Part III

DYNAMICS OF LIGHT

XVII.

[ON THE PROPAGATION OF LIGHT IN CRYSTALS*]

[1835–1838.]

[Note Book 40.]

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[*Equations of motion of an attracting or repelling system in vibration about a state of rest or motion.*]

1. Let x, y, z be the rectangular coordinates of any one point A of an attracting or repelling system at the time t; let $x+r\cos\alpha$, $y+r\cