', \quad y' = -F\beta'', \quad \text{when } z' = F' - F;$$

and

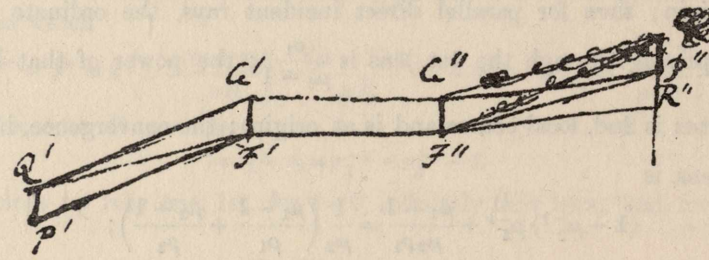
$$x'' = F\alpha', \quad y'' = F\beta', \quad \text{when } z'' = F'' + F.$$

Also

$$x'' = x' = -F(\alpha'' - \alpha'), \quad y'' = y' = -F(\beta'' - \beta'),$$

when

$$z' = F', \quad z'' = F''.$$



F', F'', focal centres;
$P'F' = F''P''$ = focal length.

If $P'F'$ be direction of incident ray, $F''P''$, parallel thereto, will be the direction of the emergent; and the last algebraic theorem shows that if $G'G''$ is parallel to $F'F''$, and if the direction of the incident ray passes through G', then the direction of the emergent will pass through G'', or will be $G''P''$, if $Q'G'$, parallel to $P'F'$, be the direction of the incident ray. *This theorem gives a very easy construction to determine the emergent ray $G''P''$, corresponding to any given incident $Q'G'$, when the points F', F'', and the focal length $F''P''$ are known; for we have only to draw $G'G''$, $F''P''$, and so determine two points G'', P'', on the sought emergent ray.*

A geometrical proof of the theorem may be had by taking $P''R'' = G''F''$ $(= G'F' = Q'P')$, so that $F''R''$ shall be parallel to $Q'F'$ and to $G''P''$. Then, because of the *position* of the point Q', the incident rays $Q'F'$, $Q'G'$ have their corresponding emergent rays parallel to each other, that is, the emergent ray corresponding to $Q'G'$ has the same *direction* as $G''P''$. But it also passes through P'', because the parallel incident rays $Q'G'$, $P'F'$ give emergent rays which meet on $R''P''$, and one of these rays is $F''P''$. Thus a certain incident *position* (Q') gives the emergent *direction* (parallel to $F''R''$), and the incident *direction* (parallel to $P'F'$) gives a certain emergent *position* (P'').

For a *single lens*,

$$F' = v_1 - \frac{tr_2}{\mu R}; \quad F'' = v_2 - \frac{tr_1}{\mu R}; \quad F = \frac{1}{(\mu - 1) R}; \quad R = r_1 - r_2 + (1 - \mu^{-1}) r_1 r_2 t;$$

and for a *combination of two*,

$$F = \frac{F_1 F_2}{F_1 + F_2 + F_1'' - F_2'}; \quad F' = F_1' + \frac{F}{F_2}(F_2' - F_1''); \quad F'' = F_2'' - \frac{F}{F_1}(F_2' - F_1'').$$

For a *telescope*,

$$F=\infty, \quad F_1+F_2=F_2'-F_1'', \quad F_2\alpha''+F_1\alpha'=0, \quad F_2\beta''+F_1\beta'=0,$$

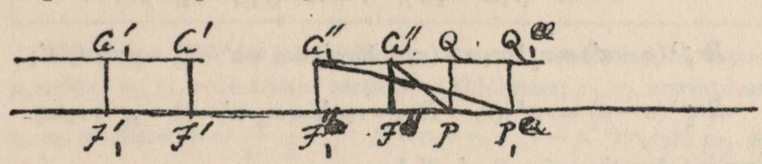
$$\text{magnifying power} = -\frac{F_1}{F_2}.$$

[9.] For the combination of two lenses, by [7.],

$$\frac{\alpha}{F}=\frac{\alpha''}{F_1}+\frac{\alpha'}{F_2}; \quad \frac{\beta}{F}=\frac{\beta''}{F_1}+\frac{\beta'}{F_2};$$

α, β belonging to the intermediate ray (between the lenses); α', β' to initial, and α'', β'' to final ray; while F_1, F_2, F are the component and resultant focal lengths.

It is easy to explain these equations, geometrically, by the aid of the construction in [8.].



Let an incident ray $G_1' G'$, parallel to the axis, take the direction $G_1'' P_1$ after passing through the 1st. lens, and the direction $G'' P$ after passing through the combination. Let F_1', F_1'' be the focal centres of the 1st. lens, and F', F'' of the combination. Then, by the theorem of [8.], applied to the 1st. lens,

$$F_1'' G_1'' = F_1' G_1';$$

and by the same theorem applied to the 2nd. lens,

$$F'' G'' = F' G';$$

but because the 1st. incident ray is parallel to the axis,

$$F' G' = F_1' G_1';$$

therefore

$$F'' G'' = F_1' G_1'';$$

that is,

$$F\alpha'' = F_1\alpha,$$

when $\alpha'=0$. In like manner, if the final ray be parallel to the axis, the intermediate and initial rays will meet the ordinates to the axis, erected at the anterior focal centres of the 2nd. lens and the combination respectively, at heights above (or below) the axis equal to each other, because equal to the height of the final ray above (or below) that axis; therefore

$$F\alpha' = F_2\alpha,$$

when $\alpha''=0$. If then we admit, as known, that a linear relation, without a constant term, exists between α, α', α'', we see that it can be only that written at the beginning of this section; and similarly for the relation between β, β', β''.

[10.] *Aberrations of Lens.*

For *a single lens*, by [2.], [4.], if α, β be the final and α', β' the initial direction cosines; μ, index; v_1, v_2, ordinates of vertices; r_1, r_2, curvatures; s_1, s_2, coefficients of $\left(\frac{x^2+y^2}{2}\right)^2$ in development of z for the two surfaces; t, thickness, $=v_2-v_1$; $R=r_1-r_2+(1-\mu^{-1})r_1r_2t$; we shall have

$$T^{(4)}=\frac{v_1}{8}(\alpha'^2+\beta'^2)^2-\frac{v_2}{8}(\alpha^2+\beta^2)^2+\frac{t}{8}\frac{(\sigma^2+\tau^2)^2}{\mu^3}$$

$$-\frac{r_1^{-1}}{4}\frac{(\sigma-\alpha')^2+(\tau-\beta')^2}{(\mu-1)^2}\left(\frac{\sigma^2+\tau^2}{\mu}-(\alpha'^2+\beta'^2)\right)+\frac{r_2^{-1}}{4}\frac{(\sigma-\alpha)^2+(\tau-\beta)^2}{(\mu-1)^2}\left(\frac{\sigma^2+\tau^2}{\mu}-(\alpha^2+\beta^2)\right)$$

$$+\frac{s_1r_1^{-4}}{4}\frac{((\sigma-\alpha')^2+(\tau-\beta')^2)^2}{(\mu-1)^3}-\frac{s_2r_2^{-4}}{4}\frac{((\sigma-\alpha)^2+(\tau-\beta)^2)^2}{(\mu-1)^3};$$

in which*

$$\sigma=R^{-1}(r_1\alpha-r_2\alpha'),\quad \tau=R^{-1}(r_1\beta-r_2\beta').$$

Hence

$$Rr_1^{-1}(\sigma-\alpha')=r_1^{-1}\{r_1\alpha-(r_2+R)\alpha'\}=\alpha-\alpha'-(1-\mu^{-1})r_2t\alpha';$$

$$Rr_2^{-1}(\sigma-\alpha)=r_2^{-1}\{(r_1-R)\alpha-r_2\alpha'\}=\alpha-\alpha'-(1-\mu^{-1})r_1t\alpha;$$

and $T^{(4)}$, as an explicit function of α, β, α', β', becomes

$$\boxed{\begin{aligned}T^{(4)}&=\tfrac{1}{8}v_1(\alpha'^2+\beta'^2)^2-\tfrac{1}{8}v_2(\alpha^2+\beta^2)^2+\tfrac{1}{8}t\mu^{-3}R^{-4}\{(r_1\alpha-r_2\alpha')^2+(r_1\beta-r_2\beta')^2\}^2\\&-\frac{r_1\mu^{-1}R^{-4}}{4(\mu-1)^2}\begin{Bmatrix}(\alpha-\alpha'-(1-\mu^{-1})r_2t\alpha')^2\\+(\beta-\beta'-(1-\mu^{-1})r_2t\beta')^2\end{Bmatrix}\begin{Bmatrix}(r_1\alpha-r_2\alpha')^2-\mu R^2(\alpha'^2+\beta'^2)\\+(r_1\beta-r_2\beta')^2\end{Bmatrix}\\&+\frac{r_2\mu^{-1}R^{-4}}{4(\mu-1)^2}\begin{Bmatrix}(\alpha-\alpha'-(1-\mu^{-1})r_1t\alpha)^2\\+(\beta-\beta'-(1-\mu^{-1})r_1t\beta)^2\end{Bmatrix}\begin{Bmatrix}(r_1\alpha-r_2\alpha')^2-\mu R^2(\alpha^2+\beta^2)\\+(r_1\beta-r_2\beta')^2\end{Bmatrix}\\&+\frac{s_1R^{-4}}{4(\mu-1)^3}\begin{Bmatrix}(\alpha-\alpha'-(1-\mu^{-1})r_2t\alpha')^2\\+(\beta-\beta'-(1-\mu^{-1})r_2t\beta')^2\end{Bmatrix}^2-\frac{s_2R^{-4}}{4(\mu-1)^3}\begin{Bmatrix}(\alpha-\alpha'-(1-\mu^{-1})r_1t\alpha)^2\\+(\beta-\beta'-(1-\mu^{-1})r_1t\beta)^2\end{Bmatrix}^2.\end{aligned}}$$

(Accordingly this expression agrees, some slight differences of notation excepted, with page 1 of my investigations begun Jan. 13th. 1832, which is stated to agree with page 32 of 7th. series of investigations respecting lenses of revolution, written in 1831.†)

If we make for abbreviation

$$\epsilon=\alpha^2+\beta^2,\quad \epsilon_{,}=\alpha\alpha'+\beta\beta',\quad \epsilon'=\alpha'^2+\beta'^2;$$

we have

$$(r_1\alpha-r_2\alpha')^2+(r_1\beta-r_2\beta')^2=r_1^2\epsilon-2r_1r_2\epsilon_{,}+r_2^2\epsilon';$$

$$(\alpha-\alpha'-(1-\mu^{-1})r_2t\alpha')^2+\&c.=\epsilon-2(1+r_2t-\mu^{-1}r_2t)\epsilon_{,}+(1+r_2t-\mu^{-1}r_2t)^2\epsilon';$$

$$(\alpha-\alpha'-(1-\mu^{-1})r_1t\alpha)^2+\&c.=\epsilon'-2(1-r_1t+\mu^{-1}r_1t)\epsilon_{,}+(1-r_1t+\mu^{-1}r_1t)^2\epsilon;$$

if then we make

$$1+r_2t-\mu^{-1}r_2t=\rho_2,\quad 1-r_1t+\mu^{-1}r_1t=\rho_1,$$

* [These relations were given in [4.]; they also follow at once from the form of $T^{(2)}$ given at the beginning of [5.], by making use of the fact that this expression has a stationary value with respect to σ and τ.]

† [We have not been able to find either of these among the Hamilton MSS.]

we shall have the following more concise expression:

$$T^{(4)} = \frac{v_1}{8}\epsilon'^2 - \frac{v_2}{8}\epsilon^2 + \frac{t}{8}\mu^{-3}R^{-4}(r_1^2\epsilon - 2r_1r_2\epsilon_, + r_2^2\epsilon')^2$$
$$- \frac{r_1\mu^{-1}R^{-4}}{4(\mu-1)^2}(\epsilon - 2\rho_2\epsilon_, + \rho_2^2\epsilon')\{r_1^2\epsilon - 2r_1r_2\epsilon_, + (r_2^2 - \mu R^2)\epsilon'\}$$
$$+ \frac{r_2\mu^{-1}R^{-4}}{4(\mu-1)^2}(\epsilon' - 2\rho_1\epsilon_, + \rho_1^2\epsilon)\{r_2^2\epsilon' - 2r_1r_2\epsilon_, + (r_1^2 - \mu R^2)\epsilon\}$$
$$+ \frac{s_1R^{-4}}{4(\mu-1)^3}(\epsilon - 2\rho_2\epsilon_, + \rho_2^2\epsilon')^2 - \frac{s_2R^{-4}}{4(\mu-1)^3}(\epsilon' - 2\rho_1\epsilon_, + \rho_1^2\epsilon)^2.$$

(Function $T^{(4)}$, for ANY SINGLE LENS of revolution *in vacuo*: μ, index; v_1, v_2, ordinates of vertices; t, thickness; r_1, r_2, curvatures; s_1, s_2, coefficients of $\left(\frac{x^2+y^2}{2}\right)^2$; $R = r_1 - r_2 + (1-\mu^{-1})r_1r_2t$; ρ_1, ρ_2, ϵ, $\epsilon_,$, ϵ', abridgments, as above.)

If we write, for abridgment,

$$T^{(4)} = Q\epsilon^2 + Q_,\epsilon\epsilon_, + Q'\epsilon\epsilon' + Q_{,,}\epsilon_,^2 + Q_,'\epsilon_,\epsilon' + Q''\epsilon'^2,$$

and substitute for R^{-1} its value $(\mu-1)F$, F being focal length, we have, by above, for *any single lens* of revolution *in vacuo*:

$$Q = -\tfrac{1}{8}v_2 + \tfrac{1}{8}t\mu^{-3}(\mu-1)^4F^4r_1^4 - \tfrac{1}{4}\mu^{-1}(\mu-1)^2F^4r_1^3 + \tfrac{1}{4}\mu^{-1}\rho_1^2r_2F^2\{(\mu-1)^2F^2r_1^2 - \mu\}$$
$$+ \tfrac{1}{4}(\mu-1)(s_1 - \rho_1^4s_2)F^4;$$
$$Q'' = \tfrac{1}{8}v_1 + \tfrac{1}{8}t\mu^{-3}(\mu-1)^4F^4r_2^4 + \tfrac{1}{4}\mu^{-1}(\mu-1)^2F^4r_2^3 - \tfrac{1}{4}\mu^{-1}\rho_2^2r_1F^2\{(\mu-1)^2F^2r_2^2 - \mu\}$$
$$- \tfrac{1}{4}(\mu-1)(s_2 - \rho_2^4s_1)F^4;$$
$$Q_, = -\tfrac{1}{2}t\mu^{-3}(\mu-1)^4F^4r_1^3r_2 + \tfrac{1}{2}\mu^{-1}(\mu-1)^2F^4r_1^2(r_2 + \rho_2r_1)$$
$$- \tfrac{1}{2}\mu^{-1}\rho_1r_2F^2\{(\mu-1)^2F^2r_1(r_1 + \rho_1r_2) - \mu\} - (\mu-1)(\rho_2s_1 - \rho_1^3s_2)F^4;$$
$$Q_,' = -\tfrac{1}{2}t\mu^{-3}(\mu-1)^4F^4r_1r_2^3 - \tfrac{1}{2}\mu^{-1}(\mu-1)^2F^4r_2^2(r_1 + \rho_1r_2)$$
$$+ \tfrac{1}{2}\mu^{-1}\rho_2r_1F^2\{(\mu-1)^2F^2r_2(r_2 + \rho_2r_1) - \mu\} + (\mu-1)(\rho_1s_2 - \rho_2^3s_1)F^4;$$
$$Q' = \tfrac{1}{4}t\mu^{-3}(\mu-1)^4F^4r_1^2r_2^2 - \tfrac{1}{4}\mu^{-1}r_1F^2\{(\mu-1)^2F^2(r_2^2 + \rho_2^2r_1^2) - \mu\}$$
$$+ \tfrac{1}{4}\mu^{-1}r_2F^2\{(\mu-1)^2F^2(r_1^2 + \rho_1^2r_2^2) - \mu\} + \tfrac{1}{2}(\mu-1)(\rho_2^2s_1 - \rho_1^2s_2)F^4;$$
$$Q_{,,} = \tfrac{1}{2}t\mu^{-3}(\mu-1)^4F^4r_1^2r_2^2 - \mu^{-1}(\mu-1)^2r_1r_2F^4(\rho_2r_1 - \rho_1r_2) + (\mu-1)(\rho_2^2s_1 - \rho_1^2s_2)F^4.$$

Hence,

$$2(2Q' - Q_{,,})F^{-2} = r_1 - r_2 + \mu^{-1}(\mu-1)^2F^2\{r_2(\rho_1^2r_2^2 + r_1^2) - r_1(\rho_2^2r_1^2 + r_2^2) + 2r_1r_2(\rho_2r_1 - \rho_1r_2)\}.$$

When $t = 0$, this last expression reduces itself to $(1-\mu^{-1})(r_1 - r_2)$; and

$$2Q' - Q_{,,} = \frac{F}{2\mu}.$$

[11.] Changing first $\mu, t, R,$ to μ_1, t_1, R_1; then changing

$$\mu_1, t_1, R_1, v_1, v_2, r_1, r_2, s_1, s_2, \rho_1, \rho_2, \epsilon, \epsilon_{,}, \epsilon',$$

to

$$\mu_2, t_2, R_2, v_3, v_4, r_3, r_4, s_3, s_4, \rho_3, \rho_4, \epsilon'', \epsilon_{,,}, \epsilon,$$

in which

$$\begin{aligned} R_2 &= r_3 - r_4 + (1-\mu_2^{-1})\, r_3 r_4 t_2, \\ t_2 &= v_4 - v_3, \\ \rho_3 &= 1 - r_3 t_2 + \mu_2^{-1} r_3 t_2, \\ \rho_4 &= 1 + r_4 t_2 - \mu_2^{-1} r_4 t_2, \\ \epsilon'' &= \alpha''^2 + \beta''^2, \\ \epsilon_{,,} &= \alpha\alpha'' + \beta\beta''; \end{aligned}$$

and adding the two results: we find for *any combination of two coaxal lenses of revolution in vacuo,*

$$\begin{aligned} T^{(4)} = {} & \tfrac{1}{8} v_1 \epsilon'^2 + \tfrac{1}{8}(v_3 - v_2)\,\epsilon^2 - \tfrac{1}{8} v_4 \epsilon''^2 \\ & + \tfrac{1}{8} t_1 \mu_1^{-3} R_1^{-4} (r_1^2 \epsilon - 2 r_1 r_2 \epsilon_{,} + r_2^2 \epsilon')^2 \\ & + \tfrac{1}{8} t_2 \mu_2^{-3} R_2^{-4} (r_3^2 \epsilon'' - 2 r_3 r_4 \epsilon_{,,} + r_4^2 \epsilon)^2 \\ & + \frac{r_1^2\epsilon - 2r_1 r_2 \epsilon_{,} + r_2^2 \epsilon'}{4\mu_1(\mu_1-1)^2 R_1^4} \{(\rho_1^2 r_2 - r_1)\,\epsilon - 2(\rho_1 r_2 - \rho_2 r_1)\,\epsilon_{,} + (r_2 - \rho_2^2 r_1)\,\epsilon'\} \\ & + \frac{r_3^2\epsilon'' - 2r_3 r_4 \epsilon_{,,} + r_4^2 \epsilon}{4\mu_2(\mu_2-1)^2 R_2^4} \{(\rho_3^2 r_4 - r_3)\,\epsilon'' - 2(\rho_3 r_4 - \rho_4 r_3)\,\epsilon_{,,} + (r_4 - \rho_4^2 r_3)\,\epsilon\} \\ & + \frac{R_1^{-2}}{4(\mu_1-1)^2} \{r_1 \epsilon' (\epsilon - 2\rho_2 \epsilon_{,} + \rho_2^2 \epsilon') - r_2 \epsilon (\epsilon' - 2\rho_1 \epsilon_{,} + \rho_1^2 \epsilon)\} \\ & + \frac{R_2^{-2}}{4(\mu_2-1)^2} \{r_3 \epsilon (\epsilon'' - 2\rho_4 \epsilon_{,,} + \rho_4^2 \epsilon) - r_4 \epsilon'' (\epsilon - 2\rho_3 \epsilon_{,,} + \rho_3^2 \epsilon'')\} \\ & + \frac{s_1 R_1^{-4}}{4(\mu_1-1)^3} (\epsilon - 2\rho_2 \epsilon_{,} + \rho_2^2 \epsilon')^2 - \frac{s_2 R_1^{-4}}{4(\mu_1-1)^3} (\epsilon' - 2\rho_1 \epsilon_{,} + \rho_1^2 \epsilon)^2 \\ & + \frac{s_3 R_2^{-4}}{4(\mu_2-1)^3} (\epsilon'' - 2\rho_4 \epsilon_{,,} + \rho_4^2 \epsilon)^2 - \frac{s_4 R_2^{-4}}{4(\mu_2-1)^3} (\epsilon - 2\rho_3 \epsilon_{,,} + \rho_3^2 \epsilon'')^2; \end{aligned}$$

(Function $T^{(4)}$ for ANY DOUBLE LENS of revolution *in vacuo.*)

which may be reduced to an explicit function of $\epsilon', \epsilon_{,}', \epsilon''$, in which

$$\epsilon_{,}' = \alpha'\alpha'' + \beta'\beta'',$$

by employing the relations

$$\alpha = F(F_1^{-1}\alpha'' + F_2^{-1}\alpha'), \quad \beta = F(F_1^{-1}\beta'' + F_2^{-1}\beta'),$$

which give

$$\begin{cases} \epsilon_{,} = F F_1^{-1} F_2^{-1} (F_2 \epsilon_{,}' + F_1 \epsilon'), \\ \epsilon_{,,} = F F_1^{-1} F_2^{-1} (F_2 \epsilon'' + F_1 \epsilon_{,}'), \\ \epsilon = F^2 F_1^{-2} F_2^{-2} (F_2^2 \epsilon'' + 2F_1 F_2 \epsilon_{,}' + F_1^2 \epsilon'). \end{cases}$$

Also,

$$R_1^{-1} = (\mu_1 - 1) F_1; \quad R_2^{-1} = (\mu_2 - 1) F_2.$$

[12.] For the case of two infinitely thin lenses, close together, we may suppose

$$0 = v_1 = v_2 = v_3 = v_4 = t_1 = t_2;$$

$$R_1 = r_1 - r_2, \quad R_2 = r_3 - r_4;$$

$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 1;$$

$$F_1' = F_1'' = F_2' = F_2'' = F' = F'' = 0;$$

$$F^{-1} = F_1^{-1} + F_2^{-1}; \quad F_1^{-1} = (\mu_1 - 1)(r_1 - r_2), \quad F_2^{-1} = (\mu_2 - 1)(r_3 - r_4);$$

and calling these last powers p_1 and p_2, ($F^{-1} = p_1 + p_2$,) we have

$$(p_1 + p_2)\alpha = p_1\alpha'' + p_2\alpha',$$

$$(p_1 + p_2)(\alpha - \alpha') = p_1(\alpha'' - \alpha'),$$

$$(p_1 + p_2)(\alpha - \alpha'') = p_2(\alpha' - \alpha''),$$

$$\epsilon - 2\epsilon_, + \epsilon' = \left(\frac{p_1}{p_1 + p_2}\right)^2 (\epsilon'' - 2\epsilon_,' + \epsilon'),$$

$$\epsilon'' - 2\epsilon_{,,} + \epsilon = \left(\frac{p_2}{p_1 + p_2}\right)^2 (\epsilon'' - 2\epsilon_,' + \epsilon'),$$

and $T^{(4)}$ becomes divisible* by $\epsilon'' - 2\epsilon_,' + \epsilon'$.

(Feb. 13th. 1844.) And if we make, for abridgment,

$$\alpha = f'\alpha' + f''\alpha'',$$

$$f' = \frac{F_1}{F_1 + F_2 - (F_2' - F_1'')}, \quad f'' = \frac{F_2}{F_1 + F_2 - (F_2' - F_1'')},$$

so that for the case here supposed $f' + f'' = 1$, then

$$\alpha - \alpha' = f''(\alpha'' - \alpha'), \quad \alpha'' - \alpha = f'(\alpha'' - \alpha'),$$

$$\epsilon - 2\epsilon_, + \epsilon' = f''^2(\epsilon'' - 2\epsilon_,' + \epsilon'), \quad \epsilon'' - 2\epsilon_{,,} + \epsilon = f'^2(\epsilon'' - 2\epsilon_,' + \epsilon');$$

and the quotient of the division of $T^{(4)}$ by $\epsilon'' - 2\epsilon_,' + \epsilon'$ is composed of the following parts

1st. $$\frac{(r_1^2\epsilon - 2r_1r_2\epsilon_, + r_2^2\epsilon')(r_2 - r_1)f''^2}{4\mu_1(\mu_1 - 1)^2(r_1 - r_2)^4} = \frac{-F^2}{4\mu_1(r_1 - r_2)}(r_1^2\epsilon - +)$$

$$= \frac{-F^2}{4\mu_1(r_1 - r_2)}\{\overline{(r_1f' - r_2)\alpha' + r_1f''\alpha''}^2 + \&c.\}$$

$$= \frac{-F^2}{4\mu_1(r_1 - r_2)}\{(\overline{r_1 - r_2}\alpha'f' + \overline{r_1\alpha'' - r_2\alpha'}f'')^2 + \&c.\}$$

$$= -\frac{F^2}{4\mu_1}\left\{(r_1 - r_2)f'^2\epsilon' + 2f'f''(r_1\epsilon_,' - r_2\epsilon') + \frac{f''^2}{r_1 - r_2}(r_1^2\epsilon'' - 2r_1r_2\epsilon_,' + r_2^2\epsilon')\right\};$$

* [This is true for every "thin" system. See Appendix, Note 26, p. 511.]

2nd. $$\frac{(r_3^2\epsilon'' - 2r_3r_4\epsilon_{,,} + r_4^2\epsilon)(r_4 - r_3)f'^2}{4\mu_2(\mu_2-1)^2(r_3-r_4)^4} = -\frac{F^2}{4\mu_2}\frac{\{(r_3\alpha'' - r_4\alpha)^2 + \&c.\}}{r_3 - r_4}$$

$$= \frac{-F^2}{4\mu_2(r_3-r_4)}\{(\overline{r_3 - r_4}f''\alpha'' + \overline{r_3\alpha'' - r_4\alpha'f'})^2 + \&c.\}$$

$$= -\frac{F^2}{4\mu_2}\left\{(r_3 - r_4)f''^2\epsilon'' + 2f'f''(r_3\epsilon'' - r_4\epsilon_,') + \frac{f'^2}{r_3 - r_4}(r_3^2\epsilon'' - 2r_3r_4\epsilon_,' + r_4^2\epsilon')\right\};$$

3rd. $$\frac{f''^2(r_1\epsilon' - r_2\epsilon)}{4(\mu_1-1)^2(r_1-r_2)^2} = \frac{F^2}{4}\{r_1\epsilon' - r_2(f'^2\epsilon' + 2f'f''\epsilon_,' + f''^2\epsilon'')\};$$

4th. $$\frac{f'^2(r_3\epsilon - r_4\epsilon'')}{4(\mu_2-1)^2(r_3-r_4)^2} = \frac{F^2}{4}\{r_3(f'^2\epsilon' + 2f'f''\epsilon_,' + f''^2\epsilon'') - r_4\epsilon''\};$$

5th. $$\frac{(s_1 - s_2)f''^4(\epsilon'' - 2\epsilon_,' + \epsilon')}{4(\mu_1-1)^3R_1^4} = \frac{\mu_1 - 1}{4}F^4(s_1 - s_2)(\epsilon'' - 2\epsilon_,' + \epsilon');$$

6th. $$\frac{(s_3 - s_4)f'^4(\epsilon'' - 2\epsilon_,' + \epsilon')}{4(\mu_2-1)^3R_2^4} = \frac{\mu_2 - 1}{4}F^4(s_3 - s_4)(\epsilon'' - 2\epsilon_,' + \epsilon').$$

Hence, in $\dfrac{4F^{-4}T^{(4)}}{\epsilon'' - 2\epsilon_,' + \epsilon'}$, the coefficient of ϵ'', (because

$$f''F^{-1} = F_1^{-1} = (\mu_1 - 1)(r_1 - r_2) = p_1,$$
$$f'F^{-1} = F_2^{-1} = (\mu_2 - 1)(r_3 - r_4) = p_2,$$
$$F^{-1} = F_1^{-1} + F_2^{-1} = p_1 + p_2,)$$

is

$$= -\frac{(\mu_1-1)^2}{\mu_1}r_1^2(r_1 - r_2) - \frac{r_3 - r_4}{\mu_2}\{(\mu_1 - 1)(r_1 - r_2) + (\mu_2 - 1)r_3\}^2$$
$$+ (\mu_1 - 1)^2(r_1 - r_2)^2(r_3 - r_2) - r_4\{(\mu_1 - 1)(r_1 - r_2) + (\mu_2 - 1)(r_3 - r_4)\}^2$$
$$+ (\mu_1 - 1)(s_1 - s_2) + (\mu_2 - 1)(s_3 - s_4);$$

the coefficient of ϵ' is

$$= -\frac{(\mu_2-1)^2}{\mu_2}r_4^2(r_3 - r_4) - \frac{r_1 - r_2}{\mu_1}\{(\mu_2 - 1)(r_3 - r_4) - (\mu_1 - 1)r_2\}^2$$
$$+ (\mu_2 - 1)^2(r_3 - r_4)^2(r_3 - r_2) + r_1\{(\mu_1 - 1)(r_1 - r_2) + (\mu_2 - 1)(r_3 - r_4)\}^2$$
$$+ (\mu_1 - 1)(s_1 - s_2) + (\mu_2 - 1)(s_3 - s_4);$$

and the coefficient of $-2\epsilon_,'$ is

$$= \frac{(\mu_1 - 1)}{\mu_1}r_1(r_1 - r_2)\{(\mu_2 - 1)(r_3 - r_4) - (\mu_1 - 1)r_2\}$$
$$- \frac{(\mu_2 - 1)}{\mu_2}r_4(r_3 - r_4)\{(\mu_1 - 1)(r_1 - r_2) + (\mu_2 - 1)r_3\}$$
$$+ (\mu_1 - 1)(\mu_2 - 1)(r_1 - r_2)(r_2 - r_3)(r_3 - r_4)$$
$$+ (\mu_1 - 1)(s_1 - s_2) + (\mu_2 - 1)(s_3 - s_4).$$

(Feb. 14th.) By above, if $v_1 = v_2 = v_3 = v_4 = 0$, that is, if the two lenses be infinitely thin and close together at the origin, then

$$T^{(4)} = (\epsilon'' - 2\epsilon_{,}' + \epsilon')(Q\epsilon'' - \tfrac{1}{2}Q_{,,}\epsilon_{,}' + Q''\epsilon');$$

in which, $4F^{-4}Q$, $4F^{-4}Q''$, and $F^{-4}Q_{,,}$ have respectively the values assigned above, as the coefficients of ϵ'', $\epsilon_{,}'$ and $-2\epsilon_{,}'$, in

$$4(\epsilon'' - 2\epsilon_{,}' + \epsilon')^{-1}F^{-4}T^{(4)}.$$

The coefficient of ϵ'' in $T^{(2)}$ is*

$$-\left(\frac{F'' + F}{2}\right) = -\tfrac{1}{2}F;$$

therefore if longitudinal aberration vanish for direct parallel incident rays, we must have†

$$-\tfrac{1}{2}F\epsilon'' + Q\epsilon''^2 = \text{const.} + \text{const.}'\sqrt{1-\epsilon''} = \text{const.}' \times (-\tfrac{1}{2}\epsilon'' - \tfrac{1}{8}\epsilon''^2),$$

$$Q = -\tfrac{1}{8}F.$$

(Compare p. 383,‡ $Q = \tfrac{1}{4}P$, $P = -\tfrac{1}{2}F$.) Hence

$$4F^{-4}Q = -\tfrac{1}{2}F^{-3}.$$

$$Q_{,} = -2Q - \tfrac{1}{2}Q_{,,};$$

if then $Q_{,} = 0$ (see same p. 383,) we have

$$4F^{-4}Q + F^{-4}Q_{,,} = 0;$$

and if $Q = -\tfrac{1}{8}F$, then also

$$F^{-4}Q_{,,} - 4F^{-4}Q = F^{-3};$$

an equation which, it is remarkable, is independent of s_1, s_2, s_3, s_4; and is divisible by F^{-1}. The quotient of this division gives,

$$(1-\mu_1^{-1})\,r_1(r_1-r_2) + \mu_2^{-1}(r_3-r_4)\,\{(\mu_1-1)(r_1-r_2) + (\mu_2-1)\,r_3\}$$

$$+ (\mu_1-1)(r_1-r_2)(r_2-r_3) + r_4\,\{(\mu_1-1)(r_1-r_2) + (\mu_2-1)(r_3-r_4)\}$$

$$= \{(\mu_1-1)(r_1-r_2) + (\mu_2-1)(r_3-r_4)\}^2;$$

that is,

$$\mu_1^{-1}p_1r_1 + \mu_2^{-1}p_2r_4 + p_1(r_2-r_3) + (p_1+p_2)\,r_4 + \frac{\mu_2^{-1}p_2(p_1+p_2)}{\mu_2-1} = (p_1+p_2)^2:$$

or finally,

$$(\mu_1^{-1}+1)\,p_1(r_1+r_2) + (\mu_2^{-1}+1)\,p_2(r_3+r_4) = (\mu_1^{-1}-\mu_2^{-1})\,p_1^2 + (\mu_2^{-1}+2)(p_1+p_2)^2.$$

(Equation (B) of Jan. 2nd. 1844.)

* [Cf. the expression at the beginning of [8.]; $F''=0$, since the focal centres of the thin combination coincide at the origin.]

† [In order that the emergent rays (corresponding to an incident system parallel to the axis) should all pass rigorously through a single point on the axis, it is necessary and sufficient that when we put $\alpha'=\beta'=0$, we should have $T=k+k'\gamma''$, where k, k' are constants.]

‡ [This reference is to the letter to Prof. Phillips.]

[**13.**] (Feb. 15th. 1844.) Since the coefficient of ϵ'' in $\dfrac{4F^{-4}T^{(4)}}{\epsilon''-2\epsilon_,'+\epsilon_{,}}$ is $4F^{-4}Q$, (for a thin double lens of revolution *in vacuo*,) while $F^{-1}=p_1+p_2$, we have*

$$4F^{-4}\left(4Q+\frac{F}{2}\right)-4(\mu_1-1)(s_1-s_2)-4(\mu_2-1)(s_3-s_4)=-(1-\mu_1^{-1})\text{ \&c.}$$

$$=-(1-m_1)p_1\left\{\overline{r_1+r_2}+\frac{m_1p_1}{1-m_1}\right\}^2-\frac{m_2^2p_2}{1-m_2}\{2p_1+p_2+(m_2^{-1}-1)(r_3+r_4)\}^2$$

$$+2p_1^2\left\{\frac{m_1p_1}{1-m_1}+\frac{m_2p_2}{1-m_2}-(r_1+r_2)+\overline{r_3+r_4}\right\}+2(p_1+p_2)^2\left\{\frac{m_2p_2}{1-m_2}-\overline{r_3+r_4}\right\}+2(p_1+p_2)^3$$

$$=-(1-m_1)p_1(r_1+r_2)^2-(1-m_2)p_2(r_3+r_4)^2-2(1+m_1)p_1^2(r_1+r_2)$$
$$-2(1+m_2)(p_2^2+2p_1p_2)(r_3+r_4)$$
$$+\frac{2-m_1^2}{1-m_1}p_1^3+\frac{2-m_2^2}{1-m_2}p_2^3+2(3+2m_2)p_1p_2(p_1+p_2);$$

this last quantity, therefore, when added to

$$4(\mu_1-1)(s_1-s_2)+4(\mu_2-1)(s_3-s_4),$$

is to give an evanescent sum, if the longitudinal aberration is to vanish, for direct parallel incident rays.

If the surfaces be all *spheric*, then $s_1=\frac{1}{2}r_1^3$, &c.; therefore

$$4(\mu_1-1)(s_1-s_2)=2(\mu_1-1)(r_1^3-r_2^3)=2p_1(r_1^2+r_1r_2+r_2^2)=\frac{p_1}{2}\{4(r_1+r_2)^2-4r_1r_2\}$$
$$=\frac{p_1}{2}\{3(r_1+r_2)^2+(r_1-r_2)^2\}=\frac{m_1p_1^3}{2(1-m_1)^2}+\tfrac{3}{2}p_1(r_1+r_2)^2,$$
$$4(\mu_2-1)(s_3-s_4)=\text{\&c.};$$

therefore for a *thin double spheric lens in vacuo*, the condition of *direct aplanaticity* is

$$\left(4F^{-4}\left(4Q+\frac{F}{2}\right)=\right)$$
$$(m_1+\tfrac{1}{2})p_1(r_1+r_2)^2+(m_2+\tfrac{1}{2})p_2(r_3+r_4)^2-2(m_2+1)p_2(p_1+p_2)(r_3+r_4)$$
$$-2p_1\{(m_1+1)p_1(r_1+r_2)+(m_2+1)p_2(r_3+r_4)\}$$
$$+\frac{4-4m_1-m_1^2+2m_1^3}{2(1-m_1)^2}p_1^3+\frac{4-4m_2-m_2^2+2m_2^3}{2(1-m_2)^2}p_2^3+2(3+2m_2)p_1p_2(p_1+p_2)=0;\quad\text{(A)}$$

while the additional condition for OBLIQUE APLANATICITY† is, by end of [**12.**],

$$\left(-4F^{-3}\left(Q_,+4Q+\frac{F}{2}\right)=\right)$$
$$(m_1+1)p_1(r_1+r_2)+(m_2+1)p_2(r_3+r_4)-(m_1-m_2)p_1^2-(m_2+2)(p_1+p_2)^2=0.\quad\text{(B)}$$

* [$m_1=\mu_1^{-1}$, $m_2=\mu_2^{-1}$; see next page.]

† [Earlier writers had used the word *aplanatic* to mean *free from spherical aberration*, the incident rays being direct. In modern usage, following E. Abbe, the word implies in addition the satisfaction of the "sine condition," which gives absence of circular coma of all orders (cf. G. C. Steward, *The Symmetrical Optical System*, p. 51). When Hamilton's conditions for direct and oblique aplanaticity are both satisfied, the system (being free from spherical aberration and coma) is aplanatic in the modern sense, to the order considered. See footnote to p. 429.]

These equations (A) and (B), which had been deduced in former investigations, were used by me on the 2nd. of January (1844), to determine the radii for the four surfaces of Mr. Phillips's double object glass.* m_1, m_2 are the reciprocals of the indices of the two component lenses; p_1, p_2, their powers;

$$\because \; p_1 = (m_1^{-1} - 1)(r_1 - r_2), \quad p_2 = (m_2^{-1} - 1)(r_3 - r_4);$$

r_1, r_2, r_3, r_4 being the four successive curvatures, positive when convex to the incident rays. The equation (A), under other forms, agrees with known results, for example, with Herschel's; the equation (B) is my new condition, for the improvement of the achromatic telescope. (See [**33**.], [**34**.].)

[**14**.] *Rays in one Diametral Plane.*

(Feb. 15th. 1844.) Let me now recapitulate, or reproduce, the most necessary part of the foregoing calculations, for the important case where the rays are supposed to be all contained *in one common diametral plane* of the instrument, which we shall take for the plane of xz.

In *any one medium*, index μ,

$$\sigma = \mu\alpha, \quad \upsilon = \mu\gamma, \quad \sigma^2 + \upsilon^2 = \mu^2, \quad \alpha\,\delta\sigma + \gamma\,\delta\upsilon = 0, \quad \frac{x'' - x'}{\alpha} = \frac{z'' - z'}{\gamma},$$

$$(x'' - x')\,\delta\sigma + (z'' - z')\,\delta\upsilon = 0;$$

or more concisely

$$\boxed{\Delta x\,\delta\sigma + \Delta z\,\delta\upsilon = 0.}$$

For *any one refraction*, at surface $\delta z = p\,\delta x$, $\Delta\sigma = -p\,\Delta\upsilon$,

$$\boxed{\Delta\sigma\,\delta x + \Delta\upsilon\,\delta z = 0.}$$

Thus, if light pass from x_0, z_0, to x_{n+1}, z_{n+1}, undergoing n refractions at x_1, z_1, .. x_n, z_n; and having its components of slowness successively σ_0, υ_0; σ_1, υ_1; .. σ_n, υ_n; we shall have

$$\left\{\begin{array}{l} (x_1 - x_0)\,\delta\sigma_0 + (z_1 - z_0)\,\delta\upsilon_0 = 0; \quad (\sigma_1 - \sigma_0)\,\delta x_1 + (\upsilon_1 - \upsilon_0)\,\delta z_1 = 0; \\ (x_2 - x_1)\,\delta\sigma_1 + (z_2 - z_1)\,\delta\upsilon_1 = 0; \quad (\sigma_2 - \sigma_1)\,\delta x_2 + (\upsilon_2 - \upsilon_1)\,\delta z_2 = 0; \\ \dots\dots\dots\dots\dots\dots \\ (x_{n+1} - x_n)\,\delta\sigma_n + (z_{n+1} - z_n)\,\delta\upsilon_n = 0; \end{array}\right.$$

therefore if

$$T_1 = x_1(\sigma_1 - \sigma_0) + z_1(\upsilon_1 - \upsilon_0),$$

$$T_2 = x_2(\sigma_2 - \sigma_1) + z_2(\upsilon_2 - \upsilon_1), \text{ \&c.,}$$

and

$$T = T_1 + T_2 + \dots + T_n,$$

we shall have

$$\delta T = x_{n+1}\,\delta\sigma_n - x_0\,\delta\sigma_0 + z_{n+1}\,\delta\upsilon_n - z_0\,\delta\upsilon_0$$

$$= \left(x_{n+1} - \frac{\alpha_n}{\gamma_n} z_{n+1}\right)\delta\sigma_n - \left(x_0 - \frac{\alpha_0}{\gamma_0} z_0\right)\delta\sigma_0.$$

If, then, we consider T as a function of σ_0 and σ_n, we shall have the two equations

$$x_0 - \frac{\alpha_0}{\gamma_0} z_0 = -\frac{\delta T}{\delta\sigma_0}, \quad x_{n+1} - \frac{\alpha_n}{\gamma_n} z_{n+1} = \frac{\delta T}{\delta\sigma_n};$$

* [See p. 384.]

of which the one may be considered as belonging to the initial, and the other to the final ray; whether these final and initial rays, or portions of one bent path of light, be *in vacuo* or in any ordinary media. And so far all is rigorous.

Now, let the surfaces be all of revolution about the axis of z, and let the course of each ray be little distant from that axis; then we may make, approximately,

$$v_i = v_i^{(0)} + v_i^{(2)} + v_i^{(4)}, \quad z_i = z_i^{(0)} + z_i^{(2)} + z_i^{(4)},$$
$$T_i = T_i^{(0)} + T_i^{(2)} + T_i^{(4)}, \quad T = T^{(0)} + T^{(2)} + T^{(4)};$$

neglecting terms small of the 6th. dimension in T.

$$T_i^{(0)} = z_i^{(0)}(v_i^{(0)} - v_{i-1}^{(0)}); \quad T_i^{(2)} = z_i^{(0)}(v_i^{(2)} - v_{i-1}^{(2)}) + z_i^{(2)}(v_i^{(0)} - v_{i-1}^{(0)}) + x_i(\sigma_i - \sigma_{i-1});$$
$$T_i^{(4)} = z_i^{(0)}(v_i^{(4)} - v_{i-1}^{(4)}) + z_i^{(2)}(v_i^{(2)} - v_{i-1}^{(2)}) + z_i^{(4)}(v_i^{(0)} - v_{i-1}^{(0)});$$

in which,

$$v_i^{(0)} = \mu_i; \quad v_i^{(2)} = -\frac{\sigma_i^2}{2\mu_i}; \quad v_i^{(4)} = -\frac{\sigma_i^4}{8\mu_i^3};$$

and we may write

$$z_i^{(0)} = v_i; \quad z_i^{(2)} = \tfrac{1}{2} r_i x_i^2; \quad z_i^{(4)} = \tfrac{1}{4} s_i x_i^4.$$

Considering T_i as an explicit function of x_i, σ_i, σ_{i-1}, we have the rigorous equations

$$0 = \frac{\delta T_i}{\delta x_i}; \quad 0 = \frac{\delta T_i}{\delta \sigma_i} + \frac{\delta T_{i+1}}{\delta \sigma_i};$$

namely n of the 1st. sort, and $n-1$ of the 2nd., to eliminate the $2n-1$ auxiliary or intermediate quantities $x_1, .. x_n, \sigma_1, .. \sigma_{n-1}$.

[15.] We have therefore $2n-1$ approximate equations of the forms:

$$\boxed{0 = \frac{\delta T_i^{(2)}}{\delta x_i}; \quad 0 = \frac{\delta (T_i^{(2)} + T_{i+1}^{(2)})}{\delta \sigma_i};}$$

which are all linear with respect to the quantities x and σ, and determine, approximately, values for $x_1, .. x_n, \sigma_1, .. \sigma_{n-1}$, as linear functions of σ_0 and σ_n; and if these values be substituted in $T_1^{(2)} + .. + T_n^{(2)}$, the result will evidently be $T^{(2)}$; as, still more evidently, $T_1^{(0)} + .. + T_n^{(0)} = T^{(0)}$. But, farther, since we have, still more nearly,

$$0 = \frac{\delta T_i^{(2)}}{\delta x_i} + \frac{\delta T_i^{(4)}}{\delta x_i}; \quad 0 = \frac{\delta}{\delta \sigma_i}(T_i'^{(2)} + T_{i+1}^{(2)} + T_i^{(4)} + T_{i+1}^{(4)});$$

therefore the errors of the approximate values, above deduced, for $x_1, .. x_n, \sigma_1, .. \sigma_{n-1}$, are small of the 3rd. dimension; the error, therefore, produced in $T_1^{(2)} + .. + T_n^{(2)}$, by the substitution of those approximate values, is small of the 6th. dimension, because it depends only on the squares and products of those small errors; consequently the substitution of the correct values of $x_1, .. x_n$, $\sigma_1, .. \sigma_{n-1}$, in $T_1^{(2)} + .. + T_n^{(2)}$, would contribute nothing to $T^{(4)}$, though it would to $T^{(6)}$; and therefore we may write

$$\boxed{T^{(0)} = T_1^{(0)} + .. + T_n^{(0)}; \quad T^{(2)} = T_1^{(2)} + .. + T_n^{(2)}; \quad T^{(4)} = T_1^{(4)} + .. + T_n^{(4)};}$$

of which expressions, indeed, the 1st. may be considered as useless, but of which the two others are very important, and in which the *approximate* values of $x_1, .. \sigma_{n-1}$ are to be used, as determined by the equations at the beginning of this section.*

* [See Appendix, Note 24, p. 507.]

By [14.],

$$T_i^{(2)} = \tfrac{1}{2} v_i (\mu_{i-1}^{-1} \sigma_{i-1}^2 - \mu_i^{-1} \sigma_i^2) + \tfrac{1}{2} (\mu_i - \mu_{i-1}) r_i x_i^2 + (\sigma_i - \sigma_{i-1}) x_i;$$

$$\because \frac{\delta T_i^{(2)}}{\delta x_i} = \sigma_i - \sigma_{i-1} + (\mu_i - \mu_{i-1}) r_i x_i;$$

$$\frac{\delta T_i^{(2)}}{\delta \sigma_i} = x_i - v_i \mu_i^{-1} \sigma_i; \quad \frac{\delta T_{i+1}^{(2)}}{\delta \sigma_i} = - x_{i+1} + v_{i+1} \mu_i^{-1} \sigma_i;$$

therefore the $2n-1$ linear equations referred to above are of the forms:

$$- r_i x_i = \frac{\sigma_i - \sigma_{i-1}}{\mu_i - \mu_{i-1}}; \quad \frac{\sigma_i}{\mu_i} = \frac{x_{i+1} - x_i}{v_{i+1} - v_i};$$

those of the 1st. form expressing, with an approximation sufficient for our purpose, the law of refraction; and those of the 2nd. form expressing the law of rectilinearity. Under these forms they might have been deduced by more elementary considerations; thus the 1st. form, being equivalent to

$$\frac{\alpha_i + r_i x_i}{\alpha_{i-1} + r_i x_i} = \frac{\mu_{i-1}}{\mu_i},$$

is easily seen to give the law of the sines, to an accuracy of the 1st. dimension, or indeed of the 2nd., inclusive. *But the foregoing analysis is important, as showing that after calculating $T^{(2)}$ with these approximate values, we need not employ more exact expressions in order to obtain $T^{(4)}$ to the required degree of accuracy, but may simply substitute the same 1st. approximate values in $T_1^{(4)} + .. + T_n^{(4)}$.*

For a *single refracting surface*, eliminating x_i, we find

$$2T_i^{(2)} = v_i (\mu_{i-1}^{-1} \sigma_{i-1}^2 - \mu_i^{-1} \sigma_i^2) - r_i^{-1} (\mu_i - \mu_{i-1})^{-1} (\sigma_i - \sigma_{i-1})^2;$$

or, more concisely,

$$- 2T_i^{(2)} = v_i \Delta \frac{\sigma_{i-1}^2}{\mu_{i-1}} + r_i^{-1} \frac{\Delta \sigma_{i-1}^2}{\Delta \mu_{i-1}}.$$

[16.] For *two successive refractions*, the linear relation between $\sigma_{i-1}, \sigma_i, \sigma_{i+1}$, may be obtained by adding the recent value of $T_i^{(2)}$ to that of $T_{i+1}^{(2)}$, and equating to 0 the differential of the sum, taken with respect to σ_i; which process gives

$$0 = (v_{i+1} - v_i) \mu_i^{-1} \sigma_i - r_i^{-1} (\mu_i - \mu_{i-1})^{-1} (\sigma_i - \sigma_{i-1}) + r_{i+1}^{-1} (\mu_{i+1} - \mu_i)^{-1} (\sigma_{i+1} - \sigma_i);$$

a result which may also easily be obtained by eliminating x_i, x_{i+1}, between the three equations

$$- r_i x_i = (\mu_i - \mu_{i-1})^{-1} (\sigma_i - \sigma_{i-1}), \quad - r_{i+1} x_{i+1} = (\mu_{i+1} - \mu_i)^{-1} (\sigma_{i+1} - \sigma_i),$$

$$\mu_i^{-1} \sigma_i = (v_{i+1} - v_i)^{-1} (x_{i+1} - x_i);$$

that is, between two equations of refraction, and one equation of rectilinearity.

For a *single lens in a single medium*, $\mu_{i+1} = \mu_{i-1}$, and if we make

$$R_i = r_i - r_{i+1} + \mu_i^{-1}(\mu_i - \mu_{i-1}) r_i r_{i+1} (v_{i+1} - v_i),$$

then

$$R_i \sigma_i = r_i \sigma_{i+1} - r_{i+1} \sigma_{i-1};$$

also

$$2T_i^{(2)} + 2T_{i+1}^{(2)} - \mu_{i-1}^{-1}(v_i \sigma_{i-1}^2 - v_{i+1}\sigma_{i+1}^2) - (\mu_i - \mu_{i-1})^{-1}(r_{i+1}^{-1}\sigma_{i+1}^2 - r_i^{-1}\sigma_{i-1}^2)$$

$$= \{(v_{i+1} - v_i)\mu_i^{-1} - (\mu_i - \mu_{i-1})^{-1}(r_i^{-1} - r_{i+1}^{-1})\}\sigma_i^2$$

$$+ 2(\mu_i - \mu_{i-1})^{-1}\{r_i^{-1}\sigma_{i-1} - r_{i+1}^{-1}\sigma_{i+1}\}\sigma_i$$

$$= r_i^{-1} r_{i+1}^{-1}(\mu_i - \mu_{i-1})^{-1}\{R_i \sigma_i^2 + 2(r_{i+1}\sigma_{i-1} - r_i \sigma_{i+1})\sigma_i\}$$

$$= - r_i^{-1} r_{i+1}^{-1}(\mu_i - \mu_{i-1})^{-1} R_i^{-1}(r_i \sigma_{i+1} - r_{i+1}\sigma_{i-1})^2;$$

and

$$(r_i \sigma_{i+1}^2 - r_{i+1}\sigma_{i-1}^2)(r_i - r_{i+1}) - (r_i \sigma_{i+1} - r_{i+1}\sigma_{i-1})^2 = - r_i r_{i+1}(\sigma_{i+1} - \sigma_{i-1})^2;$$

$$\because\ 2(T_i^{(2)} + T_{i+1}^{(2)}) = \mu_{i-1}^{-1}(v_i \sigma_{i-1}^2 - v_{i+1}\sigma_{i+1}^2) + \mu_i^{-1} R_i^{-1}(v_{i+1} - v_i)(r_i \sigma_{i+1}^2 - r_{i+1}\sigma_{i-1}^2)$$
$$- (\mu_i - \mu_{i-1})^{-1} R_i^{-1}(\sigma_{i+1} - \sigma_{i-1})^2. \quad \text{(Compare [22.].)}$$

The equation of a ray incident on this lens may be put under the approximate form

$$x_{i-1} = \alpha_{i-1}(z_{i-1} - v_i + \mu_i^{-1}\mu_{i-1} R_i^{-1} r_{i+1}\overline{v_{i+1} - v_i}) - (\mu_i - \mu_{i-1})^{-1} R_i^{-1}(\sigma_{i+1} - \sigma_{i-1});$$

and the equation of the corresponding emergent ray is, in the same order of approximation,

$$x_{i+2} = \alpha_{i+1}(z_{i+2} - v_{i+1} + \mu_i^{-1}\mu_{i+1} R_i^{-1} r_i \overline{v_{i+1} - v_i}) - (\mu_i - \mu_{i-1})^{-1} R_i^{-1}(\sigma_{i+1} - \sigma_{i-1}).$$

Hence, *if these two rays be parallel to each other*, the 1st. cuts the axis of the lens in the *focal centre*

$$z_{i-1} = v_i - \frac{\mu_{i-1} r_{i+1}}{\mu_i R_i}(v_{i+1} - v_i);$$

and the 2nd. cuts the axis in the other focal centre

$$z_{i+2} = v_{i+1} - \frac{\mu_{i+1} r_i}{\mu_i R_i}(v_{i+1} - v_i).$$

Also any incident ray has the same distance from the axis at the 1st. focal centre, as the corresponding emergent ray at the 2nd. focal centre; namely at a distance

$$= - (\mu_i - \mu_{i-1})^{-1} R_i^{-1}(\sigma_{i+1} - \sigma_{i-1}).$$

Rays incident towards &c. See [22.], [23.].

[17.] (Feb. 16th.) Defining α', γ' to be sine and cosine of inclination of incident ray to axis of z; and α'', γ'' sine and cosine of inclination of refracted ray to same axis; ν corresponding inclination of normal of refracting surface, at point of refraction; the sine of incidence will be, rigorously,

$$\alpha' \cos \nu - \gamma' \sin \nu,$$

and that of refraction will be

$$\alpha'' \cos \nu - \gamma'' \sin \nu;$$

so that the fundamental law of refraction gives, rigorously,

$$\mu' (\alpha' \cos \nu - \gamma' \sin \nu) = \mu'' (\alpha'' \cos \nu - \gamma'' \sin \nu),$$

μ' being the index of the 1st. medium, and μ'' that of the 2nd. If then we make

$$\mu'\alpha' = \sigma', \quad \mu'\gamma' = \upsilon', \quad \mu''\alpha'' = \sigma'', \quad \mu''\gamma'' = \upsilon'',$$

we have

$$(\sigma'' - \sigma') \cos \nu = (\upsilon'' - \upsilon') \sin \nu,$$

or more concisely

$$\boxed{\Delta\sigma = \Delta\upsilon \tan \nu,}$$

as an expression for the LAW OF REFRACTION. This gives

$$\boxed{\Delta\sigma\,\delta x + \Delta\upsilon\,\delta z = 0,}$$

x, z being coordinates of incidence. Also, by rectilinearity of ray between any two successive refractions,

$$\frac{\Delta x}{\alpha} = \frac{\Delta z}{\gamma}, \quad \therefore \frac{\Delta x}{\sigma} = \frac{\Delta z}{\upsilon};$$

and because

$$\mu^2 = \sigma^2 + \upsilon^2, \quad 0 = \sigma\,\delta\sigma + \upsilon\,\delta\upsilon,$$

the LAW OF RECTILINEARITY is expressed by the equation

$$\boxed{\Delta x\,\delta\sigma + \Delta z\,\delta\upsilon = 0.}$$

Hence, if we make

$$\boxed{T_i = x_i \Delta_i \sigma + z_i \Delta_i \upsilon,}$$

x_i, z_i being coordinates of ith. point of incidence, or of refraction, and Δ_i the characteristic of the change there produced, so that, more fully,

$$\Delta_i \sigma = \Delta\sigma_{i-1} = \sigma_i - \sigma_{i-1},$$

if σ_i be the value of σ after the ith. refraction, we shall have, *by the law of refraction*,

$$\delta T_i = x_i \Delta_i \delta\sigma + z_i \Delta_i \delta\upsilon;$$

and therefore, by the *law of rectilinearity*,

$$\boxed{\delta T_i = x_{i+1}\delta\sigma_i + z_{i+1}\delta\upsilon_i - x_{i-1}\delta\sigma_{i-1} - z_{i-1}\delta\upsilon_{i-1},}$$

x_{i-1}, z_{i-1} being the coordinates of any point on the ith. incident ray, and x_{i+1}, z_{i+1} being the coordinates of any point on the ith. refracted ray. If then we consider two successive refractions, we have (because $0 = \Delta x_i \delta\sigma_i + \Delta z_i \delta\upsilon_i$)

$$\delta (T_i + T_{i+1}) = x_{i+2}\delta\sigma_{i+1} + z_{i+2}\delta\upsilon_{i+1} - x_{i-1}\delta\sigma_{i-1} - z_{i-1}\delta\upsilon_{i-1}, \text{ \&c.};$$

and making $T = T_1 + T_2 + .. + T_n$, we have, for n successive refractions, the formula

$$\boxed{\delta T = x_{n+1}\delta\sigma_n + z_{n+1}\delta\upsilon_n - x_0\delta\sigma_0 - z_0\delta\upsilon_0;}$$

in which, by definition,

$$T = x_1(\sigma_1 - \sigma_0) + x_2(\sigma_2 - \sigma_1) + .. + x_n(\sigma_n - \sigma_{n-1}) + z_1(\upsilon_1 - \upsilon_0) + z_2(\upsilon_2 - \upsilon_1) + .. + z_n(\upsilon_n - \upsilon_{n-1});$$

it is therefore immediately given as an explicit homogeneous function of the 2nd. dimension, of the $2n$ coordinates of incidence, and the $2n+2$ quantities σ, υ; but the *equations of the refracting curves* give each z as a function of its own x; and the equations of the form $\sigma^2 + \upsilon^2 = \mu^2$, give, *for each medium*, υ as a function of σ; thus T may be considered as a function of the n x'es, and the $n+1$ σ's; but by the law of refraction, T is to be a maximum or minimum, or more generally to have a *stationary value* with respect to each of the x'es; and by the law of rectilinearity, it is to be stationary also with respect to $\sigma_1, .. \sigma_{n-1}$; eliminating therefore these auxiliary quantities, it will become a function of σ_0, σ_n, and we shall have the two *equations for initial and final rays*:

$$x_0 - \frac{\alpha_0}{\gamma_0} z_0 = -\frac{\delta T}{\delta \sigma_0}; \quad x_{n+1} - \frac{\alpha_n}{\gamma_n} z_{n+1} = +\frac{\delta T}{\delta \sigma_n}.$$

And the *elimination of each x, separately,* can be effected by means of the equation of the corresponding refracting curve. For that equation gives

$$z_i + x_i \tan \nu_i = f_i(\tan \nu_i); \quad T_i = \Delta_i \upsilon \,. f_i\left(\frac{\Delta_i \sigma}{\Delta_i \upsilon}\right);$$

and

$$T = \Sigma_{(i)}{}_1^n \,. \Delta_i \upsilon f_i\left(\frac{\Delta_i \sigma}{\Delta_i \upsilon}\right),$$

rigorously.

[18.] For a *refracting circle,*

$$x = -r^{-1} \sin \nu, \quad z = c - r^{-1} \cos \nu,$$

c being ordinate of centre, and r^{-1} radius, positive when convex to incident light; therefore

$$z + x \tan \nu = c - r^{-1} \sec \nu,$$

and

$$f(\tan \nu) = c - r^{-1} \sqrt{1 + (\tan \nu)^2}.$$

Hence, for ANY COMBINATION OF n REFRACTING CIRCLES, having their centres on one common axis, we have, *rigorously,*

$$T = \Sigma_{(i)}{}_1^n c_i \Delta_i \upsilon - \Sigma_{(i)}{}_1^n r_i^{-1} \Delta_i \upsilon \sqrt{1 + \left(\frac{\Delta_i \sigma}{\Delta_i \upsilon}\right)^2};$$

the radical being positive. Developing the radical as far as the 4th. power of $\Delta_i \sigma$, which we shall suppose to be small, and denoting the ordinate of the ith. vertex by

$$v_i = c_i - r_i^{-1},$$

we have, *nearly,*

$$T = \Sigma_{(i)}{}_1^n \{v_i \Delta_i \upsilon - \tfrac{1}{2} r_i^{-1} \Delta_i \upsilon^{-1} \Delta_i \sigma^2 + \tfrac{1}{8} r_i^{-1} \Delta_i \upsilon^{-3} \Delta_i \sigma^4\};$$

$\Delta_i v^{-1}$, $\Delta_i v^{-3}$, $\Delta_i \sigma^2$, $\Delta_i \sigma^4$, denoting $(\Delta_i v)^{-1}$, $(\Delta_i v)^{-3}$, $(\Delta_i \sigma)^2$, $(\Delta_i \sigma)^4$. And because, in the same order of approximation,

$$v = \mu - \tfrac{1}{2}\mu^{-1}\sigma^2 - \tfrac{1}{8}\mu^{-3}\sigma^4,$$

we have, still in the same order of approximation, T being $= \Sigma_{(i)}{}_1^n T_i$,

$$T_i = v_i \Delta_i \mu - \tfrac{1}{2} v_i \Delta_i \left(\frac{\sigma^2}{\mu}\right) - \tfrac{1}{8} v_i \Delta_i \left(\frac{\sigma^4}{\mu^3}\right) - \tfrac{1}{2} r_i^{-1} (\Delta_i \mu)^{-1} (\Delta_i \sigma)^2$$
$$- \tfrac{1}{4} r_i^{-1} (\Delta_i \mu)^{-2} (\Delta_i \sigma)^2 \Delta_i \left(\frac{\sigma^2}{\mu}\right) + \tfrac{1}{8} r_i^{-1} (\Delta_i \mu)^{-3} (\Delta_i \sigma)^4;$$

the parentheses being employed to make the notation more unambiguous. (Compare [**2**.].) We may conveniently distinguish these 6 terms of T_i as follows:

$$T_i = T_i^{(0)} + T'^{(2)}_i + T''^{(2)}_i + T'^{(4)}_i + T''^{(4)}_i + T'''^{(4)}_i = T_i^{(0)} + T_i^{(2)} + T_i^{(4)};$$

$$T_i^{(0)} = v_i \Delta_i \mu; \quad T'^{(2)}_i = -\tfrac{1}{2} v_i \Delta_i \frac{\sigma^2}{\mu}; \quad T''^{(2)}_i = -\tfrac{1}{2} r_i^{-1} (\Delta_i \mu)^{-1} (\Delta_i \sigma)^2;$$

$$T'^{(4)}_i = -\tfrac{1}{8} v_i \Delta_i \frac{\sigma^4}{\mu^3}; \quad T''^{(4)}_i = -\tfrac{1}{4} r_i^{-1} (\Delta_i \mu)^{-2} (\Delta_i \sigma)^2 \Delta_i \frac{\sigma^2}{\mu}; \quad T'''^{(4)}_i = \tfrac{1}{8} r_i^{-1} (\Delta_i \mu)^{-3} (\Delta_i \sigma)^4.$$

Also the $n-1$ intermediate σ's are to be eliminated by the $n-1$ conditions of *stationary value*, which are of the form

$$0 = \frac{\delta (T_i + T_{i+1})}{\delta \sigma_i};$$

that is, sufficiently for the calculation of T, to the accuracy of the 4th. dimension inclusive (*by the properties of stationary values*),

$$0 = \frac{\delta T_i^{(2)}}{\delta \sigma_i} + \frac{\delta T_{i+1}^{(2)}}{\delta \sigma_i}; \quad T_i^{(2)} = T'^{(2)}_i + T''^{(2)}_i; \quad T_i^{(4)} = T'^{(4)}_i + T''^{(4)}_i + T'''^{(4)}_i;$$
$$T = T^{(0)} + T^{(2)} + T^{(4)}; \quad T^{(0)} = \Sigma_i T_i^{(0)}; \quad T^{(2)} = \Sigma_i T_i^{(2)}; \quad T^{(4)} = \Sigma_i T_i^{(4)}.$$

Now,

$$\frac{\delta T'^{(2)}_i}{\delta \sigma_i} = -\frac{v_i \sigma_i}{\mu_i}; \quad \frac{\delta T'^{(2)}_{i+1}}{\delta \sigma_i} = \frac{v_{i+1} \sigma_i}{\mu_i}; \quad \therefore \frac{\delta}{\delta \sigma_i} (T'^{(2)}_i + T'^{(2)}_{i+1}) = \mu_i^{-1} (v_{i+1} - v_i) \sigma_i;$$

$$\frac{\delta T''^{(2)}}{\delta \sigma_i} = -r_i^{-1} (\Delta_i \mu)^{-1} \Delta_i \sigma; \quad \frac{\delta T''^{(2)}_{i+1}}{\delta \sigma_i} = r_{i+1}^{-1} (\Delta_{i+1} \mu)^{-1} \Delta_{i+1} \sigma;$$

therefore the equation connecting σ_i with σ_{i-1} and σ_{i+1} is the following;*

$$0 = \{\mu_i^{-1} (v_{i+1} - v_i) - r_i^{-1} (\Delta_i \mu)^{-1} - r_{i+1}^{-1} (\Delta_{i+1} \mu)^{-1}\} \sigma_i + r_i^{-1} (\Delta_i \mu)^{-1} \sigma_{i-1} + r_{i+1}^{-1} (\Delta_{i+1} \mu)^{-1} \sigma_{i+1}.$$

This *equation in differences* is of the form

$$0 = A_i \sigma_{i-1} + C_i \sigma_i + A_{i+1} \sigma_{i+1};$$

* [See also beginning of [**16**.].]

and we have $n-1$ such, namely those corresponding to $i=1, 2, .. n-1$. *When the surfaces are all close together*, then

$$C_i = -A_i - A_{i+1};$$

and the equation in differences becomes

$$A_i(\sigma_i - \sigma_{i-1}) = A_{i+1}(\sigma_{i+1} - \sigma_i) = C;$$

$$\sigma_1 - \sigma_0 = CA_1^{-1}, \quad \sigma_2 - \sigma_1 = CA_2^{-1}, \; .. \; \sigma_n - \sigma_{n-1} = CA_n^{-1};$$

$$\because \; \sigma_n - \sigma_0 = C\Sigma_{(i)}{}_1^n A_i^{-1},$$

and finally

$$\sigma_i = \sigma_0 + (\sigma_n - \sigma_0)\frac{\Sigma_{(i)}{}_1^i r_i \Delta_i \mu}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu};$$

that is,

$$\boxed{\sigma_i \Sigma_{(i)}{}_1^n r_i \Delta_i \mu = \sigma_0 \Sigma_{(i)}{}_{i+1}^n r_i \Delta_i \mu + \sigma_n \Sigma_{(i)}{}_1^i r_i \Delta_i \mu.}$$

[**19**.] [When the vertices of the refracting circles are distinct, we have]

$$\Delta(A_i \Delta_i \sigma) = -\mu_i^{-1}\sigma_i \Delta v_i = A_i'; \quad (\Delta_i \sigma = \sigma_i - \sigma_{i-1}; \quad \Delta\phi_i = \phi_{i+1} - \phi_i;) \quad A_i = r_i^{-1}(\Delta_i\mu)^{-1};$$

$$\left.\begin{array}{l} A_2\Delta_2\sigma - A_1\Delta_1\sigma = A_1' \\ A_3\Delta_3\sigma - A_2\Delta_2\sigma = A_2' \\ \quad\&\text{c.} \end{array}\right\} \; \because \; \left\{\begin{array}{l} A_3\Delta_3\sigma - A_1\Delta_1\sigma = A_1' + A_2', \\ A_4\Delta_4\sigma - A_1\Delta_1\sigma = A_1' + A_2' + A_3', \\ \quad\&\text{c.} \end{array}\right.$$

If then we make $A_i'' = \Sigma_{(i)}{}_1^i A_i'$, we have $n-1$ equations of the form

$$A_{i+1}\Delta_{i+1}\sigma = A_1\Delta_1\sigma + A_i'',$$

or of the form

$$\Delta_{i+1}\sigma = A_{i+1}^{-1}(A_1\Delta_1\sigma + A_i''),$$

i being successively $=1, 2, .. n-1$; we may also include with these the case $i=0$, by treating A_0'' as $=0$. Hence, by addition,

$$\sigma_n - \sigma_0 = \Sigma_{(i)}{}_1^n . A_i^{-1}(A_1\Delta_1\sigma + A_{i-1}'');$$

$$\because \; A_1\Delta_1\sigma_1 = \frac{\sigma_n - \sigma_0 - \Sigma_{(i)}{}_1^n . A_i^{-1}A_{i-1}''}{\Sigma_{(i)}{}_1^n . A_i^{-1}}; \quad \sigma_i - \sigma_0 = A_1\Delta_1\sigma . \Sigma_{(i)}{}_1^i A_i^{-1} + \Sigma_{(i)}{}_1^i . A_i^{-1}A_{i-1}'';$$

and

$$\left\{\begin{array}{l} (\sigma_i - \sigma_0)\,\Sigma_{(i)}{}_1^n r_i\Delta_i\mu - (\sigma_n - \sigma_0)\,\Sigma_{(i)}{}_1^i r_i\Delta_i\mu \\ = \Sigma_{(i)}{}_1^n r_i\Delta_i\mu . \Sigma_{(i)}{}_1^i r_i\Delta_i\mu A_{i-1}'' - \Sigma_{(i)}{}_1^i r_i\Delta_i\mu . \Sigma_{(i)}{}_1^n r_i\Delta_i\mu A_{i-1}''; \end{array}\right.$$

in which

$$A_{i-1}'' = -\Sigma_{(i)}{}_1^{i-1} . \mu_i^{-1}\sigma_i(v_{i+1} - v_i); \quad A_0'' = 0; \quad \Sigma_{(i)}{}_1^{\&\text{c.}} . (\&\text{c.})\,A_{i-1}'' = \Sigma_{(i)}{}_2^{\&\text{c.}} . (\&\text{c.})\,A_{i-1}''.$$

Thus, retaining the abridgments A_i, A_i', as defined above, and not neglecting any powers of the quantities A_i', we have

$$\sigma_i \Sigma_{(i)}{}_1^n A_i^{-1} = \sigma_0 \Sigma_{(i)}{}_{i+1}^n A_i^{-1} + \sigma_n \Sigma_{(i)}{}_1^i A_i^{-1}$$
$$+ \Sigma_{(i)}{}_1^n A_i^{-1} . \Sigma_{(i)}{}_1^i . A_i^{-1}\Sigma_{(i)}{}_1^{i-1} A_i' - \Sigma_{(i)}{}_1^i A_i^{-1} . \Sigma_{(i)}{}_1^n . A_i^{-1}\Sigma_{(i)}{}_1^{i-1} A_i';$$

or, making for abridgment

$$\boxed{\lambda_i \Sigma_{(i)}{}_1^n A_i^{-1} = \Sigma_{(i)}{}_1^i A_i^{-1},}$$

so that $\lambda_0 = 0, \lambda_n = 1$, we have

$$\boxed{\sigma_i = \sigma_0 + \lambda_i (\sigma_n - \sigma_0 - \Sigma_{(i)2}^{\ n} . A_i^{-1} \Sigma_{(i)1}^{\ i-1} A_i') + \Sigma_{(i)2}^{\ i} . A_i^{-1} \Sigma_{(i)1}^{\ i-1} A_i'.}$$

For example,

$$\sigma_1 = \sigma_0 + \lambda_1 (\sigma_n - \sigma_0 - \Sigma_{(i)2}^{\ n} . A_i^{-1} \Sigma_{(i)1}^{\ i-1} A_i');$$

and accordingly this agrees with $A_1 \Delta_1 \sigma$, above. Make for abridgment

$$\boxed{B_i = \Sigma_{(i)1}^{\ i} . A_i^{-1} \Sigma_{(i)1}^{\ i-1} A_i';}$$

so that

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = A_2^{-1} A_1', \quad B_3 - B_2 = A_3^{-1} (A_1' + A_2'),$$

$$B_4 - B_3 = A_4^{-1} (A_1' + A_2' + A_3'), \text{ \&c.},$$

$$B_i = A_1' (A_2^{-1} + A_3^{-1} + .. + A_i^{-1}) + A_2' (A_3^{-1} + .. + A_i^{-1}) + .. + A_{i-1}' A_i^{-1};$$

then

$$\boxed{\sigma_i = \sigma_0 + \lambda_i (\sigma_n - \sigma_0 - B_n) + B_i.}$$

To verify that this expression does in fact satisfy the equation in differences relative to σ, we may observe that it gives

$$\Delta_i \sigma = \sigma_i - \sigma_{i-1} = (\lambda_i - \lambda_{i-1}) (\sigma_n - \sigma_0 - B_n) + B_i - B_{i-1};$$

in which

$$\lambda_i - \lambda_{i-1} = A_i^{-1} (\Sigma_{(i)1}^{\ n} A_i^{-1})^{-1}; \quad B_i - B_{i-1} = A_i^{-1} \Sigma_{(i)1}^{\ i-1} A_i';$$

$$\because \ A_i \Delta_i \sigma = (\sigma_n - \sigma_0 - B_n) (\Sigma_{(i)1}^{\ n} A_i^{-1})^{-1} + \Sigma_{(i)1}^{\ i-1} A_i',$$

$$\because \ \boxed{\Delta . A_i \Delta_i \sigma = A_i',} \text{ as above.}$$

And if, in A_i', we substitute for σ_i its first approximate value, namely $\sigma_0 + \lambda_i (\sigma_n - \sigma_0)$, we shall obtain corresponding expressions for B_i, B_n, which will give for σ_i a more correct value, indeed the one which we are to employ, if we neglect the squares and products of the intervals between the successive refracting surfaces.

[20.] (Feb. 16, 1844.) If we make for abridgment

$$a_i = r_i (\mu_i - \mu_{i-1}), \quad b_i = (v_{i+1} - v_i) \mu_i^{-1} \sigma_i,$$

the linear equation between σ_{i-1}, σ_i, σ_{i+1}, assigned near the beginning of **[16.]**,* becomes

$$0 = a_{i+1}^{-1} (\sigma_{i+1} - \sigma_i) - a_i^{-1} (\sigma_i - \sigma_{i-1}) + b_i.$$

Thus,

$$\begin{cases} 0 = a_2^{-1} (\sigma_2 - \sigma_1) - a_1^{-1} (\sigma_1 - \sigma_0) + b_1, \\ 0 = a_3^{-1} (\sigma_3 - \sigma_2) - a_2^{-1} (\sigma_2 - \sigma_1) + b_2, \text{ \&c.} \end{cases}$$

Hence

$$0 = a_3^{-1} (\sigma_3 - \sigma_2) - a_1^{-1} (\sigma_1 - \sigma_0) + b_1 + b_2, \text{ \&c.};$$

* [See also **[18.]**.]

$$\because \begin{cases} \sigma_1 - \sigma_0 = a_1 a_1^{-1} (\sigma_1 - \sigma_0), \\ \sigma_2 - \sigma_1 = a_2 a_1^{-1} (\sigma_1 - \sigma_0) - a_2 b_1, \\ \sigma_3 - \sigma_2 = a_3 a_1^{-1} (\sigma_1 - \sigma_0) - a_3 (b_1 + b_2), \text{ \&c.} \end{cases}$$

$$\therefore \begin{cases} \sigma_2 - \sigma_0 = a_1^{-1} (a_1 + a_2)(\sigma_1 - \sigma_0) - a_2 b_1, \\ \sigma_3 - \sigma_0 = a_1^{-1} (a_1 + a_2 + a_3)(\sigma_1 - \sigma_0) - a_2 b_1 - a_3 (b_1 + b_2), \text{ \&c.} \end{cases}$$

$$\therefore \begin{cases} \sigma_n - \sigma_0 = a_1^{-1} (a_1 + a_2 + .. + a_n)(\sigma_1 - \sigma_0) \\ \qquad - a_2 b_1 - a_3 (b_1 + b_2) - .. - a_n (b_1 + b_2 + .. + b_{n-1}); \end{cases}$$

let

$$\lambda_i = \frac{a_1 + a_2 + .. + a_i}{a_1 + a_2 + .. + a_n}, \quad \lambda_i' = a_2 b_1 + a_3 (b_1 + b_2) + .. + a_i (b_1 + .. + b_{i-1});$$

then

$$\boxed{\sigma_i - \sigma_0 = \lambda_i (\sigma_n - \sigma_0 + \lambda_n') - \lambda_i';} \qquad \lambda_1' = 0, \quad \lambda_n = 1.$$

This equation is a rigorous result of the equation in differences relative to σ; but λ_i', λ_n' involve the intermediate σ's, which are the quantities sought. (In [**19.**] B_i was written for what is here $-\lambda_i'$.) However they involve them only as multiplied by the successive intervals between the vertices, or surfaces; if then we neglect the squares and products of these intervals, we shall have

$$b_i = (v_{i+1} - v_i) \mu_i^{-1} \{\sigma_0 + \lambda_i (\sigma_n - \sigma_0)\},$$

and thence may compute λ_i', λ_n', and ultimately σ_i.

Make for abridgment

$$(v_{i+1} - v_i) \mu_i^{-1} = d_i,$$

(since we do not employ the differential d;) then

$$b_j = d_j \sigma_j = d_j \{\sigma_0 + \lambda_j (\sigma_n - \sigma_0)\};$$

and the coefficient of d_j in σ_i is $\sigma_0 + \lambda_j (\sigma_n - \sigma_0)$ multiplied by the coefficient of b_j in $\lambda_i \lambda_n' - \lambda_i'$; which last coefficient is

$$\lambda_i (a_{j+1} + a_{j+2} + .. + a_n) - (a_{j+1} + a_{j+2} + .. + a_i).$$

This coefficient vanishes, unless $j < n$; and its last part vanishes, unless $j < i$. When multiplied by $a_1 + .. + a_n$, it becomes

$$= (a_1 + .. + a_j)(a_{i+1} + .. + a_n),$$

if $j \leqq i$; but

$$= (a_1 + .. + a_i)(a_{j+1} + .. + a_n),$$

if $j \geqq i$. (When $j < i$, we employ here the principle that

$$(A + B)(B + C) - (A + B + C) B = AC;$$

$$A = a_1 + .. + a_j; \quad B = a_{j+1} + .. + a_i; \quad C = a_{i+1} + .. + a_n.)$$

[21.] Since $T^{(2)} = \Sigma T^{(2)}_i$ is homogeneous of the second dimension,

$$2T^{(2)} = \sigma_0 \frac{\delta T^{(2)}}{\delta \sigma_0} + \sigma_n \frac{\delta T^{(2)}}{\delta \sigma_n} = \sigma_0 \frac{\delta T^{(2)}_1}{\delta \sigma_0} + \sigma_n \frac{\delta T^{(2)}_n}{\delta \sigma_n};$$

$$\left(0 = \frac{\delta T^{(2)}}{\delta \sigma_1} = \frac{\delta T^{(2)}}{\delta \sigma_2} = .. = \frac{\delta T^{(2)}}{\delta \sigma_{n-1}};\right)$$

now*

$$T^{(2)}_1 = T'^{(2)}_1 + T''^{(2)}_1, \quad T^{(2)}_n = T'^{(2)}_n + T''^{(2)}_n;$$

$$\frac{\delta T'^{(2)}_1}{\delta \sigma_0} = v_1 \mu_0^{-1} \sigma_0, \quad \frac{\delta T'^{(2)}_n}{\delta \sigma_n} = - v_n \mu_n^{-1} \sigma_n;$$

$$\frac{\delta T''^{(2)}_1}{\delta \sigma_0} = r_1^{-1} (\Delta_1 \mu)^{-1} \Delta_1 \sigma; \quad \frac{\delta T''^{(2)}_n}{\delta \sigma_n} = - r_n^{-1} (\Delta_n \mu)^{-1} \Delta_n \sigma;$$

$$\Delta_1 \sigma = \sigma_1 - \sigma_0 = \lambda_1 (\sigma_n - \sigma_0 - B_n); \quad \Delta_n \sigma = \sigma_n - \sigma_{n-1} = (1 - \lambda_{n-1})(\sigma_n - \sigma_0) + \lambda_{n-1} B_n - B_{n-1};$$

$$\lambda_1 = r_1 \Delta_1 \mu \,.\, (\Sigma_{(i)}{}_1^n r_i \Delta_i \mu)^{-1}; \quad 1 - \lambda_{n-1} = r_n \Delta_n \mu \,.\, (\Sigma_{(i)}{}_1^n r_i \Delta_i \mu)^{-1};$$

$$B_n - B_{n-1} = r_n \Delta_n \mu \,.\, \Sigma_{(i)}{}_1^{n-1} A_i';$$

$$\therefore \frac{\delta T''^{(2)}_i}{\delta \sigma_0} = \frac{\sigma_n - \sigma_0 - B_n}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu}; \quad \frac{\delta T''^{(2)}_n}{\delta \sigma_n} = - \frac{\sigma_n - \sigma_0 - B_n}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu} - \Sigma_{(i)}{}_1^{n-1} A_i';$$

$$\therefore \boxed{2T^{(2)} = v_1 \mu_0^{-1} \sigma_0^2 - v_n \mu_n^{-1} \sigma_n^2 - \frac{(\sigma_n - \sigma_0)(\sigma_n - \sigma_0 - B_n)}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu} - \sigma_n \Sigma_{(i)}{}_1^{n-1} A_i';}$$

and this equation is *rigorous*, WHATEVER MAY BE THE NUMBER OF THE REFRACTING CURVES, AND THE MAGNITUDES OF THE INTERVALS BETWEEN THEM. But because

$$\boxed{A_i' = - \mu_i^{-1} \sigma_i \Delta v_i, \quad B_n = \Sigma_{(i)}{}_2^n \,.\, r_i \Delta_i \mu \Sigma_{(i)}{}_1^{i-1} A_i',}$$

the expression just given for $2T^{(2)}$ involves, explicitly, the $n-1$ intermediate σ's, though only as multiplied by the $n-1$ corresponding intervals Δv_i. If, however, we neglect these intervals, we find, for ANY COMBINATION OF REFRACTING SURFACES CLOSE TOGETHER,

$$\boxed{2T^{(2)} = v (\mu_0^{-1} \sigma_0^2 - \mu_n^{-1} \sigma_n^2) - \frac{(\sigma_n - \sigma_0)^2}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu};}$$

so that the approximate equations of initial and final rays are respectively

$$\boxed{x_0 = \alpha_0 (z_0 - v) - \frac{\sigma_n - \sigma_0}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu}; \quad x_{n+1} = \alpha_n (z_{n+1} - v) - \frac{\sigma_n - \sigma_0}{\Sigma_{(i)}{}_1^n r_i \Delta_i \mu}.}$$

If these two rays pass through the common vertex v, then $\sigma_n = \sigma_0$; and in fact the law of refraction shows easily that in this case $\sigma = \mu \alpha$ is not changed at all. This last result must hold

* [See [18.].]

good, even when higher powers of the σ's are taken into account, provided that the intervals between the surfaces still vanish.

For a COMBINATION OF TWO REFRACTING SURFACES, not necessarily close together, we have $n = 2$,

$$\Sigma_{(i)}{}_1^{n-1} A_i' = A_1' = -\mu_1^{-1}\sigma_1\Delta v_1; \quad B_n = r_2\Delta_2\mu \,.\, A_1' \,(= B_2);$$

also

$$\sigma_1 = \sigma_0 + \lambda_1(\sigma_2 - \sigma_0 - B_2), \quad \lambda_1 = \frac{r_1\Delta_1\mu}{r_1\Delta_1\mu + r_2\Delta_2\mu},$$

$$\boxed{\{r_1\Delta_1\mu + r_2\Delta_2\mu\,(1 - \mu_1^{-1}\Delta v_1 \,.\, r_1\Delta_1\mu)\}\,\sigma_1 = \sigma_2 r_1\Delta_1\mu + \sigma_0 r_2\Delta_2\mu;}$$

that is, (compare [18.]),

$$0 = (\sigma_2 - \sigma_1)(r_2\Delta_2\mu)^{-1} - (\sigma_1 - \sigma_0)(r_1\Delta_1\mu)^{-1} + \mu_1^{-1}\sigma_1\Delta v_1;$$

which is in fact (since 0, 1, 2 may here be changed to $i-1, i, i+1$, the two surfaces being arbitrary) the old equation in differences between any three successive σ's, deduced as a particular case from its own general integral. And since

$$\frac{r_2\Delta_2\mu \,.\, (\sigma_2 - \sigma_0)}{r_1\Delta_1\mu + r_2\Delta_2\mu} - \sigma_2 = -\frac{\sigma_2 r_1\Delta_1\mu + \sigma_0 r_2\Delta_2\mu}{r_1\Delta_1\mu + r_2\Delta_2\mu} = -\left\{1 - \frac{\mu_1^{-1} r_1 r_2 \Delta v_1 \Delta_1\mu\Delta_2\mu}{r_1\Delta_1\mu + r_2\Delta_2\mu}\right\}\sigma_1,$$

the additional term introduced into $2T^{(2)}$, by B_2 and A_1', for a combination of two surfaces, is

$$+\mu_1^{-1}\Delta v_1 \,.\left\{1 - \frac{\mu_1^{-1}\Delta v_1 \,.\, r_1\Delta_1\mu \,.\, r_2\Delta_2\mu}{r_1\Delta_1\mu + r_2\Delta_2\mu}\right\}\sigma_1^2;$$

that is,

$$\frac{\mu_1^{-1}\Delta v_1 \,.\, (\sigma_2 r_1\Delta_1\mu + \sigma_0 r_2\Delta_2\mu)^2}{(r_1\Delta_1\mu + r_2\Delta_2\mu)\,\{r_1\Delta_1\mu + r_2\Delta_2\mu\,(1 - \mu_1^{-1}\Delta v_1 \,.\, r_1\Delta_1\mu)\}}.$$

But

$$(\rho_1\sigma_2 + \rho_2\sigma_0)^2 + \rho_1\rho_2(\sigma_2 - \sigma_0)^2 = (\rho_1 + \rho_2)(\rho_1\sigma_2^2 + \rho_2\sigma_0^2);$$

therefore, *for any combination of two refracting surfaces,*

$$\boxed{2T^{(2)} = v_1\mu_0^{-1}\sigma_0^2 - v_2\mu_2^{-1}\sigma_2^2 - \frac{(\sigma_2 - \sigma_0)^2 - \mu_1^{-1}\Delta v_1 \,.\, (\sigma_2^2 r_1\Delta_1\mu + \sigma_0^2 r_2\Delta_2\mu)}{r_1\Delta_1\mu + r_2\Delta_2\mu - \mu_1^{-1}\Delta v_1 \,.\, r_1\Delta_1\mu \,.\, r_2\Delta_2\mu}.}$$

[22.] By foot of [15.], the form of $T^{(2)}$ is such that, for any *combination of two successive refracting surfaces,* (considered as 1st. and 2nd. in order,)

$$\begin{aligned}-2T^{(2)} + v_1\mu_0^{-1}\sigma_0^2 - v_2\mu_2^{-1}\sigma_2^2 &= -(v_2 - v_1)\mu_1^{-1}\sigma_1^2 + r_1^{-1}(\mu_1 - \mu_0)^{-1}(\sigma_1 - \sigma_0)^2 \\ &\quad + r_2^{-1}(\mu_2 - \mu_1)^{-1}(\sigma_2 - \sigma_1)^2 \\ &= \{(r_1\Delta_1\mu)^{-1} + (r_2\Delta_2\mu)^{-1} - \mu_1^{-1}\Delta v_1\}\,\sigma_1^2 - 2\{\sigma_2(r_2\Delta_2\mu)^{-1} + \sigma_0(r_1\Delta_1\mu)^{-1}\}\,\sigma_1 \\ &\quad + \sigma_0^2(r_1\Delta_1\mu)^{-1} + \sigma_2^2(r_2\Delta_2\mu)^{-1};\end{aligned}$$

therefore, eliminating σ_1 by the condition of stationary value, we find

$$\begin{aligned}&\{-2T^{(2)} + v_1\mu_0^{-1}\sigma_0^2 - v_2\mu_2^{-1}\sigma_2^2\}\,\{(r_1\Delta_1\mu)^{-1} + (r_2\Delta_2\mu)^{-1} - \mu_1^{-1}\Delta v_1\} \\ &= \{(r_1\Delta_1\mu)^{-1} + (r_2\Delta_2\mu)^{-1} - \mu_1^{-1}\Delta v_1\}\,\{(r_1\Delta_1\mu)^{-1}\sigma_0^2 + (r_2\Delta_2\mu)^{-1}\sigma_2^2\} - \{(r_1\Delta_1\mu)^{-1}\sigma_0 + (r_2\Delta_2\mu)^{-1}\sigma_2\}^2 \\ &= (r_1\Delta_1\mu)^{-1}(r_2\Delta_2\mu)^{-1}(\sigma_2 - \sigma_0)^2 - \mu_1^{-1}\Delta v_1\,\{(r_1\Delta_1\mu)^{-1}\sigma_0^2 + (r_2\Delta_2\mu)^{-1}\sigma_2^2\};\end{aligned}$$

a result which entirely agrees with that obtained at the foot of the preceding section, by a more complicated but more general process, of which the one in the present section may serve as a verification.

Particularising farther, let $\Delta_2\mu = -\Delta_1\mu$, and therefore $\mu_2 = \mu_0$; we get the case of a *single lens in any medium*, and have, for it, after multiplying both sides by $r_1 r_2 \Delta_1 \mu$, the equation

$$\{r_1 - r_2 + \mu_1^{-1}(\mu_1 - \mu_0) r_1 r_2 (v_2 - v_1)\}\{2T^{(2)} - \mu_0^{-1} v_1 \sigma_0^2 + \mu_0^{-1} v_2 \sigma_2^2\}$$
$$= -(\mu_1 - \mu_0)^{-1}(\sigma_2 - \sigma_0)^2 + \mu_1^{-1}(v_2 - v_1)(r_1\sigma_2^2 - r_2\sigma_0^2);$$

agreeing with [16.]. If, in this equation, we change μ_0, μ_1, σ_0, σ_2, $v_2 - v_1$, to 1, μ, α_0, α_2, t, we arrive at the same result as if, in the expression at the foot of [4.], we make $\beta_0 = 0$, $\beta_2 = 0$.

Resuming the *general* COMBINATION OF ANY TWO REFRACTING SURFACES, and making, for abridgment, as in [19.],

$$A_1 = (r_1 \Delta_1 \mu)^{-1}, \quad A_2 = (r_2 \Delta_2 \mu)^{-1},$$

and

$$t = \text{thickness} = \Delta v_1 = v_2 - v_1,$$

we have the expression*

$$T^{(2)} = \tfrac{1}{2} v_1 \mu_0^{-1} \sigma_0^2 - \tfrac{1}{2} v_2 \mu_2^{-1} \sigma_2^2 - \frac{A_1 A_2 (\sigma_2 - \sigma_0)^2 - \mu_1^{-1} t (A_1 \sigma_0^2 + A_2 \sigma_2^2)}{2(A_1 + A_2 - \mu_1^{-1} t)};$$

which gives, for the *initial and final rays*, the approximate equations:

$$x_0 = \alpha_0 \left(z_0 - v_1 - \frac{\mu_0 \mu_1^{-1} t A_1}{A_1 + A_2 - \mu_1^{-1} t} \right) - \frac{A_1 A_2 (\sigma_2 - \sigma_0)}{A_1 + A_2 - \mu_1^{-1} t};$$
$$x_3 = \alpha_2 \left(z_3 - v_2 + \frac{\mu_2 \mu_1^{-1} t A_2}{A_1 + A_2 - \mu_1^{-1} t} \right) - \frac{A_1 A_2 (\sigma_2 - \sigma_0)}{A_1 + A_2 - \mu_1^{-1} t}.$$

Hence if we make for abridgment (compare [6.], [8.])

$$F' = v_1 + \frac{\mu_0 \mu_1^{-1} t A_1}{A_1 + A_2 - \mu_1^{-1} t}; \quad F'' = v_2 - \frac{\mu_2 \mu_1^{-1} t A_2}{A_1 + A_2 - \mu_1^{-1} t};$$

the points F', F'', on the axis, may be called the *two focal centres of the combination*; in this sense, among others, that *if the final direction be the same as would have been produced by a plate, then this incident ray crosses the axis in F', and the final in F''.*†

* [No additional difficulty is involved in calculating $T^{(2)}$ in a form suitable for the discussion of exdiametral rays; this form is given by changing σ_0^2 to $\sigma_0^2 + \tau_0^2$, σ_2^2 to $\sigma_2^2 + \tau_2^2$, and $(\sigma_2 - \sigma_0)^2$ to $(\sigma_2 - \sigma_0)^2 + (\tau_2 - \tau_0)^2$. The expression at the end of [4.] is a particular case of the expression obtained as above.]

† [For refraction through a plate, whose faces are perpendicular to the z-axis, we have $\sigma_0 = \sigma_1 = \sigma_2$, $\tau_0 = \tau_1 = \tau_2$, since at each refraction $\Delta\sigma = \Delta\tau = 0$. The focal centres are the principal points; see Appendix, Note 25, p. 508.]

[23.] If we also make, for abridgment,

$$F = \frac{A_1 A_2}{A_1 + A_2 - \mu_1^{-1} t},$$

then parallel initial rays have for their final focus

$$x_3 = F\sigma_0, \quad z_3 = F'' + \mu_2 F;$$

and parallel final rays have for their initial focus

$$x_0 = -F\sigma_2, \quad z_0 = F' - \mu_0 F. \qquad \text{(Compare [24.].)}$$

Suppose then that an instrument is formed by enclosing the three successive media, on the one hand within a (sufficiently large) cylinder coaxal with the two refracting surfaces of revolution; and on the other hand within two planes, sufficiently distant from those surfaces, and perpendicular to the common axis: and let this instrument be exposed, *in vacuo*, directly to a planet, so as to form in each of its two reverse positions an image within the third medium, reckoning from the planet. *These two images will have equal dimensions*; for σ_0 in the first position will be equal to the angular semi-diameter of the planet, and so will σ_2 in the second position (neglecting signs); the images will also be *both inverted*, if F be positive, or *both erect*, if F be negative; and we may call F the *focal length of the instrument*, and therefore, in a certain sense, the *focal length of the combination* also, formed by the three media and the two curved refracting surfaces. Or we may state the theorem thus: an instrument of revolution, *in vacuo*, bounded by plane surfaces externally, and containing within itself any three successive media, separated from each other by any two curved surfaces, coaxal with the instrument itself, will have its focal length $= F$, in each of its two opposite positions; (because a plane refracting surface does not alter the magnitude of an image parallel to itself;) in such a manner that in each position it will form an image of the planet with a radius $= F \times$ angular semi-diameter of planet; and this image will be inverted or erect, according as F is $>$ or < 0.

[24.] This expression* is of the form

$$\boxed{2T^{(2)} = F'\mu_0^{-1}\sigma_0^2 - F''\mu_2^{-1}\sigma_2^2 - F(\sigma_2 - \sigma_0)^2;}$$

(Combination of any two coaxal refracting surfaces.)

and the equations of the initial and final rays are, approximately,

$$\boxed{\begin{aligned} x_0 &= \alpha_0(z_0 - F') - F(\sigma_2 - \sigma_0), \\ x_3 &= \alpha_2(z_3 - F'') - F(\sigma_2 - \sigma_0). \end{aligned}}$$

These two equations will give only one relation between the initial and final directions, if z_0, z_3 are connected by the equation

$$(z_0 - F' + \mu_0 F)(z_3 - F'' - \mu_2 F) + \mu_0\mu_2 F^2 = 0;$$

and then they give

$$\boxed{\frac{x_3}{x_0} = \frac{\mu_0 F}{z_0 - F' + \mu_0 F} = \frac{F'' + \mu_2 F - z_3}{\mu_2 F}; \quad \therefore \frac{\mu_2}{z_3 - F''} = \frac{\mu_0}{z_0 - F'} + \frac{1}{F}.}$$

* [This refers to the expression obtained in [22.].]

Under the conditions expressed upon this last line, the point x_3, z_3 is the *image* of the point x_0, z_0; that is, rays having the latter for their initial, have the former for their final focus. When the initial focus is an infinitely distant point, then

$$\frac{\mu_0 x_0}{z_0 - F' + \mu_0 F} = \sigma_0;$$

and the image is

$$x_3 = F\sigma_0, \quad z_3 = F'' + \mu_2 F.$$

In like manner, when the final focus is infinitely distant,

$$\frac{\mu_2 x_3}{F'' + \mu_2 F - z_3} = -\sigma_2,$$

and the initial focus is

$$x_0 = -F\sigma_2, \quad z_0 = F' - \mu_0 F. \qquad \text{(Compare [23.].)}$$

The image of x_0, F' is x_0, F''; hence, or more simply from the equations of the rays, the ordinate (perpendicular to the axis) of the initial ray at the first *focal centre* F', is the same as the corresponding ordinate of the final ray at the second focal centre F'';* this common ordinate being equal to $-F(\sigma_2 - \sigma_0)$. It vanishes when $\sigma_2 = \sigma_0$, that is, when the final direction is the same as it would have been, if the ray with the given initial direction had passed through the same media, but through a *plate* perpendicular to the axis. (Compare foot of [22.].) This plate must in general be thus perpendicular; because, by [17.], the condition $\Delta_2\sigma + \Delta_1\sigma = 0$, gives, (to the accuracy of the first dimension,)†

$$\nu_2 \Delta_2 \mu + \nu_1 \Delta_1 \mu = 0;$$

therefore unless

$$\Delta_2 \mu + \Delta_1 \mu = \mu_2 - \mu_0 = 0,$$

that is, unless the 1st. and 3rd. media have the same index, we cannot have $\nu_2 = \nu_1$, except when each $= 0$. On the contrary,

$$\frac{\nu_2}{\nu_1} = \frac{\mu_1 - \mu_0}{\mu_1 - \mu_2};$$

an equation which determines a certain set of *prisms*, such that if any one of them enclosed the second medium, we should have $\sigma_2 = \sigma_0$. Reciprocally, if the initial ray be directed to the first focal centre, $\sigma_2 = \sigma_0$, and the recent ratio between ν_2 and ν_1 must hold good. As a verification, since (by beginning of [18.])

$$x_1 = -r_1^{-1}\nu_1, \quad x_2 = -r_2^{-1}\nu_2,$$

we ought, by the present section, to find (when $\sigma_2 = \sigma_0$),

$$\left(\frac{x_2}{x_1} =\right) \frac{r_1(\mu_1 - \mu_0)}{r_2(\mu_1 - \mu_2)} = \left(-\frac{A_2}{A_1} =\right) \frac{\mu_0}{\mu_2} \cdot \frac{v_2 - F''}{v_1 - F'};$$

* [See Appendix, Note 25, p. 508.]

† [ν denotes generally the angle between the normal to the refracting surface and the axis of the instrument; it is positive or negative according as the projection of the normal (in the sense of z increasing) on the x-axis is positive or negative.]

which accordingly is true (compare foot of [22.]); for by foot of [21.], and top of present section,

$$\mu_0^{-1}(v_1 - F') = \frac{Ft}{\mu_1} r_2(\mu_1 - \mu_2); \qquad \mu_2^{-1}(v_2 - F'') = \frac{Ft}{\mu_1} r_1(\mu_1 - \mu_0);$$

$$F^{-1} = r_1 \Delta_1 \mu + r_2 \Delta_2 \mu - \frac{t}{\mu_1} r_1 \Delta_1 \mu \,.\, r_2 \Delta_2 \mu.$$

When initial ray passes through first focal centre (and consequently final ray through second), we have

$$\frac{x_2}{-x_1} = \frac{r_1 \Delta_1 \mu}{r_2 \Delta_2 \mu};$$

therefore the ordinate of intersection of intermediate ray with axis, is

$$v_2 - \frac{r_1 \Delta_1 \mu \,.\, (v_2 - v_1)}{r_1 \Delta_1 \mu + r_2 \Delta_2 \mu} = \frac{v_1 r_1 \Delta_1 \mu + v_2 r_2 \Delta_2 \mu}{r_1 \Delta_1 \mu + r_2 \Delta_2 \mu}.$$

Hence, when $\sigma_2 = \sigma_0$, we must have

$$\frac{t \mu_1^{-1} \sigma_1 r_1 \Delta_1 \mu}{r_1 \Delta_1 \mu + r_2 \Delta_2 \mu} = x_2 = \mu_2^{-1} \sigma_2 (v_2 - F''),$$

therefore

$$\frac{\sigma_1}{\sigma_2} = \frac{\sigma_1}{\sigma_0} = F(r_1 \Delta_1 \mu + r_2 \Delta_2 \mu);$$

which accordingly agrees with the general linear relation, in [21.], between σ_0, σ_1, σ_2, since that relation may be thus written:

$$\sigma_1 = F(\sigma_2 r_1 \Delta_1 \mu + \sigma_0 r_2 \Delta_2 \mu).$$

And this last may be considered as a form for the general equation in differences, or linear equation, between any three successive σ's.

[25.] In general, by [21.], if we suppose all the intervals Δv_i to vanish except one, namely Δv_j, we shall have $A_i' = 0$, unless $i = j$; therefore

$$\Sigma_{(i)}{}_1^{n-1} A_i' = A_j' = -\mu_j^{-1} \sigma_j \Delta v_j;$$

$$B_n = A_j' \Sigma_{(i)}{}_{j+1}^{n} r_i \Delta_i \mu = A_j' (1 - \lambda_j) \Sigma_{(i)}{}_1^n r_i \Delta_i \mu;$$

therefore the part introduced by Δv_j, in $2T^{(2)}$, is *rigorously*

$$= \mu_j^{-1} \sigma_j \Delta v_j \{\sigma_n - (1 - \lambda_j)(\sigma_n - \sigma_0)\}$$

$$= \mu_j^{-1} \sigma_j \Delta v_j \{\sigma_0 + \lambda_j (\sigma_n - \sigma_0)\};$$

if, then, we neglect the square of Δv_j, this part becomes*

$$\mu_j^{-1} \sigma_j^2 \Delta v_j = \mu_j^{-1} \sigma_j^2 (v_{j+1} - v_j).$$

Adding the $n - 1$ such terms (for $j = 1, 2, .. n - 1$) to $\mu_0^{-1} \sigma_0^2 v_1 - \mu_n^{-1} \sigma_n^2 v_n$, we get

$$v_1(\mu_0^{-1} \sigma_0^2 - \mu_1^{-1} \sigma_1^2) + v_2(\mu_1^{-1} \sigma_1^2 - \mu_2^{-1} \sigma_2^2) + .. + v_n(\mu_{n-1}^{-1} \sigma_{n-1}^2 - \mu_n^{-1} \sigma_n^2),$$

* [Since, by [19.] or [20.], $\lambda_j(\sigma_n - \sigma_0)$ is approximately equal to $\sigma_j - \sigma_0$.]

that is,

$$-\Sigma_{(i)}{}_1^n v_i \Delta_i \left(\frac{\sigma^2}{\mu}\right);$$

neglecting therefore only the squares and products of the intervals between the surfaces, we have, generally,

$$T^{(2)} = -\tfrac{1}{2}\Sigma_{(i)}{}_1^n v_i \Delta_i \frac{\sigma^2}{\mu} - \frac{(\sigma_n - \sigma_0)^2}{2\Sigma_{(i)}{}_1^n r_i \Delta_i \mu}; \qquad \text{(See next section.)}$$

in which we are to make, for the present order of approximation,

$$\sigma_i \Sigma_{(i)}{}_1^n r_i \Delta_i \mu = \sigma_0 \Sigma_{(i)}{}_{i+1}^n r_i \Delta_i \mu + \sigma_n \Sigma_{(i)}{}_1^i r_i \Delta_i \mu. \qquad \text{(See foot of [18.].)}$$

Hence parallel direct incident rays are brought to a final focus z_{n+1}, such that (in the present order of approximation) (last medium being a vacuum)

$$(z_{n+1} - v_n)^{-1} = \Sigma_{(i)}{}_1^n r_i \Delta_i \mu + \Sigma_{(i)}{}_1^{n-1} \mu_i^{-1} \Delta v_i \, (\Sigma_{(i)}{}_1^i r_i \Delta_i \mu)^2.$$

For example, if there be two infinitely thin lenses, near each other, *in vacuo*; then

$$(z_5 - v_4)^{-1} = p_1 + p_2 + p_1^2 \Delta v_2,$$

Δv_2 being the interval between the two lenses, and p_1, p_2 their powers;

$$p_1 = (\mu_1 - 1)(r_1 - r_2), \quad p_2 = (\mu_3 - 1)(r_3 - r_4).$$

In fact, here, the convergence after emerging from the first lens is p_1; therefore immediately before entering the second lens, it is $(p_1^{-1} - \Delta v_2)^{-1} = p_1 + p_1^2 \Delta v_2$; to which the second lens adds the convergence p_2. In like manner, if there be l lenses, and an interval $= \lambda$ after the kth; this interval adds $\lambda(p_1 + .. + p_k)^2$ to the final convergence by my formula, because

$$\Sigma_{(i)}{}_1^{2k} r_i \Delta_i \mu = p_1 + .. + p_k;$$

and accordingly the interval λ adds $\lambda(p_1 + .. + p_k)^2$ after emerging from the kth lens. If we call $\Sigma_{(i)}{}_1^i r_i \Delta_i \mu$ the *power* of the system of the i first refractors, and denote it by F_i^{-1}, then

$$(z_{n+1} - v_n)^{-1} = F_n + \Sigma_{(i)}{}_1^{n-1} \mu_i^{-1} F_i^{-2} \Delta v_i;$$

a formula which increases the propriety of regarding F_i as the *focal length* of the system of i surfaces. (See [23.].)

[26.] (Feb. 17th, 1844.) The method, in the preceding section, of deducing the expression

$$T^{(2)} = -\tfrac{1}{2}\Sigma_{(i)}{}_1^n v_i \Delta_i \frac{\sigma^2}{\mu} - \tfrac{1}{2}(\Sigma_{(i)}{}_1^n r_i \Delta_i \mu)^{-1} (\sigma_n - \sigma_0)^2,$$

in which only the squares and products of the intervals Δv_i are neglected, from that given in [21.], namely

$$T^{(2)} = \tfrac{1}{2} v_1 \frac{\sigma_0^2}{\mu_0} - \tfrac{1}{2} v_n \frac{\sigma_n^2}{\mu_n} - \tfrac{1}{2}(\Sigma_{(i)}{}_1^n r_i \Delta_i \mu)^{-1} (\sigma_n - \sigma_0)(\sigma_n - \sigma_0 - B_n) - \tfrac{1}{2}\sigma_n \Sigma_{(i)}{}_1^{n-1} A_i';$$

which latter holds good, whatever may be the magnitudes of those intervals between the successive surfaces, and in which we had put for abridgment

$$A_i' = -\mu_i^{-1}\sigma_i \Delta v_i, \quad B_i = \Sigma_{(i)1}^{\ i} . r_i \Delta_i \mu \Sigma_{(i)1}^{\ i-1} A_i',$$

while

$$\sigma_i = \sigma_0 + B_i + \lambda_i (\sigma_n - \sigma_0 - B_n),$$

$$\lambda_i = (\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu)^{-1} \Sigma_{(i)1}^{\ i} r_i \Delta_i \mu;$$

is perhaps not inelegant in itself, and satisfactory as a rather simple result of a long and somewhat subtle analysis. But, having *thus* found the expression at the top of the present section, I *now* see that it might have been obtained in a more elementary way, as follows. By [**18.**],

$$T^{(2)} = \Sigma_{(i)1}^{\ n} T_i^{(2)}; \quad T_i^{(2)} = T'^{(2)}_i + T''^{(2)}_i; \quad T'^{(2)}_i = -\tfrac{1}{2} v_i \Delta_i \frac{\sigma^2}{\mu}; \quad T''^{(2)}_i = -\tfrac{1}{2}(r_i \Delta_i \mu)^{-1} (\Delta_i \sigma)^2;$$

and $\sigma_1, .. \sigma_{n-1}$ are to be eliminated by the $n-1$ conditions of stationary value, which are of the forms

$$0 = \frac{\delta}{\delta \sigma_i} \Sigma_{(i)1}^{\ n} (T'^{(2)}_i + T''^{(2)}_i).$$

But, because

$$\Sigma_{(i)1}^{\ n} T'^{(2)}_i = \tfrac{1}{2} v_1 \frac{\sigma_0^2}{\mu_0} - \tfrac{1}{2} v_n \frac{\sigma_n^2}{\mu_n} + \tfrac{1}{2} \Sigma_{(i)1}^{\ n-1} \frac{\sigma_i^2}{\mu_i} \Delta v_i,$$

therefore

$$\frac{\delta}{\delta \sigma_i} \Sigma_{(i)1}^{\ n} T'^{(2)}_i = \frac{\sigma_i}{\mu_i} \Delta v_i,$$

while

$$\frac{\delta}{\delta \sigma_i} \Sigma_{(i)1}^{\ n} T''^{(2)}_i = \Delta . (r_i \Delta_i \mu)^{-1} \Delta_i \sigma;$$

and thus arose the equation in differences relative to σ, already employed, in [**19.**], &c., namely

$$\boxed{0 = \Delta . (r_i \Delta_i \mu)^{-1} \Delta_i \sigma + \mu_i^{-1} \sigma_i \Delta v_i;}$$

all, so far, being rigorous, that is, such that no powers of the intervals are neglected. If, however, we now neglect the squares and products of those intervals, it is evident that, in calculating $\Sigma T'^{(2)}_i$, we may employ, for $\sigma_1, .. \sigma_{n-1}$, the approximate values furnished by altogether neglecting those intervals; because each of these intermediate σ's enters only as multiplied by one or other of these intervals. And although the intervals Δv_i do not enter explicitly into $T''^{(2)}_i$, yet, because the first approximate values of the intermediate σ's are precisely those which render this sum a *stationary* value, it follows that the employment of more correct expressions for the σ's would only add terms involving the squares and products of the Δv's. We may, therefore, not only make

$$\boxed{T^{(2)} = T'^{(2)} + T''^{(2)}; \quad T'^{(2)} = \Sigma_{(i)1}^{\ n} T'^{(2)}_i; \quad T''^{(2)} = \Sigma_{(i)1}^{\ n} T''^{(2)}_i;}$$

but may calculate *each* of these two latter sums, in the present order of approximation, by employing for the intermediate σ's the values furnished by the equation

$$\boxed{0 = \Delta . (r_i \Delta_i \mu)^{-1} \Delta_i \sigma,}$$

which gives

$$\boxed{\sigma_i - \sigma_0 = \lambda_i (\sigma_n - \sigma_0);}$$

therefore

$$\Delta_i \sigma = \frac{r_i \Delta_i \mu . (\sigma_n - \sigma_0)}{\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu}; \quad T''^{(2)}_i = -\tfrac{1}{2} r_i \Delta_i \mu . (\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu)^{-2} (\sigma_n - \sigma_0)^2;$$

therefore

$$T''^{(2)} = \Sigma_{(i)1}^{\ n} T''^{(2)}_i = -\tfrac{1}{2} (\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu)^{-1} (\sigma_n - \sigma_0)^2;$$

and therefore, finally, $T^{(2)}$ has the form quoted at the top of the present section.

[27.] (Feb. 17, 1844.) By [18.],

$$\boxed{T^{(4)} = T'^{(4)} + T''^{(4)} + T'''^{(4)};}$$

in which

$$T'^{(4)} = \Sigma_{(i)1}^{\ n} T'^{(4)}_i; \quad T''^{(4)} = \Sigma_{(i)1}^{\ n} T''^{(4)}_i; \quad T'''^{(4)} = \Sigma_{(i)1}^{\ n} T'''^{(4)}_i;$$

$$T'^{(4)}_i = -\tfrac{1}{8} v_i \Delta_i \frac{\sigma^4}{\mu^3}; \quad T''^{(4)}_i = -\tfrac{1}{4} r_i^{-1} \left(\frac{\Delta_i \sigma}{\Delta_i \mu}\right)^2 \Delta_i \frac{\sigma^2}{\mu}; \quad T'''^{(4)}_i = \tfrac{1}{8} r_i^{-1} \frac{(\Delta_i \sigma)^4}{(\Delta_i \mu)^3};$$

this last equation being relative to *spheric* surfaces. If the refracting surfaces be of *revolution* (round the axis of z), but not necessarily spheric, then we may write, (see end of [17.],)

$$z_i = v_i + \tfrac{1}{2} r_i x_i^2 + \tfrac{1}{4} s_i x_i^4, \qquad \text{neglecting } x_i^6;$$

$$\tan \nu_i = -\frac{dz_i}{dx_i} = -r_i x_i - s_i x_i^3, \quad z_i + x_i \tan \nu_i = v_i - \tfrac{1}{2} r_i x_i^2 - \tfrac{3}{4} s_i x_i^4 = f_i (\tan \nu_i),$$

$$\tfrac{1}{2} r_i^{-1} \tan \nu_i^2 = \tfrac{1}{2} r_i x_i^2 + s_i x_i^4, \quad f_i = v_i - \tfrac{1}{2} r_i^{-1} \tan \nu_i^2 + \tfrac{1}{4} s_i r_i^{-4} \tan \nu_i^4,$$

$$\boxed{T_i = \Delta_i v . f_i \left(\frac{\Delta_i \sigma}{\Delta_i v}\right) = v_i \Delta_i v - \tfrac{1}{2} r_i^{-1} (\Delta_i v)^{-1} (\Delta_i \sigma)^2 + \tfrac{1}{4} s_i r_i^{-4} (\Delta_i v)^{-3} (\Delta_i \sigma)^4;}$$

and finally, by the same kind of analysis as that in [18.],

$$\boxed{T'''^{(4)}_i = \tfrac{1}{4} s_i r_i^{-4} (\Delta_i \mu)^{-3} (\Delta_i \sigma)^4.} \qquad \text{For a } sphere,\ s_i = \tfrac{1}{2} r_i^3.$$

The expression for $T'^{(4)}_i$ gives

$$\boxed{T'^{(4)} = \tfrac{1}{8} v_1 \mu_0^{-3} \sigma_0^4 - \tfrac{1}{8} v_n \mu_n^{-3} \sigma_n^4 + \tfrac{1}{8} \Sigma_{(i)1}^{\ n-1} \mu_i^{-3} \sigma_i^4 \Delta v_i;}$$

if then we neglect the squares and products of the intervals Δv_i, we may calculate $T'^{(4)}$ by employing for $\sigma_1, .. \sigma_{n-1}$ the approximate expression

$$\sigma_i = \sigma_0 + \lambda_i (\sigma_n - \sigma_0).$$

This expression gives

$$\Delta_i \sigma = r_i \Delta_i \mu . (\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu)^{-1} (\sigma_n - \sigma_0);$$

therefore, if we neglect Δv_i,

$$T'''^{(4)}_i = \tfrac{1}{4} s_i \Delta_i \mu . (\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu)^{-4} (\sigma_n - \sigma_0)^4;$$

and, denoting by F or F_n the *focal length* of the combination, (see foot of [25.],) so that

$$F = F_n = (\Sigma_{(i)}{}_1^n r_i \Delta_i \mu)^{-1},$$

we find

$$T'''^{(4)} = \tfrac{1}{4} F^4 (\Sigma_{(i)}{}_1^n s_i \Delta_i \mu)(\sigma_n - \sigma_0)^4.$$

Supposing still the intervals to vanish, we have in like manner,

$$T''{}_i^{(4)} = -\tfrac{1}{4} r_i (\sigma_n - \sigma_0)^2 F^2 \Delta_i \frac{\sigma^2}{\mu};$$

and therefore

$$T''^{(4)} = \tfrac{1}{4} F^2 (\sigma_n - \sigma_0)^2 \left\{ \Sigma_{(i)}{}_1^{n-1} \frac{\sigma_i^2}{\mu_i} \Delta r_i + \frac{\sigma_0^2}{\mu_0} r_1 - \frac{\sigma_n^2}{\mu_n} r_n \right\}.$$

Also, $\sigma_i = \sigma_0 + FF_i^{-1}(\sigma_n - \sigma_0)$; for $\lambda_i = FF_i^{-1}$, if we write $F_i = (\Sigma_{(i)}{}_1^i r_i \Delta_i \mu)^{-1}$, according to the notation proposed at the foot of [25.]. Finally, if we still neglect the intervals between the surfaces, we have

$$T'^{(4)} = \tfrac{1}{8} v_1 \frac{\sigma_0^4}{\mu_0^3} - \tfrac{1}{8} v_n \frac{\sigma_n^4}{\mu_n^3};$$

and thus we have all the elements for calculating the aberrations of the instrument, so far as they depend on $T^{(4)}$, (in the diametral plane of xz,) by means of the following equations of the initial and final rays:

$$x_0 = \alpha_0 \left(1 + \frac{\alpha_0^2}{2}\right)(z_0 - v_1) - \frac{\delta}{\delta \sigma_0}(T''^{(2)} + T''^{(4)} + T'''^{(4)});$$

$$x_{n+1} = \alpha_n \left(1 + \frac{\alpha_n^2}{2}\right)(z_{n+1} - v_n) + \frac{\delta}{\delta \sigma_n}(T''^{(2)} + T''^{(4)} + T'''^{(4)});$$

in which (to recapitulate here all that is necessary for the present purpose), FOR ANY COMBINATION OF REFRACTING SURFACES PLACED CLOSE TOGETHER,

$$T''^{(2)} + T''^{(4)} + T'''^{(4)} = -\tfrac{1}{2} F_n (\sigma_n - \sigma_0)^2 - \tfrac{1}{4} F_n^2 (\sigma_n - \sigma_0)^2 \Sigma_{(i)}{}_1^n r_i \Delta_i \frac{\sigma^2}{\mu} + \tfrac{1}{4} F_n^4 (\sigma_n - \sigma_0)^4 \Sigma_{(i)}{}_1^n s_i \Delta_i \mu;$$

$$\sigma_i = \sigma_0 + F_n F_i^{-1}(\sigma_n - \sigma_0); \quad F_i^{-1} = \Sigma_{(i)}{}_1^i r_i \Delta_i \mu; \text{ and for spheres, } s_i = \tfrac{1}{2} r_i^3.$$

[28.] As one of the most important applications of the formulæ at the foot of the preceding section, let us consider the case of direct parallel incident rays, and determine the longitudinal aberrations of the final rays corresponding. In this case,

$$\sigma_0 = 0; \quad \sigma_i = F_n F_i^{-1} \sigma_n; \quad T''^{(2)} = -\tfrac{1}{2} F_n \sigma_n^2;$$

$$\frac{\delta}{\delta \sigma_n} T''^{(2)} = -F_n \sigma_n = -\mu_n F_n \alpha_n; \quad \tfrac{1}{2} \mu_n F_n \alpha_n^3 = \tfrac{1}{2} \mu_n^{-2} F_n \sigma_n^3 = \frac{\delta}{\delta \sigma_n} \cdot \tfrac{1}{8} \mu_n^{-2} F_n \sigma_n^4;$$

and the equation of the final ray may be put under the form

$$x_{n+1} - \alpha_n (1 + \tfrac{1}{2}\alpha_n^2)(z_{n+1} - v_n - \mu_n F_n) = \frac{\delta}{\delta\sigma_n}(\tfrac{1}{8}\mu_n^{-2} F_n \sigma_n^4 + T''^{(4)} + T'''^{(4)});$$

in which,

$$T''^{(4)} = -\tfrac{1}{4} F_n^2 \sigma_n^2 \Sigma_{(i)}{}_1^n r_i \Delta_i \frac{\sigma^2}{\mu} = \tfrac{1}{4} F_n^4 \sigma_n^4 \{\Sigma_{(i)}{}_1^{n-1} \mu_i^{-1} F_i^{-2} \Delta r_i - \mu_n^{-1} F_n^{-2} r_n\};$$

$$T'''^{(4)} = \tfrac{1}{4} F_n^4 \sigma_n^4 \Sigma_{(i)}{}_1^n s_i \Delta_i \mu;$$

also

$$\frac{\delta}{\delta\sigma_n} \cdot \tfrac{1}{4}\sigma_n^4 = \sigma_n^3 = \mu_n^3 \alpha_n^3;$$

the *equation of the final ray* will therefore be

$$\boxed{x_{n+1} = \alpha_n \gamma_n^{-1}(z_{n+1} - v_n - \mu_n F_n - L_n \alpha_n^2),}$$

if we make for abridgment

$$\boxed{L_n = -\mu_n^3 F_n^4 \{\tfrac{1}{2}\mu_n^{-2} F_n^{-3} - \mu_n^{-1} F_n^{-2} r_n + \Sigma_{(i)}{}_1^{n-1} \mu_i^{-1} F_i^{-2} \Delta r_i + \Sigma_{(i)}{}_1^n s_i \Delta_i \mu\};}$$

this last is therefore a general expression for the COEFFICIENT OF LONGITUDINAL ABERRATION for ANY COMBINATION OF REFRACTING SURFACES OF REVOLUTION CLOSE TOGETHER (but *not necessarily in vacuo*), and for *direct parallel incident rays*.

Thus, for *two surfaces, close together*,

$$L_2 = -\mu_2^3 F_2^4 \{\tfrac{1}{2}\mu_2^{-2}(r_1 \Delta_1\mu + r_2 \Delta_2\mu)^3 - \mu_2^{-1} r_2 (r_1 \Delta_1\mu + r_2\Delta_2\mu)^2$$
$$+ \mu_1^{-1}(r_1\Delta_1\mu)^2 (r_2 - r_1) + s_1 \Delta_1\mu + s_2\Delta_2\mu\};$$

in which

$$\Delta_1\mu = \mu_1 - \mu_0, \quad \Delta_2\mu = \mu_2 - \mu_1, \quad F_2^{-1} = r_1\Delta_1\mu + r_2\Delta_2\mu.$$

For a *single infinitely thin lens, in any medium*, $\mu_2 = \mu_0$, $\Delta_2\mu = -\Delta_1\mu$,

$$L_2 = -\mu_0^3 (\mu_1 - \mu_0)^{-3}(r_1 - r_2)^{-3}\left\{\tfrac{1}{2}\mu_0^{-2}(\mu_1 - \mu_0)^2 (r_1 - r_2)^2 - \mu_0^{-1}(\mu_1 - \mu_0) r_2 (r_1 - r_2)\right.$$
$$\left. - \mu_1^{-1}(\mu_1 - \mu_0) r_1^2 + \frac{s_1 - s_2}{r_1 - r_2}\right\}.$$

If the lens be *spheric*, and if we make $\mu = \mu_0^{-1}\mu_1 =$ *relative index* of lens, then [putting $\mu_0 = 1$,]

$$L_2 = -\tfrac{1}{2} F_2^3 \{(\mu - 1)^2 (r_1 - r_2)^2 - 2(\mu - 1) r_2 (r_1 - r_2) - 2(1 - \mu^{-1}) r_1^2 + r_1^2 + r_1 r_2 + r_2^2\}.$$

If the *power* be *given*, and the *aberration* a *minimum*, then $dL_2 = 0$, $dr_1 = dr_2$, therefore

$$0 = -2(\mu - 1)(r_1 - r_2) - 4(1 - \mu^{-1}) r_1 + 3(r_1 + r_2) = (2\mu + 1) r_2 + (1 - 2\mu + 4\mu^{-1}) r_1;$$

therefore

$$\frac{r_1}{r_2} = \frac{2\mu + 1}{2\mu - 1 - 4\mu^{-1}}.$$

If $\mu = \frac{3}{2}$, then $2\mu + 1 = 4$, $2\mu - 1 - 4\mu^{-1} = 2 - \frac{8}{3} = -\frac{2}{3}$, $\boxed{r_1 = -6r_2}$; thus for a *glass lens of best form* (index $\frac{3}{2}$), *both* surfaces must be *convex* or *both concave, outwards*; and the second radius = 6 times the first. The best form would be convexoplane, or concavoplane, that is, $r_2 = 0$, if

$$2\mu^2 - \mu = 4, \quad \mu = \frac{1 + \sqrt{33}}{4} = 1{,}686\,..$$

All these results respecting a single lens are well known, and have often been otherwise deduced.

For a *combination of two thin lenses close together in vacuo*,

$$F_4^{-1} = (\mu_1 - 1)(r_1 - r_2) + (\mu_3 - 1)(r_3 - r_4);$$

$$-F_4^{-4} L_4 = \tfrac{1}{2} F_4^{-3} - F_4^{-2} r_4 + F_3^{-2} \mu_3^{-1}(r_4 - r_3) + F_2^{-2}(r_3 - r_2) + F_1^{-2} \mu_1^{-1}(r_2 - r_1)$$
$$+ (\mu_1 - 1)(s_1 - s_2) + (\mu_3 - 1)(s_3 - s_2);$$

$$F_1^{-1} = (\mu_1 - 1) r_1; \quad F_2^{-1} = (\mu_1 - 1)(r_1 - r_2); \quad F_3^{-1} = (\mu_1 - 1)(r_1 - r_2) + (\mu_3 - 1) r_3.$$

Accordingly, if, in the expression near foot of [12.] for the coefficient of ϵ'' in

$$\frac{4F^{-4} T^{(4)}}{\epsilon'' - 2\epsilon_{,}' + \epsilon_{,}'},$$

we change μ_2 to μ_3, and then add $\frac{1}{2} F_4^{-3} (= \frac{1}{2} F^{-3},)$ we get the expression just now given, for

$$\boxed{-F^{-4} L_4 = F^{-4}(4Q + \tfrac{1}{2} F).}$$

We may therefore proceed to transform $-4F^{-4} L_4$, as is done in equation (A), in [13.].

[29.] For *any combination of refracting surfaces of revolution, round the common axis* (*of* z), *and close together*, we have, by [27.],

$$4F^{-4}(\sigma_n - \sigma_0)^{-2} T''^{(4)} + F^{-2}\left\{\frac{\sigma_n^2}{\mu_n} r_n - \frac{\sigma_0^2}{\mu_0} r_1\right\} = \Sigma_{(i)1}^{n-1}.\, \mu_i^{-1} \{F^{-1}\sigma_0 + F_i^{-1}(\sigma_n - \sigma_0)\}^2 \Delta r_i$$
$$= (\sigma_n - \sigma_0)^2 \Sigma_{(i)1}^{n-1} \mu_i^{-1} F_i^{-2} \Delta r_i + 2\sigma_0(\sigma_n - \sigma_0) F^{-1} \Sigma_{(i)1}^{n-1} \mu_i^{-1} F_i^{-1} \Delta r_i + \sigma_0^2 F^{-2} \Sigma_{(i)1}^{n-1}.\, \mu_i^{-1} \Delta r_i;$$

(if then we denote this expression by $A\sigma_n^2 + 2B\sigma_0\sigma_n + C\sigma_0^2$, we have

$$A + B = F^{-1} \Sigma_{(i)1}^{n-1} \mu_i^{-1} F_i^{-1} \Delta r_i;)$$

and if we neglect σ_0^2, we shall have

$$4F^{-4} T''^{(4)} = -F^{-2} \mu_n^{-1} r_n (\sigma_n^4 - 2\sigma_n^3 \sigma_0) + A(\sigma_n^4 - 4\sigma_n^3 \sigma_0) + 2(A + B)\sigma_n^3 \sigma_0$$
$$= (\Sigma_{(i)1}^{n-1} \mu_i^{-1} F_i^{-2} \Delta r_i - \mu_n^{-1} F^{-2} r_n)(\sigma_n^4 - 4\sigma_n^3 \sigma_0) + 2F^{-1}(\Sigma_{(i)1}^{n-1} \mu_i^{-1} F_i^{-1} \Delta r_i - \mu_n^{-1} F^{-1} r_n)\sigma_n^3 \sigma_0.$$

Therefore, making for simplicity $v_n = 0$, or placing the origin at the common vertex, we have $T'^{(4)} = 0$; also

$$4F^{-4} T'''^{(4)} = (\sigma_n^4 - 4\sigma_n^3 \sigma_0) \Sigma_{(i)1}^{n} s_i \Delta_i \mu;$$

$$T''^{(2)} = -\tfrac{1}{2} F(\sigma_n^2 - 2\sigma_n \sigma_0); \quad \frac{\delta}{\delta\sigma_n} T''^{(2)} = -F(\sigma_n - \sigma_0) = -\mu_n F \alpha_n + F\sigma_0;$$

$$\tfrac{1}{2}\alpha_n^3 \mu_n F = \tfrac{1}{2}\mu_n^{-2} \sigma_n^3 F = \tfrac{1}{4} F^4 \frac{\delta}{\delta\sigma_n}(\tfrac{1}{2}\mu_n^{-2} F^{-3} \sigma_n^4); \quad \sigma_n^4 = (\sigma_n^4 - 4\sigma_n^3\sigma_0) + 4\sigma_n^3 \sigma_0;$$

therefore *the approximate equation for the final ray* may be thus written:

$$x_{n+1} - F\sigma_0 - \alpha_n \gamma_n^{-1}(z_{n+1} - \mu_n F) = \tfrac{1}{4}F^4 \frac{\delta}{\delta\sigma_n}\{M(\sigma_n^4 - 4\sigma_n^3\sigma_0) + 2F^{-1}N\sigma_n^3\sigma_0\}$$
$$= F^4\{M(\sigma_n^3 - 3\sigma_n^2\sigma_0) + \tfrac{3}{2}F^{-1}N\sigma_n^2\sigma_0\};$$

in which

$$M = \tfrac{1}{2}\mu_n^{-2}F^{-3} + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}F_i^{-2}\Delta r_i - \mu_n^{-1}F^{-2}r_n + \Sigma_{(i)1}^{\ n}s_i\Delta_i\mu;$$
$$N = \mu_n^{-2}F^{-2} + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}F_i^{-1}\Delta r_i - \mu_n^{-1}F^{-1}r_n.$$

If $\sigma_0 = 0$, then $x_{n+1} = 0$ when $z_{n+1} = \mu_n F - \mu_n^3 F^4 M\alpha_n^2$; so that $\mu_n F$ is the ordinate of the *final focus*, corresponding to parallel central direct incident rays, and $L_n\alpha_n^2$ is the *aberration for marginal rays*, if $L_n = -\mu_n^3 F^4 M$; a result which agrees with the expression in last section, for the *coefficient of longitudinal aberration*. It is essential to the goodness of an object glass, that this coefficient L_n and therefore that M should (at least nearly) vanish; but even after making $M = 0$, if the coefficient N do not also vanish, and if the parallel incident rays be oblique, we shall have, *neglecting the square of that obliquity*, the following equation for a final ray:

$$x_{n+1} - F\sigma_0 - \alpha_n\gamma_n^{-1}(z_{n+1} - \mu_n F) = \tfrac{3}{2}F^3 N\sigma_n^2\sigma_0;$$

and consequently, when $z_{n+1} = \mu_n F$, we shall have

$$x_{n+1} = F\sigma_0 + \tfrac{3}{2}F^3 N\sigma_n^2\sigma_0.$$

To make the aberrations vanish, for parallel oblique incident rays (in diametral plane), we are therefore to combine the *two conditions*:

$$M = 0; \quad N = 0;$$

M and N having the values assigned above. The first condition has been deduced by other writers; the second has been added by myself, as that required for OBLIQUE APLANATICITY.*

[**30.**] For a *combination of two infinitely thin lenses, close together, in vacuo*, by preceding section,

$$N = F^{-2} + \mu_1^{-1}F_1^{-1}\Delta r_1 + F_2^{-1}\Delta r_2 + \mu_3^{-1}F_3^{-1}\Delta r_3 - F^{-1}r_4$$
$$= \{(\mu_1 - 1)(r_1 - r_2) + (\mu_3 - 1)(r_3 - r_4)\}^2 + (1 - \mu_1^{-1})r_1(r_2 - r_1) + (\mu_1 - 1)(r_1 - r_2)(r_3 - r_2)$$
$$+ \mu_3^{-1}\{(\mu_1 - 1)(r_1 - r_2) + (\mu_3 - 1)r_3\}(r_4 - r_3) - \{(\mu_1 - 1)(r_1 - r_2) + (\mu_3 - 1)(r_3 - r_4)\}r_4;$$

and if we equate this to 0, we obtain the same condition as if, near the foot of [**12.**], we change μ_2 to μ_3: and therefore are conducted to the equation (B) of [**13.**].

In general if n be an even number, and if we consider a combination of $\frac{n}{2}$ infinitely thin lenses close together *in vacuo*, we shall have $\mu_0 = \mu_2 = \mu_4 = \&c. = 1$; $\mu_1, \mu_3, ..$ will be the indices of the successive lenses, which we shall suppose to be given; and if the powers of those lenses be also given, or the differences $\Delta r_1, \Delta r_3, ..$ on which those powers depend, we shall know $F_2, F_4, .. F$;

* [These are L. Seidel's first two conditions (*Astr. Nach.* 43 (1856), 317) for the case of a thin system, but otherwise more general, since Hamilton's surfaces are not necessarily spheres. Hamilton's "condition for oblique aplanaticity" is the same as what Seidel called "Fraunhofer's condition," on account of its satisfaction by Fraunhofer's heliometer objective at Königsberg.]

therefore each of the terms of the form $\mu_i^{-1} F_i^{-2} \Delta r_i$, in M, is either a known linear function of two successive curvatures, a posterior and an anterior, namely when i is even, or else a known quadratic function of an anterior curvature, namely when i is odd: F_i^{-2} being $= (F_{i-1}^{-1} + r_i \Delta_i \mu)^2$. Also, if the surfaces be spheric, each successive pair of terms of the form $s_i \Delta_i \mu$ gives a sum, such as*

$$(\mu_1 - 1)(r_1 - r_2)(r_1^2 + r_1 r_2 + r_2^2), \quad (\mu_3 - 1)(r_3 - r_4)(r_3^2 + r_3 r_4 + r_4^2), \text{ \&c.,}$$

which is a known quadratic function of two successive curvatures, anterior and posterior, of a single lens; therefore on the whole, for *any combination of thin spheric lenses close together in vacuo, with given indices and powers, M* (in preceding section) is a *known quadratic function* of the curvatures of all the surfaces, or simply of the $\frac{n}{2}$ *anterior curvatures*, r_1, r_3, &c.; or (if we prefer to put it so) of the $\frac{n}{2}$ *sums of curvatures*, anterior and posterior, for each lens separately, namely $r_1 + r_2$, $r_3 + r_4$, &c.: while N (in the same section) is a *known linear function* of the same $\frac{n}{2}$ sought quantities. This is the principle of my calculation, referred to in [**13.**], for a THIN DOUBLE ACHROMATIC OBJECT GLASS; I determine the two sought sums $r_1 + r_2$ and $r_3 + r_4$, by the two equations, quadratic and linear, $M = 0$, $N = 0$. For a thin *triple* object glass, we should have one quadratic and one linear equation between *three* such disposable sums, and might in general introduce some other condition. (But see next section, for the dependence of the third coefficient O, on the indices and powers of the lenses.)

[**31.**] With the recent meanings of M, N, we have, for any combination of refracting surfaces of revolution close together at the origin,†

$$\begin{aligned} 4F^{-4}(\sigma_n - \sigma_0)^{-2} T^{(4)} &= F^{-2}(\mu_0^{-1}\sigma_0^2 r_1 - \mu_n^{-1}\sigma_n^2 r_n + \sigma_0^2 \Sigma_{(i)1}^{\ n-1} \mu_i^{-1} \Delta r_i) \\ &\quad + (M - \tfrac{1}{2}\mu_n^{-2} F^{-3} + \mu_n^{-1} F^{-2} r_n)(\sigma_n - \sigma_0)^2 + 2F^{-1}(N - \mu_n^{-2} F^{-2} + \mu_n^{-1} F^{-1} r_n)\sigma_0(\sigma_n - \sigma_0) \\ &= (M - \tfrac{1}{2}\mu_n^{-2} F^{-3})(\sigma_n - \sigma_0)^2 + 2F^{-1}(N - \mu_n^{-2} F^{-2})\sigma_0(\sigma_n - \sigma_0) - F^{-2}\sigma_0^2 \Sigma_{(i)1}^{\ n} r_i \Delta_i \frac{1}{\mu}; \end{aligned}$$

because

$$(\sigma_n - \sigma_0)^2 + 2\sigma_0(\sigma_n - \sigma_0) - \sigma_n^2 = -\sigma_0^2,$$

and

$$\mu_0^{-1} r_1 - \mu_n^{-1} r_n + \Sigma_{(i)1}^{\ n-1} \mu_i^{-1} \Delta r_i = -\Sigma_{(i)1}^{\ n} r_i \Delta_i \frac{1}{\mu}.$$

Also

$$-\tfrac{1}{2}(\sigma_n - \sigma_0)^2 - 2\sigma_0(\sigma_n - \sigma_0) = -\tfrac{1}{2}\sigma_n^2 - \sigma_0\sigma_n + \tfrac{3}{2}\sigma_0^2 = -\tfrac{1}{2}(\sigma_n + \sigma_0)^2 + 2\sigma_0^2;$$

therefore if we make

$$\boxed{O = 2\mu_n^{-2} F^{-1} - \Sigma_{(i)1}^{\ n} r_i \Delta_i \frac{1}{\mu},}$$

we shall have

$$\boxed{4F^{-4}(\sigma_n - \sigma_0)^{-2} T^{(4)} = M(\sigma_n - \sigma_0)^2 + 2F^{-1} N \sigma_0 (\sigma_n - \sigma_0) + F^{-2} O \sigma_0^2 - \tfrac{1}{2}\mu_n^{-2} F^{-3}(\sigma_n + \sigma_0)^2;}$$

* [Omitting a numerical factor, $\frac{1}{2}$.]

† [See top of [**29.**].]

and therefore*

$$T^{(2)}+T^{(4)}=-\tfrac{1}{2}F(\sigma_n-\sigma_0)^2+\tfrac{1}{4}F^4M(\sigma_n-\sigma_0)^4+\tfrac{1}{2}F^3N\sigma_0(\sigma_n-\sigma_0)^3+\tfrac{1}{4}F^2O\sigma_0^2(\sigma_n-\sigma_0)^2 -\tfrac{1}{8}\mu_n^{-2}F(\sigma_n^2-\sigma_0^2)^2.$$

The equations of the initial and final rays may be put under the forms:

$$x_0-\frac{\sigma_0}{\mu_0}\left(1+\tfrac{1}{2}\frac{\sigma_0^2}{\mu_0^2}\right)z_0=-\frac{\delta(T^{(2)}+T^{(4)})}{\delta\sigma_0};\quad x_{n+1}-\frac{\sigma_n}{\mu_n}\left(1+\tfrac{1}{2}\left(\frac{\sigma_n}{\mu_n}\right)^2\right)z_{n+1}=\frac{\delta(T^{(2)}+T^{(4)})}{\delta\sigma_n};$$

and consequently *the equation of the final ray* is, without neglecting any power of σ_0,†

$$x_{n+1}=F\sigma_0(1-\tfrac{1}{2}FO\sigma_0^2)+\frac{\sigma_n}{v_n}(z_{n+1}-\mu_nF+\tfrac{1}{2}\mu_n^{-1}F\sigma_0^2+\tfrac{1}{2}\mu_nF^2O\sigma_0^2) +F^4M(\sigma_n-\sigma_0)^3+\tfrac{3}{2}F^3N\sigma_0(\sigma_n-\sigma_0)^2;$$

so that, in the present order of approximation, oblique parallel indiametral incident rays are all refracted to one common focus, namely

$$X_{n+1}=F\sigma_0(1-\tfrac{1}{2}FO\sigma_0^2),\quad Z_{n+1}=\mu_nF(1-\tfrac{1}{2}FO\sigma_0^2)(1-\tfrac{1}{2}\mu_n^{-2}\sigma_0^2),$$

when the two conditions $M=0$, $N=0$, are satisfied.‡

We may also remark that if the final medium be the same as the initial, so that $\mu_n=\mu_0$, then

$$\mu_n(1-\tfrac{1}{2}\mu_n^{-2}\sigma_0^2)=v_0,$$

and

$$\frac{1}{\alpha_0}X_{n+1}=\frac{1}{\gamma_0}Z_{n+1}=\mu_nF(1-\tfrac{1}{2}FO\sigma_0^2)=\text{distance of focus from origin};$$

the direction of this distance being the same as that of the incident rays. In fact, the ray incident on the first vertex, that is, at the origin, emerges without any change, if $\mu_n=\mu_0$; but it undergoes a change of direction if μ_n be different from μ_0, because then the equation $\sigma_n=\sigma_0$ gives $\alpha_n=\mu_n^{-1}\sigma_0$. In this last, which is the more general case, we have $\alpha_n=\mu_n^{-1}\mu_0\alpha_0$;

$$\frac{1}{\alpha_n}X_{n+1}=\frac{1}{\gamma_n}Z_{n+1}=\mu_nF(1-\tfrac{1}{2}FO\sigma_0^2)=\text{distance of focus from centre of lens};$$

also

$$\tfrac{1}{2}FO\sigma_0^2=\tfrac{1}{2}\mu_n^2FO\alpha_2^n=\alpha_2^n\left(1-\tfrac{1}{2}\mu_n^2F\Sigma_{(i)}{}_1^n r_i\Delta_i\frac{1}{\mu}\right).$$

* [To obtain $T^{(2)}$, we put $v_i=0$ in the expression at the beginning of [26.].]

† [Except, of course, powers higher than the fourth in T.]

‡ [This point is the primary focus. It is seen from the identical relations of p. 456 that it is impossible to correct a thin system simultaneously for spherical aberration, coma and astigmatism. Thus Hamilton's system is astigmatic, and the condition for flatness in the locus of the primary focus, namely,

$$FO=-\mu_n^{-2},\quad\text{or}\quad \Sigma_{(i)}{}_1^n r_i\Delta_i(\mu^{-1})=3\mu_n^{-2}F^{-1},$$

differs from Petzval's condition, $\Sigma_{(i)}{}_1^n r_i\Delta_i(\mu^{-1})=0$, which is only applicable to a system corrected for astigmatism. Petzval's condition was published, without proof, in 1843 (cf. J. P. C. Southall, *Geometrical Optics* (1913), p. 439).]

For a *thin double lens in vacuo*,

$$O = \left(2 + \frac{1}{\mu_1}\right)(\mu_1 - 1)(r_1 - r_2) + \left(2 + \frac{1}{\mu_3}\right)(\mu_3 - 1)(r_3 - r_4);$$

therefore the *curvature of locus of focus* is

$$F^{-1} + O = \left(3 + \frac{1}{\mu_1}\right)(\mu_1 - 1)(r_1 - r_2) + \left(3 + \frac{1}{\mu_3}\right)(\mu_3 - 1)(r_3 - r_4);$$

and the concavity of this locus is turned towards the object glass. If ϖ be *dispersion ratio*, so that

$$(\mu_3 - 1)(r_3 - r_4) = -\varpi(\mu_1 - 1)(r_1 - r_2),$$

then, focal length multiplied by curvature of locus of focus

$$= 1 + FO = (1 - \varpi)^{-1}\{3 + \mu_1^{-1} - (3 + \mu_3^{-1})\varpi\} = 3 + \frac{1}{\mu},$$

if

$$\mu^{-1} = \frac{\mu_1^{-1} - \mu_3^{-1}\varpi}{1 - \varpi},$$

that is, if

$$\mu^{-1} = \mu_1^{-1} + \frac{\varpi}{1 - \varpi}(\mu_1^{-1} - \mu_3^{-1}).$$

Thus radius of curvature of locus of focus, for *indiametral* rays, originally parallel, is focal length

$$\div\, 3 + \mu_1^{-1} + \frac{\varpi}{1 - \varpi}(\mu_1^{-1} - \mu_3^{-1}).$$

[32.] *Herschel's second condition of aplanaticity.*

(Feb. 21st. 1844). By the preceding section, for any combination of coaxal refracting surfaces of revolution placed close together at the origin, the equations of the initial and final rays may be thus written:

$$x_0 = \frac{\sigma_0}{\mu_0}\left(1 + \frac{\sigma_0^2}{2\mu_0^2}\right) z_0 - F(\sigma_n - \sigma_0) - \frac{\delta T^{(4)}}{\delta\sigma_0};$$

$$x_{n+1} = \frac{\sigma_n}{\mu_n}\left(1 + \frac{\sigma_n^2}{2\mu_n^2}\right) z_{n+1} - F(\sigma_n - \sigma_0) + \frac{\delta T^{(4)}}{\delta\sigma_n};$$

and therefore the abscissa ($x_1 = x_2 = .. = x_n$) of incidence is nearly

$$x = -F(\sigma_n - \sigma_0).$$

Adopting as an abridgment this last expression, and supposing that a ray from $0, z_0$ is refracted to $0, z_{n+1}$, we have

$$\frac{\mu_0}{z_0} = \frac{\sigma_0(1 + \frac{1}{2}\mu_0^{-2}\sigma_0^2)}{-x + \dfrac{\delta T^{(4)}}{\delta\sigma_0}} = -\sigma_0 x^{-1} - \tfrac{1}{2}\mu_0^{-2}\sigma_0^3 x^{-1} - x^{-2}\sigma_0\frac{\delta T^{(4)}}{\delta\sigma_0},$$

$$\frac{\mu_n}{z_{n+1}} = \frac{\sigma_n(1 + \frac{1}{2}\mu_n^{-2}\sigma_n^2)}{-x - \dfrac{\delta T^{(4)}}{\delta\sigma_n}} = -\sigma_n x^{-1} - \tfrac{1}{2}\mu_n^{-2}\sigma_n^3 x^{-1} + x^{-2}\sigma_n\frac{\delta T^{(4)}}{\delta\sigma_n};$$

also

$$\sigma_n \frac{\delta T^{(4)}}{\delta \sigma_n} + \sigma_0 \frac{\delta T^{(4)}}{\delta \sigma_0} = 4T^{(4)}; \quad -(\sigma_n - \sigma_0)\, x^{-1} = F^{-1};$$

therefore

$$\boxed{\frac{\mu_n}{z_{n+1}} - \frac{\mu_0}{z_0} - \frac{1}{F} = \frac{\mu_n^{-2}\sigma_n^3 - \mu_0^{-2}\sigma_0^3}{2F(\sigma_n - \sigma_0)} + 4F^{-2}(\sigma_n - \sigma_0)^{-2}\, T^{(4)}.}$$

Also

$$\sigma_n^3 - (\mu_n \mu_0^{-1})^2 \sigma_0^3 - (\sigma_n^2 - \sigma_0^2)(\sigma_n + \sigma_0) = -\sigma_0 \sigma_n^2 + \sigma_0^2 \sigma_n + (1 - \mu_n^2 \mu_0^{-2})\, \sigma_0^3;$$

if then we neglect $\left(\frac{\sigma_0}{\sigma_n}\right)^2$, or $\left(\frac{z_{n+1}}{z_0}\right)^2$, we have

$$\frac{\mu_n}{z_{n+1}} - \frac{\mu_0}{z_0} - \frac{1}{F} = MF^2(\sigma_n^2 - 2\sigma_0\sigma_n) + 2NF\sigma_0\sigma_n - \tfrac{1}{2}\mu_n^{-2}F^{-1}\sigma_0\sigma_n;$$

in which we may make $\sigma_0 = \sigma_n \frac{\mu_0}{\mu_n} \frac{z_{n+1}}{z_0}$;* thus the conditions requisite in order that z_{n+1} may be independent of σ_n, when we neglect σ_n^4 and $\sigma_n^2 z_0^{-2}$, are

$$\boxed{M = 0, \quad N = \tfrac{1}{4}\mu_n^{-2}F^{-2};}$$

these therefore must agree with those which Herschel has proposed, for the construction of an aplanatic object glass, applicable to terrestrial objects. Accordingly they agree with those which I deduced from Herschel's formulæ, in my calculation of Jan. 2nd, 1844.

A somewhat easier though less elegant analysis would be the following. Our object is to eliminate σ_0 between the equations of the initial and final rays, after making in those equations, that is, in the two first of the present section, $x_0 = 0$, $x_{n+1} = 0$, and neglecting z_0^{-2}. We may therefore substitute for σ_0, in the second equation, its value derived from the first, namely

$$\sigma_0 = \mu_0 F z_0^{-1} \sigma_n + \mu_0 z_0^{-1} \frac{\delta T^{(4)}}{\delta \sigma_0},$$

σ_0 being treated as $= 0$ in the last term, or $T^{(4)}$ confined to the part proportional to $\sigma_0 \sigma_n^3$. In this manner we find, by the second equation,

$$\mu_n^{-1} z_{n+1} = F(1 - \mu_0 F z_0^{-1})(1 - \tfrac{1}{2}\mu_n^{-2}\sigma_n^2) - \frac{\delta T^{(4)}}{\sigma_n \delta \sigma_n} - \mu_0 F z_0^{-1} \frac{\delta T^{(4)}}{\sigma_n \delta \sigma_0};\dagger$$

σ_0 being treated as $= \mu_0 F z_0^{-1} \sigma_n$ in $\frac{\delta T^{(4)}}{\sigma_n \delta \sigma_n}$. Make then for abridgment, (see [**10.**],)

$$\boxed{T^{(4)} = Q\sigma_n^4 + Q_{,}\sigma_n^3\sigma_0 + (Q' + Q_{,,})\,\sigma_n^2\sigma_0^2 + Q_{,}'\sigma_n\sigma_0^3 + Q''\sigma_0^4,}$$

and we shall have, in the present order of approximation,

$$\boxed{z_{n+1} = \mu_n F(1 - \mu_0 F z_0^{-1}) - \mu_n(4Q + \tfrac{1}{2}\mu_n^{-2}F)\,\sigma_n^2 - \mu_n(4Q_{,} - \tfrac{1}{2}\mu_n^{-2}F)\,\mu_0 F z_0^{-1}\sigma_n^2.}$$

* [This is given by the equations of the rays to the first approximation; we have corrected an obvious error in the formula, which (in the MS.) lacks the factor σ_n, and has a minus sign.]

† [In the last term we have inserted σ_n, which is lacking in the MS.]

To destroy the aberration for direct parallel incident rays, we are to make $4Q + \frac{1}{2}\mu_n^{-2}F = 0$, (compare [12.],) that is, by preceding section, $M = 0$; and to destroy also the aberration for rays proceeding from a distant point on the axis, we are to employ the condition $4Q_{,} - \frac{1}{2}\mu_n^{-2}F = 0$, that is, $N = \frac{1}{4}\mu_n^{-2}F^{-2}$, as above. Herschel's second condition is *always incompatible with mine.**

[33.] *Summary of Calculations for deducing* (A) *and* (B).

With respect to my own form of a thin double object glass, the chief *calculations* are the following.

$$T_i = r_i^{-1}\Delta_i v \,.\left\{1 - \sqrt{1 + \left(\frac{\Delta_i\sigma}{\Delta_i v}\right)^2}\right\}, \textit{ rigorously};\dagger$$

therefore *approximately*,

$$T_i = T_i^{(2)} + T''^{(4)}_i + T'''^{(4)}_i,$$

in which

$$\left(v = \mu - \frac{\sigma^2}{2\mu},\right) \quad \left(T_i^{(2)} + T''^{(4)}_i = -\frac{(\Delta_i\sigma)^2}{2r_i\Delta_i v},\right)$$

$$T_i^{(2)} = -\frac{(\Delta_i\sigma)^2}{2r_i\Delta_i\mu}, \quad T''^{(4)}_i = -\frac{r_i^{-1}}{4}\left(\frac{\Delta_i\sigma}{\Delta_i\mu}\right)^2\Delta_i\frac{\sigma^2}{\mu},$$

$$T'''^{(4)}_i = \tfrac{1}{8}r_i^{-1}(\Delta_i\mu)^{-3}(\Delta_i\sigma)^4; \quad T = T^{(2)} + T''^{(4)} + T'''^{(4)};$$

$$T^{(2)} = \Sigma_{(i)1}^{\ n}T_i^{(2)}; \quad T''^{(4)} = \Sigma_{(i)1}^{\ n}T''^{(4)}_i; \quad T'''^{(4)} = \Sigma_{(i)1}^{\ n}T'''^{(4)}_i;$$

$$0 = \frac{\delta}{\delta\sigma_i}(T_i^{(2)} + T_{i+1}^{(2)}) = -\frac{\Delta_i\sigma}{r_i\Delta_i\mu} + \frac{\Delta_{i+1}\sigma}{r_{i+1}\Delta_{i+1}\mu}, \quad \because \ \Delta_i\sigma = Cr_i\Delta_i\mu,$$

$$C = F(\sigma_n - \sigma_0), \quad F^{-1} = \Sigma_{(i)1}^{\ n}r_i\Delta_i\mu; \quad T_i^{(2)} = -\tfrac{1}{2}F^2 r_i\Delta_i\mu(\sigma_n - \sigma_0)^2,$$

$$T^{(2)} = -\tfrac{1}{2}F(\sigma_n - \sigma_0)^2; \quad T''^{(4)}_i = -\tfrac{1}{4}r_iF^2(\sigma_n - \sigma_0)^2\Delta_i\frac{\sigma^2}{\mu}; \quad T'''^{(4)}_i = \tfrac{1}{8}r_i^3\Delta_i\mu F^4(\sigma_n - \sigma_0)^4;$$

$$F^{-1}\sigma_i = F^{-1}\sigma_0 + F_i^{-1}(\sigma_n - \sigma_0), \text{ if } F_i^{-1} = \Sigma_{(i)1}^{\ i}r_i\Delta_i\mu;$$

$$\Sigma_{(i)1}^{\ n}r_i\Delta_i\phi = r_n\phi_n - r_1\phi_0 - \Sigma_{(i)1}^{\ n-1}\phi_i\Delta r_i;$$

$$T''^{(4)} = \tfrac{1}{4}F^4(\sigma_n - \sigma_0)^2\left\{-F^{-2}r_n\frac{\sigma_n^2}{\mu_n} + F^{-2}r_1\frac{\sigma_0^2}{\mu_0} + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}(F^{-1}\sigma_0 + F_i^{-1}\overline{\sigma_n - \sigma_0})^2\Delta r_i\right\};$$

$$T^{(4)} = Q\sigma_n^4 + Q_{,}\sigma_n^3\sigma_0 + (Q' + Q_{,,})\sigma_n^2\sigma_0^2 + Q_{,}'\sigma_n\sigma_0^3 + Q''\sigma_0^4;$$

$$x_{n+1} = \mu_n^{-1}\sigma_n(1 + \tfrac{1}{2}\mu_n^{-2}\sigma_n^2)z_{n+1} - F(\sigma_n - \sigma_0) + \frac{\delta T^{(4)}}{\delta\sigma_n}$$

$$= F\sigma_0 + Q_{,}'\sigma_0^3 + \mu_n^{-1}\sigma_n(1 + \tfrac{1}{2}\mu_n^{-2}\sigma_n^2)(z_{n+1} - \mu_nF + 2\mu_n\overline{Q' + Q_{,,}}\sigma_0^2)$$

$$+ (4Q + \tfrac{1}{2}\mu_n^{-2}F)\sigma_n^3 + 3Q_{,}\sigma_0\sigma_n^2;$$

* [L. Seidel (*Astr. Nach.* 43 (1856), 328) also remarked that these conditions were incompatible in the case of a telescope objective.]

† [Cf. [18.].]

$$4Q + \tfrac{1}{2}\mu_n^{-2}F = 0, \quad Q_{,} = 0; \quad \because \text{ neglecting* } \sigma_0^2,$$

$$\begin{aligned}4T^{(4)}F^{-4}(\sigma_n - \sigma_0)^{-2} &= -\tfrac{1}{2}\mu_n^{-2}F^{-3}\sigma_n(\sigma_n + 2\sigma_0)\\ &= -F^{-2}r_n\mu_n^{-1}\sigma_n^2 + \Sigma_{(i)1}^{\ \ n-1}\mu_i^{-1}F_i^{-2}\Delta r_i\sigma_n(\sigma_n - 2\sigma_0)\\ &\quad + 2F^{-1}\Sigma_{(i)1}^{\ \ n-1}\mu_i^{-1}F_i^{-1}\Delta r_i\,\sigma_n\sigma_0 + \tfrac{1}{2}\sigma_n(\sigma_n - 2\sigma_0)\,\Sigma_{(i)1}^{\ \ n}r_i^3\Delta_i\mu\\ &= -\tfrac{1}{2}\mu_n^{-2}F^{-3}\sigma_n(\sigma_n - 2\sigma_0) - 2\mu_n^{-2}F^{-3}\sigma_n\sigma_0;\end{aligned}$$

$$\boxed{\begin{aligned}0 &= \tfrac{1}{2}\mu_n^{-2}F^{-3} + \Sigma_{(i)1}^{\ \ n-1}\mu_i^{-1}F_i^{-2}\Delta r_i + \tfrac{1}{2}\Sigma_{(i)1}^{\ \ n}r_i^3\Delta_i\mu - \mu_n^{-1}r_nF^{-2};\\ 0 &= \mu_n^{-2}F^{-2} + \Sigma_{(i)1}^{\ \ n-1}\mu_i^{-1}F_i^{-1}\Delta r_i - \mu_n^{-1}r_nF^{-1}.\end{aligned}}\ \dagger$$

So far, we have made no supposition respecting the five‡ indices, but have only supposed the four surfaces to be spheric, and close together; we might even extend the two resulting equations to any system of four coaxal surfaces of revolution, close together, by changing, in the first equation, $\tfrac{1}{2}\Sigma_{(i)1}^{\ \ n}r_i^3\Delta_i\mu$ to $\Sigma_{(i)1}^{\ \ n}s_i\Delta_i\mu$. But if we now suppose $n=4$, $\mu_0=\mu_2=\mu_4=1$, $\mu_1=\mu'$, $\mu_3=\mu''$, $F^{-1}=p$, $F_2^{-1}=p'$, $p-p'=p''$, (μ', μ'' the indices, and p', p'' the powers of the two component lenses,) then

$$r_1 - r_2 = \frac{p'}{\mu'-1}, \quad r_3 - r_4 = \frac{p''}{\mu''-1};$$

$$\mu_1^{-1}F_1^{-2}\Delta r_1 = -\frac{(\mu'-1)}{\mu'}p'r_1^2, \quad \mu_2^{-1}F_2^{-2}\Delta r_2 = p'^2(r_3 - r_2),$$

$$\mu_3^{-1}F_3^{-2}\Delta r_3 = -\frac{p''}{\mu''(\mu''-1)}\{p' + (\mu''-1)r_3\}^2;$$

$$\mu_1^{-1}F_1^{-1}\Delta r_1 = -\frac{p'r_1}{\mu'}; \quad \mu_2^{-1}F_2^{-1}\Delta r_2 = p'(r_3 - r_2); \quad \mu_3^{-1}F_3^{-1}\Delta r_3 = -\frac{p''}{\mu''(\mu''-1)}\{p' + (\mu''-1)r_3\};$$

and the two conditions become:

$$\boxed{\begin{aligned}0 &= \tfrac{1}{2}(p'+p'')^3 + \tfrac{1}{2}p'(r_1^2 + r_1r_2 + r_2^2) + \tfrac{1}{2}p''(r_3^2 + r_3r_4 + r_4^2) - \frac{\mu'-1}{\mu'}p'r_1^2\\ &\quad + p'^2(r_3 - r_2) - \frac{\mu''^{-1}p''}{\mu''-1}\{p' + (\mu''-1)r_3\}^2 - p^2r_4; \qquad (1)\\ 0 &= (p'+p'')^2 - \frac{p'r_1}{\mu'} + p'(r_3 - r_2) - \frac{p''}{\mu''(\mu''-1)}\{p' + (\mu''-1)r_3\} - pr_4; \qquad (2)\end{aligned}}$$

* [The defect of astigmatism depends on σ_0^2 (see [48.] or p. 378), and does not occur when σ_0^2 is neglected.]

† [These formulæ for the correction of spherical aberration and coma, for any infinitely thin system of spherical refracting surfaces, possess the advantage of involving only the fundamental data of the instrument (curvatures and indices). In this, although otherwise less complete and general, they possess an advantage over the conditions of L. Seidel (*Astr. Nach.* 43 (1856)). The forms of Seidel's conditions for a thin system will be found in J. P. C. Southall's *Geometrical Optics* (1913), p. 470, where there follow interesting historical references to other general methods. Although Hamilton's argument here appears to apply only to rays in one diametral plane (for which of course the phenomenon of coma, geometrically described in No. XIX, does not present itself), Hamilton gives later, in [46.], the extension of the argument. The essential fact underlying the step from two to three dimensions is that, when we put $\tau=0$, $\tau'=0$ in the general expression for $T^{(4)}$, the coefficients Q, $Q_{,}$ (unlike Q', $Q_{,,}$) remain still the coefficients of distinct terms, and may therefore be evaluated by the consideration of indiametral rays alone.]

‡ [In the foregoing argument there is no numerical limit to the number of surfaces involved.]

in which, $p = p' + p''$; and

$$r_1 - r_2 = \frac{p'}{\mu' - 1}, \quad r_3 - r_4 = \frac{p''}{\mu'' - 1}.$$

Hence (A) and (B) of [**13.**].

[**34.**] *Development of the equations* (A) *and* (B).

In fact, if we make

$$2r_1 = r_1 + r_2 + \frac{p'}{\mu' - 1}, \quad 2r_2 = r_1 + r_2 - \frac{p'}{\mu' - 1},$$

$$2r_3 = r_3 + r_4 + \frac{p''}{\mu'' - 1}, \quad 2r_4 = r_3 + r_4 - \frac{p''}{\mu'' - 1}, \quad \mu'^{-1} = m', \quad \mu''^{-1} = m'',$$

we have

$$4\,(r_1^2 + r_1 r_2 + r_2^2)\,p' = 3\,(r_1 + r_2)^2\,p' + \frac{m'^2 p'^3}{(1 - m')^2};$$

$$4p''\,(r_3^2 + r_3 r_4 + r_4^2) = 3\,(r_3 + r_4)^2\,p'' + \frac{m''^2 p''^3}{(1 - m'')^2};$$

$$-\,8\,(1 - m')\,p' r_1^2 = -\,2\,(1 - m')\,p'\,(r_1 + r_2)^2 - 4m'p'^2\,(r_1 + r_2) - \frac{2m'^2 p'^3}{1 - m'};$$

$$8p'^2\,(r_3 - r_2) = 4p'^2\left(-\,(r_1 + r_2) + r_3 + r_4 + \frac{m'p'}{1 - m'} + \frac{m''p''}{1 - m''}\right);$$

$$-\,\frac{8p''}{1 - m''}\,\{m''p' + (1 - m'')\,r_3\}^2 = -\,\frac{2p''}{1 - m''}\,\{(1 - m'')\,(r_3 + r_4) + m''\,(2p' + p'')\}^2$$

$$= -\,2\,(1 - m'')\,p''\,(r_3 + r_4)^2 - 4m''p''\,(2p' + p'')\,(r_3 + r_4) - \frac{2m''^2 p''}{1 - m''}\,(2p' + p'')^2;$$

$$-\,8p^2 r_4 = -\,4\,(p' + p'')^2\left(r_3 + r_4 - \frac{m''p''}{1 - m''}\right);$$

in the sum of which 6 terms and of $4\,(p' + p'')^3$, the coefficient of $(r_1 + r_2)^2$ is

$$3p' - 2\,(1 - m')\,p' = (2m' + 1)\,p';$$

that of $(r_3 + r_4)^2$,

$$3p'' - 2\,(1 - m'')\,p'' = (2m'' + 1)\,p'';$$

of $r_1 + r_2$,

$$-\,4m'p'^2 - 4p'^2 = -\,4\,(m' + 1)\,p'^2;$$

of $r_3 + r_4$,

$$4p'^2 - 4m''p''\,(2p' + p'') - 4\,(p' + p'')^2 = -\,4\,(m'' + 1)\,p''\,(p'' + 2p');$$

of p'^3,

$$\frac{m'^2}{(1 - m')^2} - \frac{2m'^2}{1 - m'} + \frac{4m'}{1 - m'} + 4$$

$$= \frac{m'^2 - 2\,(1 - m')\,m'^2 + 4m'\,(1 - m') + 4\,(1 - m')^2}{(1 - m')^2} = \frac{4 - 4m' - m'^2 + 2m'^3}{(1 - m')^2};$$

of p''^3,

$$\frac{m''^2}{(1 - m'')^2} - \frac{2m''^2}{1 - m''} + \frac{4m''}{1 - m''} + 4 = \frac{4 - 4m'' - m''^2 + 2m''^3}{(1 - m'')^2};$$

and the remaining terms are

$$\frac{4m''p'^2p''}{1-m''} - \frac{8m''^2p'p''(p'+p'')}{1-m''} + \frac{4m''p'p''(p'+2p'')}{1-m''} + 12p'p''(p'+p'')$$
$$= (8m''+12)p'p''(p'+p'');$$

so that the equation (1), of last section, when multiplied by 4, becomes, (halving all the recent results),

$$\begin{aligned} 0 = {} & (m'+\tfrac{1}{2})p'(r_1+r_2)^2 + (m''+\tfrac{1}{2})p''(r_3+r_4)^2 - 2(m''+1)p''(p'+p'')(r_3+r_4) \\ & - 2p'\{(m'+1)p'(r_1+r_2) + (m''+1)p''(r_3+r_4)\} \\ & + \frac{4-4m'-m'^2+2m'^3}{2(1-m')^2}p'^3 + \frac{4-4m''-m''^2+2m''^3}{2(1-m'')^2}p''^3 \\ & + 2(2m''+3)p'p''(p'+p''): \end{aligned} \qquad \text{(A)}$$

which differs from the equation (A) of [13.], only by the substitution of m', m'', p', p'', for m_1, m_2, p_1, p_2, that is, for the reciprocals of the indices, and for the powers, of the two lenses; r_1, r_2, r_3, r_4, being still the curvatures* of the four successive spheric surfaces, positive when convex to the incident light.

Again, if, in the double of the second member of equation (2) of last section, we change μ', μ'', to m'^{-1}, m''^{-1}, and $2r_1$, $2r_2$, $2r_3$, $2r_4$ to their values at the top of the present section, we find that the coefficient of r_1+r_2 is $-m'p'-p'$; of r_3+r_4, $p'-m''p''-(p'+p'')=-(m''+1)p''$; and the remaining terms are

$$2(p'+p'')^2 - \frac{m'^2p'^2}{1-m'} + \frac{m'p'^2}{1-m'} + \frac{m''p'p''}{1-m''} - \frac{m''^2p''(p''+2p')}{1-m''} + \frac{m''(p''+p')p''}{1-m''}$$
$$= (2+m')p'^2 + (2+m'')(p''^2+2p'p'') = (m'-m'')p'^2 + (2+m'')(p'+p'')^2;$$

therefore (2) gives

$$(m'+1)p'(r_1+r_2) + (m''+1)p''(r_3+r_4) = (m'-m'')p'^2 + (m''+2)(p'+p'')^2. \qquad \text{(B)}$$

Equation (B) gives the value of the second line of equation (A); it also gives $(r_1+r_2)^2$ as a quadratic function of r_3+r_4; and thus it enables us easily to transform (A) into an ordinary quadratic equation relative to r_3+r_4, after solving which, we can find r_1+r_2, and thus r_1, r_2, r_3, r_4, because

$$r_1 - r_2 = \frac{m'p'}{1-m'}; \quad r_3 - r_4 = \frac{m''p''}{1-m''}.$$

[35.] *Comparison with Herschel.*

My equations (A) and (B) (are intended to) serve for the construction of a thin double object glass, of which the aberrations in the diametral plane shall vanish, for oblique parallel incident rays, if the square of the obliquity of those rays be neglected. Herschel aimed to construct one of which the aberrations should vanish, for rays incident from a distant point in the axis, when the square of the nearness of that point is neglected. By the theory given in [32.], my formulæ will be adapted to this latter problem, by merely changing $2p^2$ to $\frac{3}{2}p^2$, that is, by subtracting half

* [That is, reciprocals of the radii.]

the square of the power of the double lens from the second member of the equation (B) of [34.], without making any change in the equation (A). But as Herschel assigns equations between the two anterior curvatures, r_1 and r_3, we must, for the purpose of comparison, change r_2 to $r_1 - \frac{m'p'}{1-m'}$, and r_4 to $r_3 - \frac{m''p''}{1-m''}$; and then $(r_1+r_2)^2$ becomes

$$4r_1^2 - \frac{4m'p'r_1}{1-m'} + \frac{m'^2p'^2}{(1-m')^2},$$

and $(r_3+r_4)^2$ becomes

$$4r_3^2 - \frac{4m''p''r_3}{1-m''} + \frac{m''^2p''^2}{(1-m'')^2};$$

consequently, in (A) thus altered (as to its form), the coefficient of r_1^2 is $2(2m'+1)p'$; of r_3^2, $2(2m''+1)p''$; of r_1,

$$-\frac{2m'(2m'+1)p'^2}{1-m'} - 4(1+m')p'^2 = -\frac{2(m'+2)p'^2}{1-m'};$$

of r_3,

$$-\frac{2m''(2m''+1)p''^2}{1-m''} - 4(m''+1)p''(p''+2p') = -\frac{2(m''+2)p''^2}{1-m''} - 8(m''+1)p'p'';$$

of p'^3,

$$\frac{m'^2(m'+\frac{1}{2})}{(1-m')^2} + \frac{2m'(m'+1)}{1-m'} + \frac{4-4m'-m'^2+2m'^3}{2(1-m')^2}$$

$$= \tfrac{1}{2}(1-m')^{-2}\{m'^2(2m'+1) + 4m'(1-m'^2) + 4 - 4m' - m'^2 + 2m'^3\}$$

$$= \frac{2}{(1-m')^2};$$

of p''^3,

$$\frac{m''^2(m''+\frac{1}{2})}{(1-m'')^2} + \frac{2m''(m''+1)}{1-m''} + \frac{4-4m''-m''^2+2m''^3}{2(1-m'')^2} = \frac{2}{(1-m'')^2};$$

of p'^2p'', $2(2m''+3)$; and of $p'p''^2$,

$$\frac{4m''(m''+1)}{1-m''} + 2(2m''+3) = \frac{2(m''+3)}{1-m''};$$

the equation (A) becomes therefore, after being halved,

$$\begin{aligned} 0 = (2m'+1)p'r_1^2 + (2m''+1)p''r_3^2 - \frac{(m'+2)p'^2r_1}{1-m'} - \frac{(m''+2)p''^2r_3}{1-m''} \\ -4(m''+1)p'p''r_3 + \frac{p'^3}{(1-m')^2} + \frac{p''^3}{(1-m'')^2} + (2m''+3)p'^2p'' + \frac{(m''+3)p'p''^2}{1-m''}; \end{aligned} \qquad (A')$$

which accordingly agrees with Herschel's equation (v), *Light*, art. 313, if we adapt that equation to our present notation, by changing the symbols L', L'', μ', μ'', R', R''', to p', p'', m'^{-1}, m''^{-1}, r_1, r_3, after taking care to read the last term of (v) as $\frac{2+3\mu''}{\mu''}L'^2L''$, as was remarked to Mr. Phillips in my letter of Jan. 3d. 1844; see p. 385 [of present volume]. In fact, it is easy to assure ourselves by mental calculations, that with this correction of the press, the equation (v) is a consequence of the earlier equation (u), on the same page 391, of *Light*. And I must own that the equation (A′), in the present section, is of a somewhat simpler form than the equation (A) in the preceding section. With respect to Herschel's other equation, it must be deduced from (B), by changing $2(p'+p'')^2$ to $\frac{3}{2}(p'+p'')^2$, as mentioned above; doubling therefore, for simplicity, and transposing, we get for coefficient of p'^2,

$$-\frac{2m'(m'+1)}{1-m'}-(3+2m')=-\frac{3+m'}{1-m'};$$

of p''^2, $-\frac{3+m''}{1-m''}$; and of $2p'p''$, $-(3+2m'')$; that is, we obtain the equation

$$0=4(m'+1)p'r_1+4(m''+1)p''r_3-\frac{3+m'}{1-m'}p'^2-\frac{3+m''}{1-m''}p''^2-2(3+2m'')p'p'';$$

which accordingly agrees with Herschel's formula (f), art. 469; or with my (B), by changing first member to $(p'+p'')^2$. (See p. 385 [of present volume].)

[36.] *Deduction of* (A′) *and* (B′) *from* (1) *and* (2) *of* [33.].

In [**33**.], I have given a summary of all the calculations required for deducing the two equations, quadratic and linear, between the curvatures of a thin double spheric lens *in vacuo*, which will render it aplanatic for parallel incident indiametral rays of small obliquity: namely those marked (1) and (2), near the foot of the section just referred to. In [**34**.], I gave the calculations required for transforming these equations into the two marked (A) and (B), between r_1+r_2 and r_3+r_4; and in [**35**.], eliminated r_2 and r_4. It would however have been simpler to have begun by performing this last elimination. Equation (1) being put under the form:

$$0=(p'+p'')^3+p'(r_1^2+r_1r_2+r_2^2)+p''(r_3^2+r_3r_4+r_4^2)-2(1-m')p'r_1^2+2p'^2(r_3-r_2)$$
$$-\frac{2p''}{1-m''}\{m''p'+(1-m'')r_3\}^2-2(p'+p'')^2r_4,$$

(under which form it results very easily from the analysis of [**33**.],) if we change r_2 and r_4 to their values in the preceding section, namely

$$r_2=r_1-\frac{m'p'}{1-m'},\quad r_4=r_3-\frac{m''p''}{1-m''},$$

we find, for the coefficient of r_1^2,

$$3p'-2(1-m')p'=(1+2m')p';$$

of r_3^2,

$$3p''-2(1-m'')p''=(1+2m'')p'';$$

of r_1,

$$-\frac{3m'p'^2}{1-m'}-2p'^2=-\frac{2+m'}{1-m'}p'^2;$$

of r_3,

$$-\frac{3m''p''^2}{1-m''}+2p'^2-4m''p'p''-2(p'+p'')^2=-\frac{2+m''}{1-m''}p''^2-4(1+m'')p'p'';$$

of p'^3,

$$1+\left(\frac{m'}{1-m'}\right)^2+\frac{2m'}{1-m'}=\left(1+\frac{m'}{1-m'}\right)^2=(1-m')^{-2};$$

of p''^3,

$$1+\left(\frac{m''}{1-m''}\right)^2+\frac{2m''}{1-m''}=(1-m'')^{-2};$$

of p'^2p'',

$$3 - \frac{2m''^2}{1-m''} + \frac{2m''}{1-m''} = 3 + 2m'';$$

and of $p'p''^2$,

$$3 + \frac{4m''}{1-m''} = \frac{3+m''}{1-m''};$$

the equation (A′), in the preceding section, is therefore thus deduced, with great ease, from the equation (1) of [**33.**].

In like manner if we substitute for r_2, r_4, their values in the equation [namely, (2) of [**33.**], with the signs changed,]

$$0 = m'p'r_1 + p'(r_2 - r_3) + \frac{m''p''}{1-m''}\{m''p' + (1-m'')r_3\} + (p'+p'')r_4 - (p'+p'')^2,$$

we find for the coefficient of r_1, $(m'+1)p'$; of r_3, $(m''+1)p''$; of p'^2,

$$-\frac{m'}{1-m'} - 1 = -\frac{1}{1-m'};$$

of p''^2,

$$-\frac{m''}{1-m''} - 1 = -\frac{1}{1-m''};$$

and of $p'p''$,

$$\frac{m''^2}{1-m''} - \frac{m''}{1-m''} - 2 = -(2+m'');$$

therefore my condition (2) may be put under the form:

$$(m'+1)p'r_1 + (m''+1)p''r_3 = \frac{p'^2}{1-m'} + \frac{p''^2}{1-m''} + (m''+2)p'p''. \qquad \text{(B′)}$$

Accordingly this last equation might be obtained from Herschel's formula (f), or from the equivalent formula at the foot of the preceding section, by changing the first member from 0 to $(p'+p'')^2$, that is, to the square of the power of the compound lens, and reducing. But it seems to be convenient, as a summary of what is most necessary in the way of calculation for *my* purpose, to annex this section to [**33.**]; and that we may have both equations in one view, I shall here copy the other:

$$\begin{aligned} 0 &= (2m'+1)p'r_1^2 + (2m''+1)p''r_3^2 \\ &\quad - \frac{m'+2}{1-m'}p'^2r_1 - \frac{m''+2}{1-m''}p''^2r_3 - 4(m''+1)p'p''r_3 \\ &\quad + \frac{p'^3}{(1-m')^2} + \frac{p''^3}{(1-m'')^2} + \frac{m''+3}{1-m''}p''^2p' + (2m''+3)p'^2p''. \end{aligned} \qquad \text{(A′)}$$

[**37.**] *Focal Lengths and Aberrations of a System of Refracting Surfaces of Revolution, close together at the origin.*

(Feb. 22d, 1844.) By [**32.**],

$$\frac{1}{x^2}\left(\frac{\mu_n}{z_{n+1}} - \frac{\mu_0}{z_0} - \frac{1}{F}\right) = \tfrac{1}{2}\frac{\mu_n}{z_{n+1}^3} - \tfrac{1}{2}\frac{\mu_0}{z_0^3} + 4F^{-4}(\sigma_n - \sigma_0)^{-4}T^{(4)};$$

and by [**33.**], making, by [**32.**],

$$\frac{\sigma_n}{\sigma_n - \sigma_0} = \frac{\mu_n F}{z_{n+1}}, \quad \frac{\sigma_0}{\sigma_n - \sigma_0} = \frac{\mu_0 F}{z_0}, \quad \left(\text{for central rays, } \frac{\mu_n F}{z_{n+1}} - \frac{\mu_0 F}{z_0} = 1,\right)$$

we have

$$4F^{-4}(\sigma_n - \sigma_0)^{-4}T''^{(4)} = -\frac{\mu_n r_n}{z_{n+1}^2} + \frac{\mu_0 r_1}{z_0^2} + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}\left(\frac{\mu_0}{z_0} + F_i^{-1}\right)^2 \Delta r_i;$$

$$4F^{-4}(\sigma_n - \sigma_0)^{-4}T'''^{(4)} = \Sigma_{(i)1}^{\ n} s_i \Delta_i \mu, \quad = \tfrac{1}{2}\Sigma_{(i)1}^{\ n} r_i^3 \Delta_i \mu, \text{ if surfaces be spheric};$$

therefore, *the equation determining the focal lengths and aberrations of the system* is

$$\frac{\mu_n}{z_{n+1}} - \frac{\mu_0}{z_0} - \frac{1}{F} = x^2\left\{\tfrac{1}{2}\frac{\mu_n}{z_{n+1}^3} - \tfrac{1}{2}\frac{\mu_0}{z_0^3} - \frac{\mu_n r_n}{z_{n+1}^2} + \frac{\mu_0 r_1}{z_0^2} + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}\left(\frac{\mu_0}{z_0} + F_i^{-1}\right)^2 \Delta r_i + \Sigma_{(i)1}^{\ n} s_i \Delta_i \mu\right\};$$

in which $\mu_0, .. \mu_i, .. \mu_n$ are the indices of the $n+1$ successive media; $r_1, .. r_n$ are the curvatures of the n successive surfaces; $s_1, .. s_n$ the *a-parabolicities*, or the coefficients of $\left(\frac{x^2+y^2}{2}\right)^2$ in the developments of the z's; $\Delta_i \mu = \mu_i - \mu_{i-1}$, $F_i^{-1} = \Sigma_{(i)1}^{\ i} r_i \Delta_i \mu$, $F = F_n$; $\Delta r_i = r_{i+1} - r_i$; x is the semi-aperture, or the common coordinate, perpendicular to the axis, of all the near points of incidence or refraction; z_0 is the ordinate of intersection of the initial ray with the common axis of revolution, and z_{n+1} is the ordinate of the intersection of the final ray with that axis.

In the second member of the formula, we may change $\frac{\mu_n}{z_{n+1}}$ to $\frac{\mu_0}{z_0} + \frac{1}{F}$; and then that member takes the form

$$(\lambda_0 + \mu_0\lambda_1 z_0^{-1} + \mu_0^2\lambda_2 z_0^{-2} + \mu_0^3\lambda_3 z_0^{-3})\,x^2;$$

in which the coefficients have the values:

$$\begin{cases} \lambda_0 = \tfrac{1}{2}\mu_n^{-2}F^{-3} - \mu_n^{-1}F^{-2}r_n + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}F_i^{-2}\Delta r_i + \Sigma_{(i)1}^{\ n} s_i \Delta_i \mu; \\ \lambda_1 = \tfrac{3}{2}\mu_n^{-2}F^{-2} - 2\mu_n^{-1}F^{-1}r_n + 2\Sigma_{(i)1}^{\ n-1}\mu_i^{-1}F_i^{-1}\Delta r_i; \\ \lambda_2 = \tfrac{3}{2}\mu_n^{-2}F^{-1} - \mu_n^{-1}r_n + \mu_0^{-1}r_1 + \Sigma_{(i)1}^{\ n-1}\mu_i^{-1}\Delta r_i = \tfrac{3}{2}\mu_n^{-2}F^{-1} - \Sigma_{(i)1}^{\ n} r_i \Delta_i(\mu^{-1}); \\ \lambda_3 = \tfrac{1}{2}(\mu_n^{-2} - \mu_0^{-2}). \end{cases}$$

Comparing these expressions with the definitions given in [**29.**], [**31.**], of M, N, O, we have the relations

$$\lambda_0 = M; \quad \lambda_1 = 2N - \tfrac{1}{2}\mu_n^{-2}F^{-2}; \quad \lambda_2 = O - \tfrac{1}{2}\mu_n^{-2}F^{-1}.$$

In this manner therefore we might find again that the conditions for the construction of Herschel's object glass, being $\lambda_0=0$, $\lambda_1=0$, are

$$M=0, \quad N=\tfrac{1}{4}\mu_n^{-2}F^{-2},$$

as in [**32.**]. (Mine are $M=0$, $N=0$, by [**29.**].)

[**38.**] For a *single surface*, $n=1$, $F^{-1}=r_1(\mu_1-\mu_0)$,

$$\begin{cases} \lambda_0=\tfrac{1}{2}\mu_1^{-2}(\mu_1-\mu_0)^3r_1^3-\mu_1^{-1}(\mu_1-\mu_0)^2r_1^3+(\mu_1-\mu_0)s_1; \\ \lambda_1=\tfrac{3}{2}\mu_1^{-2}(\mu_1-\mu_0)^2r_1^2-2\mu_1^{-1}(\mu_1-\mu_0)r_1^2; \\ \lambda_2=\tfrac{3}{2}\mu_1^{-2}(\mu_1-\mu_0)r_1+\mu_1^{-1}\mu_0^{-1}(\mu_1-\mu_0)r_1; \\ \lambda_3=\tfrac{1}{2}(\mu_1^{-2}-\mu_0^{-2}); \end{cases}$$

that is,

$$2\mu_1^2(\mu_1-\mu_0)^{-1}r_1^{-3}\lambda_0=(\mu_1-\mu_0)^2-2\mu_1(\mu_1-\mu_0)+2\mu_1^2r_1^{-3}s_1=\mu_0^2, \text{ for a sphere};$$

$$2\mu_1^2\mu_0(\mu_1-\mu_0)^{-1}r_1^{-2}\lambda_1=3\mu_0(\mu_1-\mu_0)-4\mu_0\mu_1=-\mu_0(\mu_1+3\mu_0);$$

$$2\mu_1^2\mu_0^2(\mu_1-\mu_0)^{-1}r_1^{-1}\lambda_2=3\mu_0^2+2\mu_0\mu_1=\mu_0(2\mu_1+3\mu_0);$$

$$2\mu_1^2\mu_0^3(\mu_1-\mu_0)^{-1}\lambda_3=-\mu_0(\mu_1+\mu_0);$$

also

$$\mu_0^2(r_1-z_0^{-1})^3-\mu_0\mu_1z_0^{-1}(r_1-z_0^{-1})^2=\mu_0(r_1-z_0^{-1})^2\{-(\mu_1+\mu_0)z_0^{-1}+\mu_0r_1\};$$

therefore the formula for a refraction at a single spheric surface is

$$\boxed{\frac{\mu_1}{z_2}-\frac{\mu_0}{z_0}-(\mu_1-\mu_0)r_1=\frac{\mu_0(\mu_1-\mu_0)}{\mu_1^2}(r_1-z_0^{-1})^2\{-(\mu_1+\mu_0)z_0^{-1}+\mu_0r_1\}\frac{x^2}{2}.}$$

This accordingly agrees with Herschel's formulæ, namely

$$f=(1-m)R+mD; \quad \Delta f=\frac{m(1-m)}{2}(R-D)^2\{mR-(1+m)D\}y^2.$$

(In a paper in the *Phil. Mag.* for October, 1841 [No. XIII of the present volume, equation (26.)], I deduced, for the aberration of a single refracting spheric surface, an equation which, in the present notation, is

$$\text{first member (as above)}=\frac{\mu_0\mu_1}{\mu_1-\mu_0}(r_1-z_0^{-1}-z_2^{-1})(z_2^{-1}-z_0^{-1})^2\frac{x^2}{2}.$$

Accordingly,

$$\mu_1(z_2^{-1}-r_1)=\mu_0(z_0^{-1}-r_1)=\frac{\mu_0\mu_1}{\mu_1-\mu_0}(z_0^{-1}-z_2^{-1}).)$$

For a *lens in vacuo*, $n=2$, $\mu_0=\mu_2=1$, $\mu_1=\mu$,

$$F_1^{-1}=r_1(\mu-1), \quad F_2^{-1}=F^{-1}=(r_1-r_2)(\mu-1),$$

$$\mu_1^{-1}F_1^{-2}\Delta r_1=-\mu^{-1}(\mu-1)r_1^2F^{-1}, \quad \mu_1^{-1}F_1^{-1}\Delta r_1=-\mu^{-1}r_1F^{-1},$$

$$-\Sigma_{(i)1}^{\ 2}r_i\Delta_i\frac{1}{\mu}=-r_1(\mu^{-1}-1)-r_2(1-\mu^{-1})=\mu^{-1}F^{-1};$$

therefore

$$\begin{cases} 2F\lambda_0 = F^{-2} - 2r_2F^{-1} - 2\mu^{-1}(\mu-1)\,r_1^2 + 2\left(\dfrac{s_1-s_2}{r_1-r_2}\right); \\ 2F\lambda_1 = 3F^{-1} - 4r_2 - 4\mu^{-1}r_1 = -\left(3 - 3\mu + \dfrac{4}{\mu}\right)r_1 - (3\mu+1)\,r_2; \\ 2F\lambda_2 = 3 + 2\mu^{-1}; \qquad \lambda_3 = 0. \end{cases}$$

If the lens be spheric, the coefficient of r_1^2 in $2\mu F\lambda_0$ is

$$\mu(\mu-1)^2 - 2(\mu-1) + \mu = 2 - 2\mu^2 + \mu^3;$$

that of r_1r_2 is

$$-2(\mu-1)^2\mu - 2(\mu-1)\mu + \mu = -2\mu^3 + 2\mu^2 + \mu;$$

and that of r_2^2 is

$$\mu\{(\mu-1)^2 + 2(\mu-1) + 1\} = \mu^3.$$

Hence, for a *single infinitely thin spheric lens in vacuo*, with curvatures r_1, r_2, index μ, focal length F, we have the equation

$$\frac{1}{z_3} - \frac{1}{z_0} - \frac{1}{F} = \frac{x^2}{2\mu F}\left\{\begin{array}{l}(2 - 2\mu^2 + \mu^3)\,r_1^2 + (\mu + 2\mu^2 - 2\mu^3)\,r_1r_2 + \mu^3 r_2^2 \\ -((4 + 3\mu - 3\mu^2)\,r_1 + (\mu + 3\mu^2)\,r_2)\,z_0^{-1} + (2+3\mu)\,z_0^{-2}\end{array}\right\};$$

agreeing with Herschel's formula. (Compare [**28.**].)

(Feb. 23d, 1844.) For a *thin double spheric lens in vacuo*, we have*

$$\frac{1}{z_5} - \frac{1}{z_0} - \frac{1}{F} = x^2\{M + (2N - \tfrac{1}{2}F^{-2})\,z_0^{-1} + (O - \tfrac{1}{2}F^{-1})\,z_0^{-2}\};$$

in which

$$F^{-1} = (\mu'-1)(r_1 - r_2) + (\mu''-1)(r_3 - r_4),$$

and M, N, O have been already developed [pp. 429, 430]. Thus $M\,(= -F^{-4}L = F^{-4}(4Q + \tfrac{1}{2}F))$ is the quarter of the function of indices and curvatures, which is equated to zero in (A) of [**13.**], or [**34.**]; or it is the half of the function equated to zero in (A′), of [**35.**], [**36.**]. (As a verification, when $r_1 = r_2 = r_3 = 0$, $z_0^{-1} = 0$, we thus get

$$L = -F^4M = -\frac{\tfrac{1}{2}F}{(1-m'')^2} = -\tfrac{1}{2}\mu''^2 r_4^2 F^3;$$

which agrees with the expression for a single plano-spheric lens, exposed to parallel rays, namely, by this section,

$$\frac{1}{z_3} = \frac{1}{F} + \tfrac{1}{2}\frac{\mu^2 r_2^2 x^2}{F}.)$$

For the same double lens, $2N$ is the first member of (B) of [**13.**], with the signs changed; or it is the second member minus the first member of (B) of [**34.**]; or N is second member minus first member of (B′) of [**36.**]. Finally, by [**31.**],

$$O = 2F^{-1} + m'p' + m''p''.$$

In general, for ANY COMBINATION OF THIN LENSES IN VACUO, (spheric or not,)

$$\lambda_2' = \tfrac{3}{2}F^{-1} + \Sigma mp = \Sigma(\tfrac{3}{2} + m)\,p.$$

* [This is derived directly from the general result of the preceding section, on substituting for the λ's their values in terms of M, N, O (p. 441). Changing z_5 to z_{n+1}, the result is applicable to any system of thin lenses (spheric or not) close together *in vacuo*.]

[39.] *Foci and Aberrations for oblique parallel initial rays.*

(Feb. 22d, 1844.) By [37.], for *any combination of refracting surfaces of revolution close together in vacuo at the origin*, the focal lengths and aberrations may be determined by the formula

$$\boxed{\frac{1}{z_{n+1}}=\frac{1}{z_0}+\frac{1}{F}+\left(\lambda_0+\frac{\lambda_1}{z_0}+\frac{\lambda_2}{z_0^2}\right)x^2;}$$

in which, with the notation of [29.], [31.],

$$\lambda_0=M;\quad \lambda_1=2N-\tfrac{1}{2}F^{-2};\quad \lambda_2=O-\tfrac{1}{2}F^{-1}.$$

The initial ray passes through the points $0, z_0$ and $x, 0$; the final through $x, 0$ and $0, z_{n+1}$. (The final x, 0 is not exactly enough coincident with the initial x, 0; see below, and [40.].)* Hence the equation of the initial ray may be put under the form

$$\frac{\xi_0}{x}+\frac{\zeta_0}{z_0}=1;$$

and that of the final under the form

$$\frac{\xi_{n+1}}{x}+\frac{\zeta_{n+1}}{z_{n+1}}=1;$$

if ξ_0, ζ_0 be the general or current coordinates of the one, and ξ_{n+1}, ζ_{n+1} those of the other. Now let the initial rays be given to have a small and common inclination to the axis; then $\frac{x}{z_0}$ is small and constant $=-\alpha_0$; or $\frac{1}{z_0}=-\frac{\alpha_0}{x}$, α_0 being a quantity of which we shall neglect the square. Then the formula gives

$$\frac{x}{z_{n+1}}=\frac{x}{F}+\lambda_0x^3-\alpha_0(1+\lambda_1x^2);$$

therefore

$$\xi_{n+1}=x-\frac{x}{z_{n+1}}\zeta_{n+1}=x+\alpha_0\zeta_{n+1}(1+\lambda_1x^2)-\left(\frac{x}{F}+\lambda_0x^3\right)\zeta_{n+1};$$

such, then, is, approximately, the *equation of the final ray* from the point x, 0, if the initial ray be parallel to $\xi_0=\alpha_0\zeta_0$, and if α_0 be very small. For example, the final ray from 0, 0 is $\xi_{n+1}=\alpha_0\zeta_{n+1}$, that is, light passes through the common vertex with an unchanged direction. Also if $x=F\alpha_0$, then the principal part of the inclination of the final ray vanishes.

Now consider the intersection of any other final ray with that from the vertex. We have, for this intersection,

$$0=1+\zeta_{n+1}\left(\lambda_1\alpha_0x-\frac{1}{F}-\lambda_0x^2\right);$$

that is,

$$\zeta_{n+1}=F(1+F\lambda_0x^2-F\lambda_1\alpha_0x)^{-1}=F-F^2\lambda_0x^2+F^2\lambda_1\alpha_0x.$$

This conclusion is not exact enough, owing to the differences of the x's of intersection of the initial and final rays with the axis of x, which is perpendicular to the axis of the system at the common vertex. See [40.]. See also the investigation resumed and completed in [41.], and by other methods in [43.], [44.].*

* [These remarks were inserted subsequently.]

When $M=0$, $N=0$, and $\mu_n=\mu_0=1$, we have, on the one hand, by [37.], the relation*

(a) $$\frac{1}{z_{n+1}}=\frac{1}{z_0}+\frac{1}{F}-\frac{1}{2}\frac{x^2}{F^2 z_0}+(O-\tfrac{1}{2}F^{-1})\left(\frac{x}{z_0}\right)^2;$$

F being the focal length of the combination of surfaces of revolution, supposed to be close together *in vacuo*, and to be constructed according to my two conditions for the destruction of aberration; O is a certain other constant, namely, by [31.], $2F^{-1}-\Sigma_{(i)}{}_1^n r_i \Delta_i \frac{1}{\mu}$; z_0, z_{n+1} are the ordinates of intersection of the initial and final rays with the axis of the combination, or of z, the common vertex being taken for origin, and x is the semiaperture. Under the same conditions, by [31.],

$$T=-\tfrac{1}{2}F(\alpha_n-\alpha_0)^2+\tfrac{1}{4}F^2 O\alpha_0^2(\alpha_n-\alpha_0)^2-\tfrac{1}{8}F(\alpha_n^2-\alpha_0^2)^2;$$

therefore the equations of the initial and final rays are respectively

(b) $$x_0-\alpha_0(1+\tfrac{1}{2}\alpha_0^2)z_0+F(\alpha_n-\alpha_0)=\tfrac{1}{2}F^2 O\alpha_0(\alpha_n-\alpha_0)(2\alpha_0-\alpha_n)-\tfrac{1}{2}F\alpha_0(\alpha_n^2-\alpha_0^2);$$

(c) $$x_{n+1}-\alpha_n(1+\tfrac{1}{2}\alpha_n^2)z_{n+1}+F(\alpha_n-\alpha_0)=\tfrac{1}{2}F^2 O\alpha_0^2(\alpha_n-\alpha_0)-\tfrac{1}{2}F\alpha_n(\alpha_n^2-\alpha_0^2).$$

To show that (a) is consistent with (b) and (c), we may observe that the two last equations give, when $x_0=0$, $x_{n+1}=0$,

$$\frac{F(\alpha_n-\alpha_0)}{z_0}=\alpha_0\{1+\tfrac{1}{2}\alpha_0^2+\tfrac{1}{2}FO\alpha_0(2\alpha_0-\alpha_n)-\tfrac{1}{2}\alpha_0(\alpha_n+\alpha_0)\},$$

$$\frac{F(\alpha_n-\alpha_0)}{z_{n+1}}=\alpha_n\{1+\tfrac{1}{2}\alpha_n^2+\tfrac{1}{2}FO\alpha_0^2-\tfrac{1}{2}\alpha_n(\alpha_n+\alpha_0)\}$$
$$=\alpha_n(1-\tfrac{1}{2}\alpha_0\alpha_n+\tfrac{1}{2}FO\alpha_0^2);$$

therefore

(d) $$z_{n+1}^{-1}-z_0^{-1}-F^{-1}=-\tfrac{1}{2}F^{-1}\alpha_0\alpha_n+O\alpha_0^2;$$

the error being of the 4th dimension. Now, to the accuracy of the 1st dimension, or indeed of the 2nd, inclusive, we have

$$x=-F(\alpha_n-\alpha_0);\quad \alpha_0=-\frac{x}{z_0};\quad \alpha_n=-\frac{x}{z_{n+1}};$$

therefore

$$O\alpha_0^2=O\left(\frac{x}{z_0}\right)^2,\quad -\tfrac{1}{2}F^{-1}\alpha_0\alpha_n=-\tfrac{1}{2}F^{-1}x^2 z_0^{-1}z_{n+1}^{-1}=-\tfrac{1}{2}F^{-1}x^2 z_0^{-1}(z_0^{-1}+F^{-1});$$

therefore (d) transforms itself into (a); and reciprocally, (a) may be changed to (d).

[40.] (*Foci for oblique rays.*)

Now, the equation (c) expresses that if α_0 be given, all the final rays pass through the common focus

(e) $$X_{n+1}=\alpha_0 F(1-\tfrac{1}{2}FO\alpha_0^2),\quad Z_{n+1}=(1-\tfrac{1}{2}\alpha_0^2)F(1-\tfrac{1}{2}FO\alpha_0^2);$$

(compare [31.];†) and I wish to see whether we could deduce the existence and position of this common point or focus of the final rays, for a given small obliquity of the parallel initial rays, from the equation (a) or (d).

* [The x of this formula was defined, without ambiguity, in [32.], p. 432.]
† [The initial and final rays are at present considered to be *in vacuo*.]

That there is nearly such a common focus for the final rays, when the initial rays have been oblique to the axis but parallel to each other, may be proved even from the equation of *focal lengths* (not aberrations) for direct rays, namely

(f) $$z_{n+1}^{-1} = z_0^{-1} + F^{-1}.$$

For the relation

$$X_{n+1} = \frac{x(Z_{n+1} - z_{n+1})}{-z_{n+1}} = x\{1 - (z_0^{-1} + F^{-1}) Z_{n+1}\},$$

is satisfied for all values of x and z_0 which are in a constant ratio to each other, namely $x = -\alpha_0 z_0$, by supposing $Z_{n+1} = F$, $X_{n+1} = \alpha_0 F$. Thus, the law of the formation of approximate oblique foci, for parallel (and indeed for diverging or converging) initial rays, may be deduced from the law (f) of the approximate foci for diverging (or converging) initial rays. In fact, by the law (f), we can so far *trace the course of a given initial ray*, as to determine, with only an error of the 2nd dimension, the intersection of the final ray with the axis of z, and with only an error of the 3rd dimension in the intersection of the same ray with the axis of x, (the refracting surfaces being close together at the origin;) we can therefore determine the coordinates X and Z of this intersection of two final rays with each other, with only an error of the 3rd dimension (at most) in X, and of the 2nd dimension in Z.

Thus, if the initial rays diverge from or converge to X_0, Z_0, we have the two equations

(g) $$X_0 = x(1 - Z_0 z_0^{-1}), \quad X_{n+1} = x(1 - Z_{n+1} z_{n+1}^{-1});$$

therefore

$$X_{n+1} Z_{n+1}^{-1} - X_0 Z_0^{-1} = x(Z_{n+1}^{-1} - Z_0^{-1} - F^{-1});$$

and this will be satisfied independently of x, by establishing the following equations, which contain the theory of *images*:

(h) $$Z_{n+1}^{-1} = Z_0^{-1} + F^{-1}; \quad X_{n+1} Z_{n+1}^{-1} = X_0 Z_0^{-1}.$$

But although the equation (a) determines for a given initial ray the intersection of the final ray with the axis of z, so as to leave only an error of the 4th dimension, yet because that equation leaves us still liable to commit an error of the 3rd dimension with respect to the intersection of the same final ray with the axis of x, or the point where it emerges from the last refracting surface, we are liable, till farther information is procured respecting this last point, to commit an error of the 3rd dimension relatively to X, and therefore one of the 2nd dimension relatively to Z, of the intersection of two final rays with each other. We must not therefore expect to *deduce*, though we may perhaps *verify*, the existence of the focus (e), with the accuracy required above, by means of the equation (a) alone.

We must therefore combine with (a) another formula, derived from (b) and (c), for the change of x, at the common tangent to all the surfaces, that is, when $z_0 = 0$, $z_{n+1} = 0$; namely

(i) $$\Delta x = \tfrac{1}{2} F (\alpha_n - \alpha_0)^2 \{FO\alpha_0 - (\alpha_n + \alpha_0)\};$$

or, in the same order of approximation, (see foot of preceding section,)

(j) $$\Delta x = \tfrac{1}{2} F^{-1} x^3 \{(2 - FO) z_0^{-1} + F^{-1}\};$$

in which, [see [**31.**]]

$$2 - FO = \frac{\Sigma_{(i)1}^{\ n} r_i \Delta_i \dfrac{1}{\mu}}{\Sigma_{(i)1}^{\ n} r_i \Delta_i \mu}.$$

[41.] (*Foci for oblique rays.*)

Resuming therefore the investigations begun in [39.], with respect to the intersection, which we shall now call X_{n+1}, Z_{n+1}, of any two final rays corresponding to any two parallel oblique incident rays, or rays for which $\frac{x}{z_0}$ is constant, and employing the two formulæ (a) and (j); we have, as an equation of a final ray, the following:

(k) $$\frac{X_{n+1}}{x+\Delta x} + \frac{Z_{n+1}}{z_{n+1}} = 1;$$

that is,

(l) $$X_{n+1}\{1 - \tfrac{1}{2}F^{-1}x^2(\overline{2-FO}\,z_0^{-1} + F^{-1})\}$$
$$+ xZ_{n+1}\{z_0^{-1} + F^{-1} - \tfrac{1}{2}F^{-2}x^2z_0^{-1} + (O - \tfrac{1}{2}F^{-1})x^2z_0^{-2}\} = x.$$

Differentiating this equation with respect to x, but treating xz_0^{-1} as constant, we find

(m) $$-\tfrac{1}{2}X_{n+1}F^{-1}(\overline{2-FO}\,xz_0^{-1} + 2F^{-1}x) + Z_{n+1}\{F^{-1} - F^{-2}x^2z_0^{-1} + (O - \tfrac{1}{2}F^{-1})x^2z_0^{-2}\} = 1;$$

so that we are led to try to satisfy the system of equations

$$-\tfrac{1}{2}X_{n+1}F^{-1}(2 - FO)\,xz_0^{-1} + Z_{n+1}\{F^{-1} + (O - \tfrac{1}{2}F^{-1})\,x^2z_0^{-2}\} = 1;$$

and

$$X_{n+1}(1 - \tfrac{1}{2}F^{-2}x^2) + Z_{n+1}xz_0^{-1}(1 - \tfrac{1}{2}F^{-2}x^2) = 0,$$

that is,

$$X_{n+1} + Z_{n+1}xz_0^{-1} = 0;$$

which can in fact be satisfied (in our present order of approximation) by supposing

(n) $$\begin{cases} Z_{n+1} = \{F^{-1} + \tfrac{1}{2}(F^{-1} + O)\,x^2z_0^{-2}\}^{-1} = F(1 - \tfrac{1}{2}x^2z_0^{-2})(1 - \tfrac{1}{2}FOx^2z_0^{-2}); \\ \text{and} \quad X_{n+1} = -xz_0^{-1}Z_{n+1}; \end{cases}$$

that is, if we make $-xz_0^{-1} = \tan\sin^{-1}\alpha_0$, and therefore $1 - \tfrac{1}{2}x^2z_0^{-2} = \cos\sin^{-1}\alpha_0$,

(o) $$\frac{X_{n+1}}{\alpha_0} = \frac{Z_{n+1}}{1 - \tfrac{1}{2}\alpha_0^2} = F(1 - \tfrac{1}{2}FO\alpha_0^2);$$

formulæ which agree with (e).

The formulæ (a) and (j) are therefore sufficient to show that, under the conditions which have conducted to them, namely, those denoted already by

(p) $$M = 0, \quad N = 0, \quad \mu_n = \mu_0 = 1,$$

the aberration of the system is destroyed, for oblique parallel indiametral incident rays: which is one part of the theory of my object glass.

(Feb. 23d.) For any combination of coaxal refracting surfaces of revolution, placed close together at the origin, we have, by [31.], instead of the expression (j) at the foot of the preceding section, the following, in which Δx is the total change of abscissa of intersection of the ray with the axis of x, that is, with the common tangent to the surfaces:

$$\Delta x = \frac{\delta T}{\delta\sigma_0} + \frac{\delta T}{\delta\sigma_n} = \tfrac{1}{2}F^3N(\sigma_n - \sigma_0)^3 + \tfrac{1}{2}F^2O\sigma_0(\sigma_n - \sigma_0)^2 - \tfrac{1}{2}\mu_n^{-2}F(\sigma_n + \sigma_0)(\sigma_n - \sigma_0)^2;$$

therefore

$$\Delta x = \tfrac{1}{2}F^3(N - \mu_n^{-2}F^{-2})(\sigma_n - \sigma_0)^3 + \tfrac{1}{2}F(FO - 2\mu_n^{-2})\,\sigma_0(\sigma_n - \sigma_0)^2$$

$$= -\frac{x^3}{2}\left\{N - \mu_n^{-2}F^{-2} + (O - 2\mu_n^{-2}F^{-1})\frac{\mu_0}{z_0}\right\}^*$$

$$= \frac{x^3}{2}\left\{-\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}F_i^{-1}\Delta r_i + \mu_n^{-1}F_n^{-1}r_n + \mu_0 z_0'^{-1}\Sigma_{(i)}{}_1^{n} r_i\Delta_i\frac{1}{\mu}\right\};$$

z_0' being the ordinate of the point where the first incident ray crosses the axis of the system.† From geometrical considerations, I think that this ought to be equal to

$$\frac{x^3}{2}\Sigma_{(i)}{}_1^{n} r_i\Delta_i\frac{1}{z'},$$

z_i' being the ordinate of the point where the ith refracted ray crosses the axis of the system. If so, we must have

$$\Delta_1\frac{1}{z'} = \mu_1^{-1}F_1^{-1} + \mu_0 z_0^{-1}\Delta_1\frac{1}{\mu} = r_1\left(1 - \frac{\mu_0}{\mu_1}\right) + z_0^{-1}\left(\frac{\mu_0}{\mu_1} - 1\right),$$

that is,

$$\frac{\mu_1}{z_1'} = \frac{\mu_0}{z_0'} + r_1(\mu_1 - \mu_0),$$

which is true; and also,

$$r_n\Delta_n\frac{1}{z'} = \mu_{n-1}^{-1}F_{n-1}^{-1}(r_{n-1} - r_n) + \mu_n^{-1}F_n^{-1}r_n - \mu_{n-1}^{-1}F_{n-1}^{-1}r_{n-1} + \mu_0 z_0^{-1}r_n(\mu_n^{-1} - \mu_{n-1}^{-1}),$$

that is,

$$z_n'^{-1} - z_{n-1}'^{-1} = \mu_n^{-1}F_n^{-1} - \mu_{n-1}^{-1}F_{n-1}^{-1} + \mu_0 z_0^{-1}(\mu_n^{-1} - \mu_{n-1}^{-1}),$$

that is,

$$z_n'^{-1} - \mu_n^{-1}(F_n^{-1} + \mu_0 z_0^{-1}) = z_1'^{-1} - \mu_1^{-1}(F_1^{-1} + \mu_0 z_0^{-1}) = 0,$$

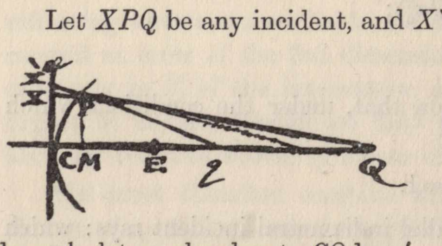
or,

$$\mu_n z_n'^{-1} = \mu_0 z_0^{-1} + F_n^{-1},$$

which is also true.

[42.]‡ (*Foci for oblique rays.*)

Let XPQ be any incident, and $X'Pq$ the corresponding refracted ray, CXX' being a tangent to the refracting surface CP at the vertex C, and E, on the axis CqQ, being the centre of curvature. Let PM, as in Herschel's figure on last page,§ be a perpendicular let fall from the point of incidence P, on the axis CQ; but let us, as in the notation already employed in my own investigations on recent pages of this book, denote PM by x_i; CE by r_i^{-1}; let us also denote CQ by z'_{i-1}; and Cq by z_i'. Then, $\overline{CM} = \frac{1}{2}r_i x_i^2$, nearly;



$$\frac{\overline{CM}}{\overline{MQ}} = \tfrac{1}{2}z_{i-1}'^{-1}r_i x_i^2 = \frac{\overline{CX} - \overline{MP}}{\overline{MP}};$$

* [Since $\dfrac{\sigma_0}{\sigma_n - \sigma_0} = \dfrac{\mu_0 F}{z_0}$; see [37.].]

† [$z_0' = z_0$.]

‡ [The method of the characteristic function is not used in [42.] to [45.] inclusive.]

§ [We have omitted the page referred to, and a few others, headed "Comparison with Herschel."]

therefore

$$\overline{CX} - \overline{MP} = \tfrac{1}{2} z_{i-1}'^{-1} r_i x_i^3,$$

neglecting x_i^5; similarly

$$\overline{CX'} - \overline{MP} = \tfrac{1}{2} z_i'^{-1} r_i x_i^3;$$

therefore

$$\overline{XX'} = \overline{CX'} - \overline{CX} = \tfrac{1}{2}(z_i'^{-1} - z_{i-1}'^{-1}) r_i x_i^3,$$

neglecting x_i^5. Hence if we denote $\overline{CX}$ by x'_{i-1}, and $\overline{CX'}$ by $x'_{i-1} + \Delta_i x'$, we have

$$\boxed{\Delta_i x' = r_i \frac{x_{i-1}'^3}{2} \Delta_i \frac{1}{z'};}$$

in which, as in former investigations,

$$\Delta_i \frac{1}{z'} = z_i'^{-1} - z_{i-1}'^{-1}.$$

Let therefore the first incident ray cut the axis of x, on the common tangent CX to all the refracting surfaces of revolution at their vertex, in a point of which the abscissa, on that tangent, is x_0', or simply x'. The last refracted ray will cut the same axis, or tangent, CX, in a point of which the abscissa is

$$\boxed{x_n' = x' + \tfrac{1}{2} x'^3 \Sigma_{(i)}{}_1^n r_i \Delta_i \frac{1}{z'}.}$$

But

$$z_i'^{-1} = \mu_i^{-1}(\mu_0 z_0'^{-1} + F_i^{-1}),$$

if we neglect the aberrations which have no effect in the present investigation (relatively to x_n'); z_0' being here the same as z_0. Hence, denoting for abridgment z_0' by z', we have for the abscissa x'' of intersection of the final ray with the axis of x, the expression:

$$x'' = x' + \frac{x'^3}{2}\left\{\frac{\mu_0}{z'} \Sigma_{(i)}{}_1^n r_i \Delta_i \frac{1}{\mu} + \Sigma_{(i)}{}_1^n r_i \Delta_i \left(\frac{1}{\mu} F^{-1}\right)\right\};$$

in which we are to consider F_0^{-1} as equal to zero, because

$$\Delta_1 \frac{1}{z'} = z_1'^{-1} - z_0'^{-1} = \mu_0 z_0'^{-1} \Delta_1 \frac{1}{\mu} + \mu_1^{-1} F_1^{-1}.$$

Thus,

$$\Sigma_{(i)}{}_1^n r_i \Delta_i (\mu^{-1} F^{-1}) = r_n \mu_n^{-1} F_n^{-1} - \Sigma_{(i)}{}_1^{n-1} \mu_i^{-1} F_i^{-1} \Delta r_i;$$

and by employing the symbols λ of [**37.**], this last expression becomes

$$-\tfrac{1}{2}\lambda_1 + \tfrac{3}{4}\mu_n^{-2} F^{-2}.$$

With the same symbols,

$$\Sigma_{(i)}{}_1^n r_i \Delta_i \left(\frac{1}{\mu}\right) = -\lambda_2 + \tfrac{3}{2}\mu_n^{-2} F^{-1};$$

thus we may write, for any *combination of coaxal refracting surfaces of revolution, placed close together at the origin,*

$$\begin{cases} x''^{-1} = x'^{-1} + \dfrac{x'}{4}\left\{\lambda_1 + 2\lambda_2 \dfrac{\mu_0}{z'} - \tfrac{3}{2}\mu_n^{-2} F^{-1}\left(F^{-1} + 2\dfrac{\mu_0}{z'}\right)\right\}; \\ z''^{-1} = \mu_n^{-1}\left(\dfrac{\mu_0}{z'} + \dfrac{1}{F}\right) + \mu_n^{-1} x'^2 \left(\lambda_0 + \lambda_1 \dfrac{\mu_0}{z'} + \lambda_2 \left(\dfrac{\mu_0}{z'}\right)^2 + \lambda_3 \left(\dfrac{\mu_0}{z'}\right)^3\right); \end{cases}$$

x', 0 and 0, z' being the coordinates of the two points in which the initial ray crosses the axes of x and z; and x'', 0 and 0, z'' being the coordinates of the two points where the final ray crosses the same axes; so that the equations of these rays may be thus written:

$$\boxed{X' x'^{-1} + Z' z'^{-1} = 1; \quad X'' x''^{-1} + Z'' z''^{-1} = 1;}$$

μ_0 being initial, and μ_n final index; $\dfrac{1}{F} = \Sigma_{(i)}{}_1^n r_i \Delta_i \mu$, as before, and λ_0, λ_1, λ_2, λ_3 having the values assigned in [37.].

And, in the order of approximation to which we have hitherto confined ourselves, we may investigate all circumstances respecting the arrangement of any system of *indiametral* rays, by the help of these last systems of equations.

[43.] (*Foci for oblique rays.*)

Thus, if the initial rays be parallel, x' will bear a constant ratio to z'; and to find the intersection of two infinitely near refracted rays, we may differentiate the equation of the refracted ray with respect to x', treating $\dfrac{x'}{z'}$ as constant; and thus we obtain, after multiplying by x',

$$\boxed{\frac{X''}{x'} + \frac{\mu_0}{\mu_n z'}\left(1 + \lambda_3 \left(\frac{\mu_0 x'}{z'}\right)^2\right) Z'' = \frac{X''}{4}(\lambda_1 - \tfrac{3}{2}\mu_n^{-2} F^{-2})\, x' + Z''\left(\lambda_1 \frac{\mu_0}{\mu_n}\frac{x'}{z'} + 2\lambda_0 \frac{x'}{\mu_n}\right) x'.}$$

Neglecting at first small terms, we have the two equations

$$\frac{X''}{x} = 1 - \mu_n^{-1} Z'' \left(\frac{\mu_0}{z'} + \frac{1}{F}\right), \quad \frac{X''}{x'} = -\mu_n^{-1} Z'' \frac{\mu_0}{z'},$$

which give as approximate coordinates of the intersection of the two infinitely near rays,

$$\boxed{X'' = -\frac{\mu_0 x' F}{z'}; \quad Z'' = \mu_n F;}$$

then substituting these values in the terms of the 2nd dimension, in the equation in the present section, and in that obtained by subtracting it from the one in the preceding section, namely

$$1 = \frac{Z''}{\mu_n F} + \frac{X'' x'}{2}(\lambda_1 - \tfrac{3}{2}\mu_n^{-2} F^{-2}) + \frac{X'' x'}{2}\frac{\mu_0}{z'}(\lambda_2 - \tfrac{3}{2}\mu_n^{-2} F^{-1})$$
$$+ \mu_n^{-1} Z'' x'^2 \left(3\lambda_0 + 2\frac{\lambda_1 \mu_0}{z'} + \frac{\lambda_2 \mu_0^2}{z'^2}\right),$$

we find 1st.,

$$\frac{Z''}{\mu_n F} = 1 + \frac{\mu_0 x'^2 F}{2z'}(\lambda_1 - \tfrac{3}{2}\mu_n^{-2} F^{-2}) + \frac{\mu_0^2 x'^2 F}{2z'^2}(\lambda_2 - \tfrac{3}{2}\mu_n^{-2} F^{-1})$$

$$- x'^2 F\left(3\lambda_0 + 2\frac{\lambda_1 \mu_0}{z'} + \frac{\lambda_2 \mu_0^2}{z'^2}\right),$$

so that this expression will involve one term varying with x'^2, and another with $\frac{x'^2}{z'}$, $\left(\frac{x'}{z'}\text{ being constant,}\right)$ unless the two following conditions are satisfied:

$$\lambda_0 = 0; \quad \lambda_1 = -\tfrac{1}{2}\mu_n^{-2} F^{-2};$$

that is, as before,

$$M = 0, \quad N = 0.$$

But 2nd., when these two conditions are satisfied, then, (see [**37.**],)

$$\frac{Z''}{\mu_n F} = 1 - \tfrac{1}{2}\left(\frac{\mu_0 x'}{z'}\right)^2 F(\lambda_2 + \tfrac{3}{2}\mu_n^{-2} F^{-1}) = 1 - \tfrac{1}{2}\left(\frac{\mu_0 x'}{\mu_n z'}\right)^2 (1 + \mu_n^2 FO)$$

$$= (1 - \tfrac{1}{2}\alpha_n^2)(1 - \tfrac{1}{2}FO\,\sigma_0^2),$$

if

$$\frac{\alpha_0}{1 - \frac{1}{2}\alpha_0^2} = -\frac{x'}{z'}, \quad \alpha_n = \frac{\mu_0 \alpha_0}{\mu_n}, \quad \sigma_0 = \mu_0 \alpha_0;$$

and, by the first equation in the present section,

$$X'' = -\frac{x' \mu_0 F}{z'}(1 - \tfrac{1}{2}\alpha_0^2)(1 - \tfrac{1}{2}\mu_0^2 FO\,\alpha_0^2) = \mu_n \alpha_n F(1 - \tfrac{1}{2}FO\,\sigma_0^2),$$

$$\sqrt{X''^2 + Z''^2} = \mu_n F(1 - \tfrac{1}{2}FO\,\sigma_0^2), \quad \text{as in [31.].}$$

We find therefore, in this way also, namely from the connexion between the equations of an initial and a final ray, given in the last section, or from the *expression for* x'', combined with that for z'', as depending on x' and z', that the additional condition, besides $\lambda_0 = 0$, or $M = 0$, necessary in order that, for parallel oblique initial rays, the final rays in the diametral plane may converge to one common focus, is

$$\lambda_1 = -\tfrac{1}{2}\mu_n^{-2} F^{-2}, \quad \text{or} \quad N = 0, \quad \text{as before.}$$

Nor have we in this last method of investigating these conditions $M = 0$, $N = 0$, employed the function T any farther than as in deducing the expressions of [37.] for $\lambda_0, \lambda_1, \lambda_2, \lambda_3$; which might, however, have been deduced by other methods, for example, by that which Herschel uses.

The theory of my object glass, therefore, (at least so far as indiametral rays are concerned,) and the fundamental equations which construct it, might have been deduced, although less elegantly, without the introduction of my characteristic function T.

[44.] (*Foci for oblique rays.*)

Without differentiating the EQUATION OF THE FINAL RAY, given in [42.], let us put it under the form

$$0=\left\{X''+\frac{Z''}{\mu_n}\frac{\mu_0 x'}{z'}\left(1+\lambda_3\left(\frac{\mu_0 x'}{z'}\right)^2\right)\right\}+x'\left\{-1+\frac{Z''}{\mu_n}\left(F^{-1}+\lambda_2\left(\frac{\mu_0 x'}{z'}\right)^2\right)+\frac{X''}{2}\frac{\mu_0 x'}{z'}\left(\lambda_2-\tfrac{3}{2}\mu_n^{-2}F^{-1}\right)\right\}$$
$$+\frac{x'^2}{4}\left\{\left(X''+4\frac{Z''}{\mu_n}\frac{\mu_0 x'}{z'}\right)\lambda_1-\tfrac{3}{2}X''\mu_n^{-2}F^{-2}\right\}+x'^3\frac{Z''}{\mu_n}\lambda_0;$$

and then we see that if it is to be satisfied for given values of $\frac{x'}{z'}$, X'', Z'', while x' remains undetermined, we must have

$$\frac{Z''}{\mu_n F}=1-\tfrac{1}{2}\left(F\lambda_2+\tfrac{3}{2}\mu_n^{-2}\right)\left(\frac{\mu_0 x'}{z'}\right)^2; \quad \frac{X''}{F}=-\frac{\mu_0 x'}{z'}\left\{1+\left(\lambda_3-\tfrac{1}{2}F\lambda_2-\tfrac{3}{4}\mu_n^{-2}\right)\left(\frac{\mu_0 x'}{z'}\right)^2\right\};$$

$$\lambda_1=-\tfrac{1}{2}\mu_n^{-2}F^{-2}; \quad \lambda_0=0.$$

And if we now employ the expressions, (see [37.]),

$$\lambda_0=M; \quad \lambda_1=2N-\tfrac{1}{2}\mu_n^{-2}F^{-2}; \quad \lambda_2=O-\tfrac{1}{2}\mu_n^{-2}F^{-1}; \quad \lambda_3=\tfrac{1}{2}\left(\mu_n^{-2}-\mu_0^{-2}\right);$$

we arrive (as before) at the conditions

$$M=0, \quad N=0,$$

and at the coordinates

$$X''=\mu_n F\alpha_n\left(1-\tfrac{1}{2}\mu_0^2 FO\alpha_0^2\right),$$
$$Z''=\mu_n F\left(1-\tfrac{1}{2}\alpha_n^2\right)\left(1-\tfrac{1}{2}\mu_0^2 FO\alpha_0^2\right),$$

in which

$$\alpha_0=-\frac{x'}{z'}\left(1-\frac{1}{2}\frac{x'^2}{z'^2}\right), \quad \alpha_n=\frac{\mu_0\alpha_0}{\mu_n};$$

therefore

$$\sqrt{X''^2+Z''^2}=\mu_n F\left(1-\tfrac{1}{2}\mu_0^2 FO\alpha_0^2\right),$$

as before. (See [31.].)

(Feb. 24th.) The *equation of the final ray*, given at the top of this section, is accurate to the 3rd dimension inclusive; and if the initial rays be parallel to each other, although oblique to the axis, then $\frac{x'}{z'}$ is constant, but x' is variable. Now x' is the distance of the initial ray from the origin, that is, from the common vertex of the n refracting surfaces, measured upon their common tangent; if then we make $x'=0$, without making $\frac{x'}{z'}=0$, we shall obtain the *equation of the final ray which corresponds to the ray incident at the vertex*, namely:

$$X''=-\frac{Z''}{\mu_n}\frac{\mu_0 x'}{z'}\left(1+\lambda_3\left(\frac{\mu_0 x'}{z'}\right)^2\right).$$

Substituting for λ_3 its value, $\lambda_3 = \frac{1}{2}(\mu_n^{-2} - \mu_0^{-2})$, and making

$$\alpha_0 = -\frac{x'}{z'}\left(1 - \tfrac{1}{2}\left(\frac{x'}{z'}\right)^2\right), \quad \alpha_n = \frac{\mu_0 \alpha_0}{\mu_n},$$

this equation becomes

$$\boxed{X'' = \alpha_n Z'' (1 + \tfrac{1}{2}\alpha_n^2),}$$

as might have been expected, because the ray incident at the vertex emerges there, with that change of direction (if μ_n be different from μ_0) which is expressed by the equation $\alpha_n = \frac{\mu_0 \alpha_0}{\mu_n}$. If we next inquire what is *the intersection of* THIS *final ray, with any other*, corresponding to any value of x' different from zero, we are to suppress the part independent of x' in the equation at the top of this section, and then divide by x', (treating always $\frac{x'}{z'}$ as constant,) and we thus find:

$$0 = -1 + \frac{Z''}{\mu_n}\left(F^{-1} + \lambda_2\left(\frac{\mu_0 x'}{z'}\right)^2\right) + \frac{X''}{2}\frac{\mu_0 x'}{z'}(\lambda_2 - \tfrac{3}{2}\mu_n^{-2}F^{-1})$$

$$+ \frac{x'}{4}\left\{\left(X'' + 4\frac{Z''}{\mu_n}\frac{\mu_0 x'}{z'}\right)\lambda_1 - \tfrac{3}{2}X''\mu_n^{-2}F^{-2}\right\} + x'^2\frac{Z''}{\mu_n}\lambda_0;$$

that is, in the same order of approximation, (neglecting *here* terms of 3rd dimension,)

$$1 = \frac{Z''}{\mu_n}\left\{F^{-1} + \lambda_2\left(\frac{\mu_0 x'}{z'}\right)^2 - \tfrac{1}{2}\left(\frac{\mu_0 x'}{z'}\right)^2(\lambda_2 - \tfrac{3}{2}\mu_n^{-2}F^{-1})\right\} + \frac{3x'}{4}\frac{\mu_0 x'}{z'}\frac{Z''}{\mu_n}(\lambda_1 + \tfrac{1}{2}\mu_n^{-2}F^{-2}) + \lambda_0 x'^2\frac{Z''}{\mu_n};$$

that is,

$$\boxed{\frac{\mu_n}{Z''} = \frac{1}{F} + \tfrac{1}{2}(\lambda_2 + \tfrac{3}{2}\mu_n^{-2}F^{-1})\left(\frac{\mu_0 x'}{z'}\right)^2 + \frac{3x'}{4}(\lambda_1 + \tfrac{1}{2}\mu_n^{-2}F^{-2})\left(\frac{\mu_0 x'}{z'}\right) + \lambda_0 x'^2.}$$

[45.] (*Foci for oblique rays.*)

If in the next place we seek the intersection of the ray emerging from the vertex, with that which emerges from an infinitely near point, the incident rays having been parallel; we are to make $x' = 0$, $\left(\text{but not } \frac{x'}{z'} = 0,\right)$ in the formula just now given, and we find

$$\boxed{\frac{\mu_n}{Z''} = \frac{1}{F} + \tfrac{1}{2}(\lambda_2 + \tfrac{3}{2}\mu_n^{-2}F^{-1})\left(\frac{\mu_0 x'}{z'}\right)^2,}$$

as a formula which determines the *central focus for oblique rays* in the diametral plane of an instrument, composed of any number of coaxal refracting surfaces of revolution, placed close together *in vacuo** at the origin. This central focus is the point X'', Z'', of which the coordinates are given in the upper half of the preceding section.

* [The restriction of being *in vacuo* is not actually made.]

Let the ordinate of this central focus of oblique rays, or the Z'' determined by the last formula, be called, for a moment, $Z\backslash''$; then, the formula at the end of the preceding section becomes:

$$\frac{\mu_n}{Z''} = \frac{\mu_n}{Z\backslash''} + \frac{3x'}{4}(\lambda_1 + \tfrac{1}{2}\mu_n^{-2}F^{-2})\left(\frac{\mu_0 x'}{z'}\right) + \lambda_0 x'^2;$$

and gives

$$Z'' = Z\backslash'' -$$

so that there are in general *two kinds of indiametral aberration*, for parallel incident rays; one kind depending solely on the semiaperture x', and answering to the term $\lambda_0 x'^2$ in the expression for $\frac{\mu_n}{Z''}$; the other kind depending partly on that semiaperture x', and partly on the obliquity or inclination of the incident rays, of which the tangent is $-\frac{x'}{z'}$. If *both* these aberrations, or parts of aberration, are to vanish, we must have not only $\lambda_0 = 0$, which is the most usual and recognised condition, but also

$$\lambda_1 + \tfrac{1}{2}\mu_n^{-2}F^{-2} = 0;$$

and thus are still again conducted, by a slightly different path, to the same two conditions already several times (in this book) assigned by me, for the construction of an aplanatic object glass.

If the *most* usual condition of aplanaticity, namely $\lambda_0 = 0$, be satisfied, but not mine; or if, (which is indeed a case of the last supposition,) the two conditions of Herschel are satisfied, namely $\lambda_0 = 0$, $\lambda_1 = 0$; then the *longitudinal* aberration for oblique parallel rays involves a term proportional to the semiaperture x', and changing sign therewith; so that the corresponding term of *lateral* aberration is of *one constant sign, independent of the sign of the semiaperture* x', being indeed proportional to $x'^2\left(\frac{x'}{z'}\right)$, while $\frac{x'}{z'}$ depends only on the inclination of the initial rays to the axis of the instrument. In fact when Z'' has the value above assigned for the central focus of oblique rays, the aberration of X'', measured from the ray which issues at the vertex, is, if $\lambda_0 = 0$, expressed by the formula:

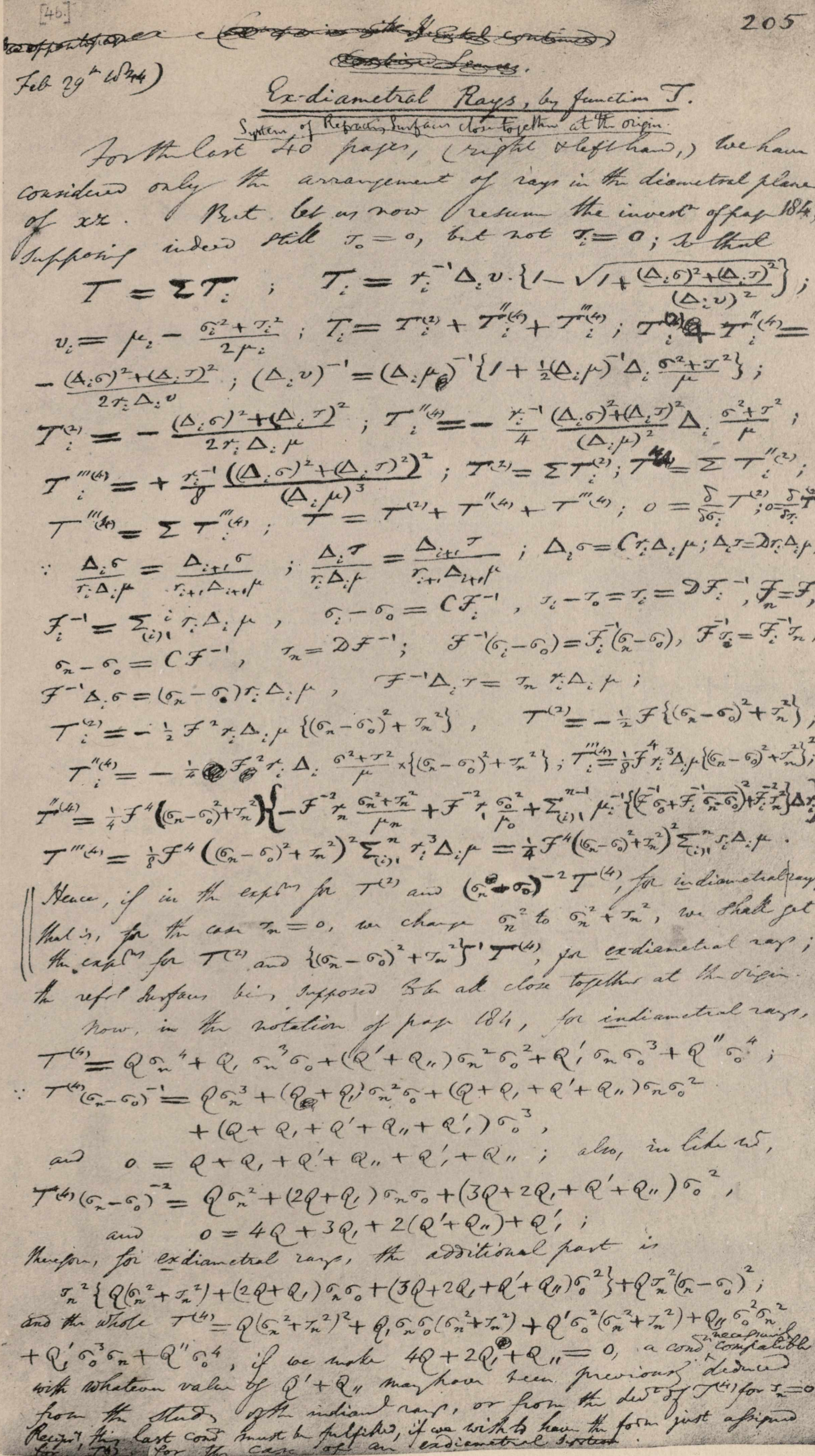
$$\boxed{X'' + \frac{Z''}{\mu_n}\frac{\mu_0 x'}{z'}\left(1 - \tfrac{1}{2}\frac{x'^2}{z'^2}\right)\left(1 + \tfrac{1}{2}\frac{\mu_0^2}{\mu_n^2}\frac{x'^2}{z'^2}\right) = -\tfrac{3}{4}\frac{\mu_0 x'}{z'}F(\lambda_1 + \tfrac{1}{2}\mu_n^{-2}F^{-2})x'^2.}$$

[46.] *Ex-diametral rays, by function T.*

System of Refracting Surfaces, close together at the origin.

(Feb. 29th, 1844.) For the last 40 pages, (right and left hand,)* we have considered only the arrangement of rays in the diametral plane of xz. But let us now resume the investigation

* [That is, of the note book. Seven of these pages, devoted to a comparison of Hamilton's results with those of Herschel on spherical aberration, are not reproduced here, and one page, headed "Foci for oblique rays. Caustic curve.", is blank. The investigations on rays in a diametral plane, as here reproduced, are contained in sections [14.] to [45.], inclusive.]



[46.] 205

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Feb 29th 1844)

Ex-diametral Rays, by function T.

System of Refracting Surfaces close together at the origin.

For the last 40 pages, (right & left hand,) we have considered only the arrangement of rays in the diametral plane of xz. But let us now resume the investn of page 184, supposing indeed still $\tau_0 = 0$, but not $\tau_i = 0$; so that

$$T = \Sigma T_i \; ; \quad T_i = r_i^{-1} \Delta_i v \cdot \left\{ 1 - \sqrt{1 + \frac{(\Delta_i \sigma)^2 + (\Delta_i \tau)^2}{(\Delta_i v)^2}} \right\};$$

$$v_i = \mu_i - \frac{\sigma_i^2 + \tau_i^2}{2\mu_i}; \quad T_i = T_i^{(2)} + T_i^{''(4)} + T_i^{'''(4)}; \quad T_i^{(2)} + T_i^{''(4)} = - \frac{(\Delta_i\sigma)^2 + (\Delta_i\tau)^2}{2 r_i \Delta_i v}; \quad (\Delta_i v)^{-1} = (\Delta_i \mu)^{-1} \left\{ 1 + \tfrac{1}{2} (\Delta_i \mu)^{-1} \Delta_i \frac{\sigma^2 + \tau^2}{\mu} \right\};$$

$$T_i^{(2)} = - \frac{(\Delta_i\sigma)^2 + (\Delta_i\tau)^2}{2 r_i \Delta_i \mu}; \quad T_i^{''(4)} = - \frac{r_i^{-1}}{4} \frac{(\Delta_i\sigma)^2 + (\Delta_i\tau)^2}{(\Delta_i\mu)^2} \Delta_i \frac{\sigma^2+\tau^2}{\mu};$$

$$T_i^{'''(4)} = + \frac{r_i^{-1}}{8} \frac{\left((\Delta_i\sigma)^2 + (\Delta_i\tau)^2\right)^2}{(\Delta_i\mu)^3}; \quad T^{(2)} = \Sigma T_i^{(2)}; \quad T^{''(4)} = \Sigma T_i^{''(4)};$$

$$T^{'''(4)} = \Sigma T_i^{'''(4)}; \quad T = T^{(2)} + T^{''(4)} + T^{'''(4)}; \quad 0 = \frac{\delta}{\delta\sigma_i} T^{(2)}; \quad 0 = \frac{\delta T^{(2)}}{\delta\tau_i};$$

$$\therefore \frac{\Delta_i\sigma}{r_i \Delta_i \mu} = \frac{\Delta_{i+1}\sigma}{r_{i+1}\Delta_{i+1}\mu}; \quad \frac{\Delta_i\tau}{r_i\Delta_i\mu} = \frac{\Delta_{i+1}\tau}{r_{i+1}\Delta_{i+1}\mu}; \quad \Delta_i\sigma = C r_i \Delta_i \mu; \quad \Delta_i \tau = D r_i \Delta_i \mu;$$

$$F_i^{-1} = \Sigma_{(i)1}^{i} r_i \Delta_i \mu, \quad \sigma_i - \sigma_0 = C F_i^{-1}, \quad \tau_i - \tau_0 = \tau_i = D F_i^{-1}, \quad F_n = F,$$

$$\sigma_n - \sigma_0 = C F^{-1}, \quad \tau_n = D F^{-1}; \quad F^{-1}(\sigma_i - \sigma_0) = F_i^{-1}(\sigma_n - \sigma_0), \quad F^{-1}\tau_i = F_i^{-1}\tau_n;$$

$$F^{-1}\Delta_i\sigma = (\sigma_n - \sigma_0) r_i \Delta_i \mu, \quad F^{-1}\Delta_i\tau = \tau_n r_i \Delta_i \mu;$$

$$T_i^{(2)} = -\tfrac{1}{2} F^2 r_i \Delta_i \mu \left\{(\sigma_n - \sigma_0)^2 + \tau_n^2\right\}, \quad T^{(2)} = -\tfrac{1}{2} F \left\{(\sigma_n - \sigma_0)^2 + \tau_n^2\right\};$$

$$T_i^{''(4)} = -\tfrac{1}{4} F_i^2 r_i \Delta_i \frac{\sigma^2 + \tau^2}{\mu} \times \left\{(\sigma_n - \sigma_0)^2 + \tau_n^2\right\}; \quad T_i^{'''(4)} = \tfrac{1}{8} F^4 r_i^3 \Delta_i \mu \left\{(\sigma_n - \sigma_0)^2 + \tau_n^2\right\}^2;$$

$$T^{''(4)} = \tfrac{1}{4} F^4 \left((\sigma_n - \sigma_0)^2 + \tau_n^2\right) \left\{ -F^{-2} r_n \frac{\sigma_n^2 + \tau_n^2}{\mu_n} + F^{-2} r_1 \frac{\sigma_0^2}{\mu_0} + \Sigma_{(i)1}^{n-1} \mu_i^{-1} \left\{ \left(F^{-1}\sigma_0 + F_i^{-1}(\sigma_n - \sigma_0)\right)^2 + F_i^{-2}\tau_n^2 \right\} \Delta r_i \right\};$$

$$T^{'''(4)} = \tfrac{1}{8} F^4 \left((\sigma_n - \sigma_0)^2 + \tau_n^2\right)^2 \Sigma_{(i)1}^{n} r_i^3 \Delta_i \mu = \tfrac{1}{4} F^4 \left((\sigma_n - \sigma_0)^2 + \tau_n^2\right)^2 \Sigma_{(i)1}^{n} s_i \Delta_i \mu.$$

Hence, if in the expns for $T^{(2)}$ and $(\sigma_n - \sigma_0)^{-2} T^{(4)}$, for indiametral rays, that is, for the case $\tau_n = 0$, we change σ_n^2 to $\sigma_n^2 + \tau_n^2$, we shall get the expns for $T^{(2)}$ and $\{(\sigma_n - \sigma_0)^2 + \tau_n^2\}^{-1} T^{(4)}$, for exdiametral rays; the refg surfaces being supposed to be all close together at the origin.

Now, in the notation of page 184, for indiametral rays,

$$T^{(4)} = Q\sigma_n^4 + Q_1 \sigma_n^3 \sigma_0 + (Q' + Q_{11})\sigma_n^2\sigma_0^2 + Q'_1 \sigma_n \sigma_0^3 + Q''\sigma_0^4;$$

$$\therefore T^{(4)}(\sigma_n - \sigma_0)^{-1} = Q\sigma_n^3 + (Q + Q_1)\sigma_n^2\sigma_0 + (Q + Q_1 + Q' + Q_{11})\sigma_n\sigma_0^2 + (Q + Q_1 + Q' + Q_{11} + Q'_1)\sigma_0^3,$$

and $0 = Q + Q_1 + Q' + Q_{11} + Q'_1 + Q''$; also, in like w,

$$T^{(4)}(\sigma_n - \sigma_0)^{-2} = Q\sigma_n^2 + (2Q + Q_1)\sigma_n\sigma_0 + (3Q + 2Q_1 + Q' + Q_{11})\sigma_0^2,$$

and $0 = 4Q + 3Q_1 + 2(Q' + Q_{11}) + Q'_1$;

therefore, for exdiametral rays, the additional part is

$$\tau_n^2 \left\{ Q(\sigma_n^2 + \tau_n^2) + (2Q + Q_1)\sigma_n\sigma_0 + (3Q + 2Q_1 + Q' + Q_{11})\sigma_0^2 \right\} + Q\tau_n^2(\sigma_n - \sigma_0)^2;$$

and the whole $T^{(4)} = Q(\sigma_n^2 + \tau_n^2)^2 + Q_1\sigma_n\sigma_0(\sigma_n^2 + \tau_n^2) + Q'\sigma_0^2(\sigma_n^2 + \tau_n^2) + Q_{11}\sigma_0^2\sigma_n^2 + Q'_1\sigma_0^3\sigma_n + Q''\sigma_0^4$, if we make $4Q + 2Q_1 + Q_{11} = 0$, a condn necessarily compatible with whatever value of $Q' + Q_{11}$ may have been previously deduced from the study of the indiaml rays, or from the devt of $T^{(4)}$ for $\tau_n = 0$. Recipry, this last condn must be fulfilled, if we wish to have the form just assigned for $T^{(4)}$ for the case of an exdiametral system.

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(See pp. 454–6)

of [**33.**], supposing indeed still $\tau_0=0$,* but not $\tau_i=0$; so that

$$T=\Sigma T_i;\quad T_i=r_i^{-1}\Delta_i\upsilon\,.\left\{1-\sqrt{1+\frac{(\Delta_i\sigma)^2+(\Delta_i\tau)^2}{(\Delta_i\upsilon)^2}}\right\};$$

$$\upsilon_i=\mu_i-\frac{\sigma_i^2+\tau_i^2}{2\mu_i};$$

$$T_i=T_i^{(2)}+T''^{(4)}_i+T'''^{(4)}_i;\quad T_i^{(2)}+T''^{(4)}_i=-\frac{(\Delta_i\sigma)^2+(\Delta_i\tau)^2}{2r_i\Delta_i\upsilon};$$

$$(\Delta_i\upsilon)^{-1}=(\Delta_i\mu)^{-1}\left\{1+\tfrac{1}{2}(\Delta_i\mu)^{-1}\Delta_i\frac{\sigma^2+\tau^2}{\mu}\right\};$$

$$T_i^{(2)}=-\frac{(\Delta_i\sigma)^2+(\Delta_i\tau)^2}{2r_i\Delta_i\mu};\quad T''^{(4)}_i=-\frac{r_i^{-1}}{4}\frac{(\Delta_i\sigma)^2+(\Delta_i\tau)^2}{(\Delta_i\mu)^2}\Delta_i\frac{\sigma^2+\tau^2}{\mu};\quad T'''^{(4)}_i=\frac{r_i^{-1}}{8}\frac{((\Delta_i\sigma)^2+(\Delta_i\tau)^2)^2}{(\Delta_i\mu)^3};$$

$$T^{(2)}=\Sigma T_i^{(2)};\quad T''^{(4)}=\Sigma T''^{(4)}_i;\quad T'''^{(4)}=\Sigma T'''^{(4)}_i;\quad T=T^{(2)}+T''^{(4)}+T'''^{(4)};$$

$$0=\frac{\delta}{\delta\sigma_i}T^{(2)};\quad 0=\frac{\delta}{\delta\tau_i}T^{(2)};$$

$$\because\ \frac{\Delta_i\sigma}{r_i\Delta_i\mu}=\frac{\Delta_{i+1}\sigma}{r_{i+1}\Delta_{i+1}\mu};\quad \frac{\Delta_i\tau}{r_i\Delta_i\mu}=\frac{\Delta_{i+1}\tau}{r_{i+1}\Delta_{i+1}\mu};\quad \Delta_i\sigma=Cr_i\Delta_i\mu;\quad \Delta_i\tau=Dr_i\Delta_i\mu;$$

$$F_i^{-1}=\Sigma_{(i)}{}_1^i r_i\Delta_i\mu,\quad \sigma_i-\sigma_0=CF_i^{-1},\quad \tau_i-\tau_0=\tau_i=DF_i^{-1},\quad F_n=F,$$

$$\sigma_n-\sigma_0=CF^{-1},\quad \tau_n=DF^{-1};\quad F^{-1}(\sigma_i-\sigma_0)=F_i^{-1}(\sigma_n-\sigma_0),\quad F^{-1}\tau_i=F_i^{-1}\tau_n;$$

$$F^{-1}\Delta_i\sigma=(\sigma_n-\sigma_0)\,r_i\Delta_i\mu,\quad F^{-1}\Delta_i\tau=\tau_n r_i\Delta_i\mu;$$

$$T_i^{(2)}=-\tfrac{1}{2}F^2r_i\Delta_i\mu\,\{(\sigma_n-\sigma_0)^2+\tau_n^2\},\quad T^{(2)}=-\tfrac{1}{2}F\,\{(\sigma_n-\sigma_0)^2+\tau_n^2\};$$

$$T''^{(4)}_i=-\tfrac{1}{4}F^2r_i\Delta_i\frac{\sigma^2+\tau^2}{\mu}\,.\,\{(\sigma_n-\sigma_0)^2+\tau_n^2\},\quad T'''^{(4)}_i=\tfrac{1}{8}F^4r_i^3\Delta_i\mu\,\{(\sigma_n-\sigma_0)^2+\tau_n^2\}^2;$$

$$T''^{(4)}=\tfrac{1}{4}F^4\left((\sigma_n-\sigma_0)^2+\tau_n^2\right)\left\{-F^{-2}r_n\frac{\sigma_n^2+\tau_n^2}{\mu_n}+F^{-2}r_1\frac{\sigma_0^2}{\mu_0}+\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}\{(F^{-1}\sigma_0+F_i^{-1}\overline{\sigma_n-\sigma_0})^2+F_i^{-2}\tau_n^2\}\,\Delta r_i\right\};$$

$$T'''^{(4)}=\tfrac{1}{8}F^4\left((\sigma_n-\sigma_0)^2+\tau_n^2\right)^2\Sigma_{(i)}{}_1^n r_i^3\Delta_i\mu=\tfrac{1}{4}F^4\left((\sigma_n-\sigma_0)^2+\tau_n^2\right)^2\Sigma_{(i)}{}_1^n s_i\Delta_i\mu.\dagger$$

Hence, if in the expressions [in [**33.**]] for $T^{(2)}$ and $(\sigma_n-\sigma_0)^{-2}T^{(4)}$, for *indiametral* rays, that is, for the case $\tau_n=0$, we change σ_n^2 to $\sigma_n^2+\tau_n^2$, we shall get the expressions for $T^{(2)}$ and

$$\{(\sigma_n-\sigma_0)^2+\tau_n^2\}^{-1}T^{(4)},$$

for *exdiametral* rays; the refracting surfaces being supposed to be all close together at the origin.

Now, in the notation of [**33.**], for *indiametral* rays,

$$T^{(4)}=Q\sigma_n^4+Q_{,}\sigma_n^3\sigma_0+(Q'+Q_{,,})\,\sigma_n^2\sigma_0^2+Q_{,}'\sigma_n\sigma_0^3+Q''\sigma_0^4;$$

therefore

$$T^{(4)}(\sigma_n-\sigma_0)^{-1}=Q\sigma_n^3+(Q+Q_{,})\,\sigma_n^2\sigma_0+(Q+Q_{,}+Q'+Q_{,,})\,\sigma_n\sigma_0^2+(Q+Q_{,}+Q'+Q_{,,}+Q_{,}')\,\sigma_0^3,$$

* [The incident rays are assumed to be parallel to the plane $y=0$.]

† [This latter expression applies to surfaces of revolution in general; the former to spheres only.]

and*

$$0 = Q + Q_{,} + Q' + Q_{,,} + Q_{,}' + Q'';$$

also, in like manner,

$$T^{(4)} (\sigma_n - \sigma_0)^{-2} = Q\sigma_n^2 + (2Q + Q_{,}) \sigma_n \sigma_0 + (3Q + 2Q_{,} + Q' + Q_{,,}) \sigma_0^2,$$

and*

$$0 = 4Q + 3Q_{,} + 2 (Q' + Q_{,,}) + Q_{,}';$$

therefore, for *exdiametral* rays, the additional part is

$$\tau_n^2 \{Q (\sigma_n^2 + \tau_n^2) + (2Q + Q_{,}) \sigma_n \sigma_0 + (3Q + 2Q_{,} + Q' + Q_{,,}) \sigma_0^2\} + Q\tau_n^2 (\sigma_n - \sigma_0)^2;$$

and the whole

$$T^{(4)} = Q (\sigma_n^2 + \tau_n^2)^2 + Q_{,} \sigma_n \sigma_0 (\sigma_n^2 + \tau_n^2) + Q' \sigma_0^2 (\sigma_n^2 + \tau_n^2) + Q_{,,} \sigma_0^2 \sigma_n^2 + Q_{,}' \sigma_0^3 \sigma_n + Q'' \sigma_0^4,$$

if we make*

$$4Q + 2Q_{,} + Q_{,,} = 0,$$

a condition necessarily compatible with whatever value of $Q' + Q_{,,}$ may have been previously deduced from the study of the indiametral rays, or from the development of $T^{(4)}$ for $\tau_n = 0$. Reciprocally, this last condition must be fulfilled, if we wish to have the form just assigned for $T^{(4)}$, for the case of an exdiametral system.

[47.] The three conditions*

$$Q + Q_{,} + Q' + Q_{,,} + Q_{,}' + Q'' = 0,$$
$$4Q + 3Q_{,} + 2 (Q' + Q_{,,}) + Q_{,}' = 0,$$
$$4Q + 2Q_{,} + Q_{,,} = 0,$$

are doubtless those required for the divisibility of the expression

$$T^{(4)} = Q (\sigma^2 + \tau^2)^2 + Q_{,} \sigma_0 \sigma (\sigma^2 + \tau^2) + Q' \sigma_0^2 (\sigma^2 + \tau^2) + Q_{,,} \sigma_0^2 \sigma^2 + Q_{,}' \sigma_0^3 \sigma + Q'' \sigma_0^4$$

by $\sigma^2 + \tau^2 - 2\sigma_0 \sigma + \sigma_0^2$. It may be instructive to verify this divisibility, and to assign an expression for the quotient.

Retaining Q, $Q_{,}$, Q', and expressing $Q_{,,}$, $Q_{,}'$, Q'' by these, we have

$$Q_{,,} = -4Q - 2Q_{,};$$
$$Q_{,}' = 4Q + Q_{,} - 2Q';$$
$$Q'' = -Q + Q'.$$

But

$$(\sigma^2 + \tau^2)^2 - 4\sigma_0^2 \sigma^2 + 4\sigma_0^3 \sigma - \sigma_0^4 = (\sigma^2 + \tau^2 - 2\sigma_0 \sigma + \sigma_0^2)(\sigma^2 + \tau^2 + 2\sigma_0 \sigma - \sigma_0^2);$$

$$\sigma_0 \sigma (\sigma^2 + \tau^2) - 2\sigma_0^2 \sigma^2 + \sigma_0^3 \sigma = (\sigma^2 + \tau^2 - 2\sigma_0 \sigma + \sigma_0^2) \sigma_0 \sigma;$$

$$\sigma_0^2 (\sigma^2 + \tau^2) - 2\sigma_0^3 \sigma + \sigma_0^4 = (\sigma^2 + \tau^2 - 2\sigma_0 \sigma + \sigma_0^2) \sigma_0^2;$$

the division therefore succeeds, and the quotient is

$$Q (\sigma^2 + \tau^2) + (2Q + Q_{,}) \sigma_0 \sigma - (Q - Q') \sigma_0^2.$$

* [These three relations between the coefficients in the general expression for any instrument of revolution,

$$T^{(4)} = Q\epsilon^2 + Q_{,} \epsilon\epsilon_{,} + Q' \epsilon\epsilon' + Q_{,,} \epsilon_{,}^2 + Q_{,}' \epsilon_{,} \epsilon' + Q'' \epsilon'^2,$$

are consequences of the condition that the system is thin and situated at the origin, $T^{(4)}$ being then divisible by $\epsilon - 2\epsilon_{,} + \epsilon'$; see Appendix, Note 26, p. 511.]

Comparing then this last expression with that given by the last section for

$$(\sigma^2+\tau^2-2\sigma_0\sigma+\sigma_0^2)^{-1}T^{(4)},$$

(in which we have written σ, τ, for σ_n, τ_n,) we find

$$\left\{\begin{aligned} &4F^{-4}Q=-F^{-2}\mu_n^{-1}r_n+\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}F_i^{-2}\Delta r_i+\Sigma_{(i)}{}_1^{n}s_i\Delta_i\mu;\\ &2F^{-4}(2Q+Q_,)=\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}F_i^{-1}(F^{-1}-F_i^{-1})\Delta r_i-\Sigma_{(i)}{}_1^{n}s_i\Delta_i\mu;\\ &4F^{-4}(Q'-Q)=F^{-2}\mu_0^{-1}r_1+\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}(F^{-1}-F_i^{-1})^2\Delta r_i+\Sigma_{(i)}{}_1^{n}s_i\Delta_i\mu;\end{aligned}\right.$$

and consequently

$$2F^{-3}(4Q+Q_,)=-F^{-1}\mu_n^{-1}r_n+\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}F_i^{-1}\Delta r_i;$$
$$2F^{-3}(2Q'+Q_,)=\ F^{-1}\mu_0^{-1}r_1+\Sigma_{(i)}{}_1^{n-1}\mu_i^{-1}(F^{-1}-F_i^{-1})\Delta r_i;$$

therefore finally,

$$4F^{-2}(2Q+Q_,+Q')=-\mu_n^{-1}r_n+\mu_0^{-1}r_1+\Sigma_{i)}{}_1^{n-1}\mu_i^{-1}\Delta r_i=-\Sigma_{(i)}{}_1^{n}r_i\Delta_i\frac{1}{\mu}.$$

[48.]* The equations of a final ray are

$$\left\{\begin{aligned} &x_{n+1}-\frac{\sigma_n}{\mu_n}\left(1+\frac{\sigma_n^2+\tau_n^2}{2\mu_n^2}\right)z_{n+1}+F(\sigma_n-\sigma_0)=\frac{\delta T^{(4)}}{\delta\sigma_n};\\ &y_{n+1}-\frac{\tau_n}{\mu_n}\left(1+\frac{\sigma_n^2+\tau_n^2}{2\mu_n^2}\right)z_{n+1}+F\tau_n=\frac{\delta T^{(4)}}{\delta\tau_n};\end{aligned}\right.$$

in which

$$\left\{\begin{aligned} &\frac{\delta T^{(4)}}{\delta\sigma_n}=4Q\sigma_n(\sigma_n^2+\tau_n^2)+Q_,\sigma_0(3\sigma_n^2+\tau_n^2)+2(Q'+Q_{,,})\sigma_0^2\sigma_n+Q_,'\sigma_0^3;\\ &\frac{\delta T^{(4)}}{\delta\tau_n}=4Q\tau_n(\sigma_n^2+\tau_n^2)+2Q_,\sigma_0\sigma_n\tau_n+2Q'\sigma_0^2\tau_n;\end{aligned}\right.$$

if then we make, as in [33.],

$$4Q=-\tfrac{1}{2}\mu_n^{-2}F,\quad Q_,=0,$$

the *equations of a final ray* will become

$$\boxed{\begin{aligned} &x_{n+1}-F\sigma_0-Q_,'\sigma_0^3=\sigma_n\left(1+\frac{\sigma_n^2+\tau_n^2}{2\mu_n^2}\right)\left\{\frac{z_{n+1}}{\mu_n}-F+2(Q'+Q_{,,})\sigma_0^2\right\};\\ &y_{n+1}=\tau_n\left(1+\frac{\sigma_n^2+\tau_n^2}{2\mu_n^2}\right)\left\{\frac{z_{n+1}}{\mu_n}-F+2Q'\sigma_0^2\right\}.\end{aligned}}$$

Hence, under these conditions, and in the present order of approximation, the final rays all intersect the two following FOCAL LINES:

* [In modern terminology, this section contains a discussion of astigmatism in an instrument of revolution, corrected for spherical aberration and coma by the relations $4Q=-\frac{1}{2}\mu_n^{-2}F$, $Q_,=0$. See also p. 378. The fact that the system is thin does not enter essentially until the last few lines, although of course it is necessary in the case of a thick system to employ different origins in the initial and final media in order to have for $T^{(2)}$ the simple form assigned in [46.]. The origins must in fact be chosen at the principal points (points of unit lateral magnification); the distances of the principal foci beyond and in front of these points will be $\mu_n F$ and $\mu_0 F$.]

Ist. $\quad x_{n+1} = F\sigma_0 + Q_,'\sigma_0^3, \quad z_{n+1} = \mu_n F - 2\mu_n (Q' + Q_{,,})\sigma_0^2;$

IInd. $\quad y_{n+1} = 0, \quad z_{n+1} = \mu_n F - 2\mu_n Q'\sigma_0^2.$

The ordinate z of the IInd, minus that of the Ist, is equal to $2\mu_n Q_{,,}\sigma_0^2$; in which, by preceding section,

$$Q_{,,} = -4Q - 2Q_,;$$

hence, by present section,

$$Q_{,,} = \tfrac{1}{2}\mu_n^{-2} F,$$

and the interval between the two focal lines is therefore equal to

$$\boxed{\mu_n^{-1} F\sigma_0^2,}$$

= distance of IInd beyond Ist.

[49.] *Combination of Two Lenses.*

(*Indiametral Rays.*)

(April 4th, 1844.) By [11.], changing $\epsilon, \epsilon', \epsilon_,, \epsilon_,', \epsilon_{,,}, \epsilon''$, to $\alpha^2, \alpha'^2, \alpha\alpha', \alpha'\alpha'', \alpha\alpha'', \alpha''^2$, we obtain *for any combination of two coaxal lenses of revolution in vacuo, and for indiametral rays,*

$$\boxed{\begin{aligned} T^{(4)} = {} & \tfrac{1}{8}v_1\alpha'^4 + \tfrac{1}{8}(v_3 - v_2)\alpha^4 - \tfrac{1}{8}v_4\alpha''^4 \\ & + \tfrac{1}{8}t_1\mu_1^{-3}R_1^{-4}(r_1\alpha - r_2\alpha')^4 + \tfrac{1}{8}t_2\mu_2^{-3}R_2^{-4}(r_3\alpha'' - r_4\alpha)^4 \\ & + \frac{(r_1\alpha - r_2\alpha')^2}{4\mu_1(\mu_1 - 1)^2 R_1^4}\{r_2(\rho_1\alpha - \alpha')^2 - r_1(\alpha - \rho_2\alpha')^2\} \\ & + \frac{(r_3\alpha'' - r_4\alpha)^2}{4\mu_2(\mu_2 - 1)^2 R_2^4}\{r_4(\rho_3\alpha'' - \alpha)^2 - r_3(\alpha'' - \rho_4\alpha)^2\} \\ & + \frac{r_1\alpha'^2(\alpha - \rho_2\alpha')^2 - r_2\alpha^2(\rho_1\alpha - \alpha')^2}{4(\mu_1 - 1)^2 R_1^2} + \frac{r_3\alpha^2(\alpha'' - \rho_4\alpha)^2 - r_4\alpha''^2(\rho_3\alpha'' - \alpha)^2}{4(\mu_2 - 1)^2 R_2^2} \\ & + \frac{s_1(\alpha - \rho_2\alpha')^4 - s_2(\rho_1\alpha - \alpha')^4}{4(\mu_1 - 1)^3 R_1^4} + \frac{s_3(\alpha'' - \rho_4\alpha)^4 - s_4(\rho_3\alpha'' - \alpha)^4}{4(\mu_2 - 1)^3 R_2^4}; \end{aligned}}$$

in which, v_1, v_2, v_3, v_4 are the ordinates of the four vertices; r_1, r_2, r_3, r_4 the four curvatures, positive when surfaces are convex to incident light; s_1, s_2, s_3, s_4 coefficients of $\left(\frac{x^2}{2}\right)^2$ in the developments of z (each y being zero); μ_1, μ_2 indices of the two lenses; $\alpha', \alpha, \alpha''$ inclinations of initial, intermediate, and final rays (each *in vacuo*) to the axis of the combination; t_1, t_2, thicknesses, so that $t_1 = v_2 - v_1$, $t_2 = v_4 - v_3$;

$$R_1 = r_1 - r_2 + (1 - \mu_1^{-1})r_1r_2t_1; \quad R_2 = r_3 - r_4 + (1 - \mu_2^{-1})r_3r_4t_2;$$

$$\rho_1 = 1 - r_1t_1 + \mu_1^{-1}r_1t_1 = 1 - (1 - \mu_1^{-1})r_1t_1; \quad \rho_2 = 1 + r_2t_1 - \mu_1^{-1}r_2t_1 = 1 + (1 - \mu_1^{-1})r_2t_1;$$

$$\rho_3 = 1 - r_3t_2 + \mu_2^{-1}r_3t_2 = 1 - (1 - \mu_2^{-1})r_3t_2; \quad \rho_4 = 1 + r_4t_2 - \mu_2^{-1}r_4t_2 = 1 + (1 - \mu_2^{-1})r_4t_2;$$

$$\alpha = f'\alpha' + f''\alpha'';$$

$$f' = F(\mu_2 - 1) R_2; \quad f'' = F(\mu_1 - 1) R_1;$$

$$F^{-1} = (\mu_1 - 1) R_1 + (\mu_2 - 1) R_2 + \lambda (\mu_1 - 1)(\mu_2 - 1) R_1 R_2;$$

$$\lambda = v_2 - v_3 - \frac{r_1 t_1}{\mu_1 R_1} + \frac{r_4 t_2}{\mu_2 R_2};$$

$$T^{(4)} = Q\alpha''^4 + Q_{,}\alpha'\alpha''^3 + (Q' + Q_{,,})\alpha'^2\alpha''^2 + Q_{,}'\alpha'^3\alpha'' + Q''\alpha'^4.$$

Hence,

$$\boxed{\begin{aligned} Q = {} & \tfrac{1}{8}(v_3 - v_2) f''^4 - \tfrac{1}{8} v_4 + \tfrac{1}{8} t_1 \mu_1^{-3} R_1^{-4} r_1^4 f''^4 + \tfrac{1}{8} t_2 \mu_2^{-3} R_2^{-4} (r_3 - r_4 f'')^4 \\ & + \frac{r_1^2 (r_2 \rho_1^2 - r_1) f''^4}{4\mu_1 (\mu_1 - 1)^2 R_1^4} + \frac{(r_3 - r_4 f'')^2 \{r_4 (\rho_3 - f'')^2 - r_3 (1 - \rho_4 f'')^2\}}{4\mu_2 (\mu_2 - 1)^2 R_2^4} \\ & - \frac{r_2 \rho_1^2 f''^4}{4(\mu_1 - 1)^2 R_1^2} + \frac{r_3 f''^2 (1 - \rho_4 f'')^2 - r_4 (\rho_3 - f'')^2}{4(\mu_2 - 1)^2 R_2^2} + \frac{(s_1 - \rho_1^4 s_2) f''^4}{4(\mu_1 - 1)^3 R_1^4} \\ & + \frac{\{s_3 (1 - \rho_4 f'')^4 - s_4 (\rho_3 - f'')^4\}}{4(\mu_2 - 1)^3 R_2^4}; \end{aligned}}$$

$$\begin{aligned} Q_{,} = {} & \tfrac{1}{2}(v_3 - v_2) f''^3 f' + \tfrac{1}{2} t_1 \mu_1^{-3} R_1^{-4} f''^3 r_1^3 (f' r_1 - r_2) - \tfrac{1}{2} t_2 \mu_2^{-3} R_2^{-4} f' r_4 (r_3 - f'' r_4)^3 \\ & + \frac{r_1 (r_2 \rho_1^2 - r_1) f''^3}{2\mu_1 (\mu_1 - 1)^2 R_1^4} (f' r_1 - r_2) + \frac{r_1^2 f''^3 R_1^{-4}}{2\mu_1 (\mu_1 - 1)^2} (r_2 \rho_1 (f' \rho_1 - 1) - r_1 (f' - \rho_2)) + \&\text{c.} = \end{aligned}$$

the $Q_{,}^{(1)} + Q_{,}^{(2)}$ of the next section. So far the 14 quantities μ_1, μ_2, v_1, v_2, v_3, v_4, r_1, r_2, r_3, r_4 s_1, s_2, s_3, s_4, remain entirely arbitrary; the two component lenses are not necessarily spheric, nor thin, nor close together.

[50.] The first differential coefficient of $T^{(4)}$ with respect to α, is

$$\begin{aligned} & \tfrac{1}{2}(v_3 - v_2)\alpha^3 + \tfrac{1}{2} t_1 \mu_1^{-3} R_1^{-4} r_1 (r_1 \alpha - r_2 \alpha')^3 - \tfrac{1}{2} t_2 \mu_2^{-3} R_2^{-4} r_4 (r_3 \alpha'' - r_4 \alpha)^3 \\ & + \frac{r_1 (r_1 \alpha - r_2 \alpha')}{2\mu_1 (\mu_1 - 1)^2 R_1^4} \{r_2 (\rho_1 \alpha - \alpha')^2 - r_1 (\alpha - \rho_2 \alpha')^2\} \\ & - \frac{r_4 (r_3 \alpha'' - r_4 \alpha)}{2\mu_2 (\mu_2 - 1)^2 R_2^4} \{r_4 (\rho_3 \alpha'' - \alpha)^2 - r_3 (\alpha'' - \rho_4 \alpha)^2\} \\ & + \frac{(r_1 \alpha - r_2 \alpha')^2}{2\mu_1 (\mu_1 - 1)^2 R_1^4} \{r_2 \rho_1 (\rho_1 \alpha - \alpha') - r_1 (\alpha - \rho_2 \alpha')\} \\ & - \frac{(r_3 \alpha'' - r_4 \alpha)^2}{2\mu_2 (\mu_2 - 1)^2 R_2^4} \{r_4 (\rho_3 \alpha'' - \alpha) - r_3 \rho_4 (\alpha'' - \rho_4 \alpha)\} \\ & + \tfrac{1}{2}(\mu_1 - 1)^{-2} R_1^{-2} \{r_1 \alpha'^2 (\alpha - \rho_2 \alpha') - r_2 \alpha (\rho_1 \alpha - \alpha')^2 - r_2 \rho_1 \alpha^2 (\rho_1 \alpha - \alpha')\} \\ & + \tfrac{1}{2}(\mu_2 - 1)^{-2} R_2^{-2} \{r_4 \alpha''^2 (\rho_3 \alpha'' - \alpha) + r_3 \alpha (\alpha'' - \rho_4 \alpha)^2 - r_4 \rho_4 \alpha^2 (\alpha'' - \rho_4 \alpha)\} \\ & + (\mu_1 - 1)^{-3} R_1^{-4} \{s_1 (\alpha - \rho_2 \alpha')^3 - \rho_1 s_2 (\rho_1 \alpha - \alpha')^3\} \\ & + (\mu_2 - 1)^{-3} R_2^{-4} \{s_4 (\rho_3 \alpha'' - \alpha)^3 - \rho_4 s_3 (\alpha'' - \rho_4 \alpha)^3\}; \end{aligned}$$

and if, in this, we make $\alpha'=0$, $\alpha=f''\alpha''$, and then divide by α''^3, and multiply by f', we find, as one part of $Q_,$,

$$Q_,^{(1)}=\tfrac{1}{2}f'\left\{(v_3-v_2)f''^3+t_1\mu_1^{-3}R_1^{-4}r_1^4f''^3-t_2\mu_2^{-3}R_2^{-4}r_4(r_3-f''r_4)^3\right.$$
$$+\frac{r_1^2f''^3(r_2\rho_1^2-r_1)}{\mu_1(\mu_1-1)^2R_1^4}-\frac{r_4(r_3-f''r_4)\{r_4(\rho_3-f'')^2-r_3(1-f''\rho_4)^2\}}{\mu_2(\mu_2-1)^2R_2^4}$$
$$+\frac{r_1^2f''^3(r_2\rho_1^2-r_1)}{\mu_1(\mu_1-1)^2R_1^4}-\frac{(r_3-f''r_4)^2\{r_4(\rho_3-f'')-r_3\rho_4(1-f''\rho_4)\}}{\mu_2(\mu_2-1)^2R_2^4}$$
$$-\frac{2r_2\rho_1^2f''^3}{(\mu_1-1)^2R_1^2}+\frac{r_4(\rho_3-f'')+r_3f''(1-f''\rho_4)^2-r_4\rho_4f''^2(1-f''\rho_4)}{(\mu_2-1)^2R_2^2}$$
$$\left.+\frac{2(s_1-\rho_1^4s_2)f''^3}{(\mu_1-1)^3R_1^4}+2\left(\frac{s_4(\rho_3-f'')^3-s_3\rho_3(1-f''\rho_4)^3}{(\mu_2-1)^3R_2^4}\right)\right\}.$$

The other part of $Q_,$ is to be found by first taking the differential coefficient of $T^{(4)}$ with respect to α', and making $\alpha'=0$, which gives, so far,

$$-\tfrac{1}{2}t_1\mu_1^{-3}R_1^{-4}r_2r_1^3\alpha^3-\frac{r_1r_2(r_2\rho_1^2-r_1)\alpha^3}{2\mu_1(\mu_1-1)^2R_1^4}-\frac{r_1^2\alpha^3(r_2\rho_1-r_1\rho_2)}{2\mu_1(\mu_1-1)^2R_1^4}$$
$$+\frac{r_2\rho_1\alpha^3}{2(\mu_1-1)^2R_1^2}-\frac{(\rho_2s_1-\rho_1^3s_2)\alpha^3}{(\mu_1-1)^3R_1^4};$$

and then, by making $\alpha=f''\alpha''$, and dividing by α''^3, we obtain

$$Q_,^{(2)}=-\tfrac{1}{2}f''^3R_1^{-4}\left\{t_1r_2r_1^3\mu_1^{-3}+\frac{r_1r_2(r_2\rho_1^2-r_1)+r_1^2(r_2\rho_1-r_1\rho_2)}{\mu_1(\mu_1-1)^2}\right.$$
$$\left.-\frac{r_2\rho_1R_1^2}{(\mu_1-1)^2}+\frac{2(\rho_2s_1-\rho_1^3s_2)}{(\mu_1-1)^3}\right\};$$

and finally,*

$$Q_,=Q_,^{(1)}+Q_,^{(2)}.$$

[There follow a few pages devoted to the case in which one of the two lenses is infinitely thin, and in contact with the other; the investigation then ends.]

* [The method employed is the following: write $T^{(4)}=(G(a',a,a''))_{a=f'a'+f''a''}=H(a',a'')$; then

$$Q_,=\frac{1}{a''^3}\left(\frac{\partial H}{\partial a'}\right)_{a'=0}.$$

But $$\frac{\partial H}{\partial a'}=\left(f'\frac{\partial G}{\partial a}+\frac{\partial G}{\partial a'}\right)_{a=f'a'+f''a''},\quad \left(\frac{\partial H}{\partial a'}\right)_{a'=0}=\left(f'\frac{\partial G}{\partial a}+\frac{\partial G}{\partial a'}\right)_{a'=0,\ a=f''a''};$$

therefore $Q_,=Q_,^{(1)}+Q_,^{(2)}$, where

$$Q_,^{(1)}=\frac{f'}{a''^3}\left(\frac{\partial G}{\partial a}\right)_{a'=0,\ a=f''a''},\quad Q_,^{(2)}=\frac{1}{a''^3}\left(\frac{\partial G}{\partial a'}\right)_{a'=0,\ a=f''a''}.]$$

[A method for the computation of the aberration coefficients in the general instrument of revolution, following Hamilton's method and notation, will be found in the Appendix, Note 27, p. 512.]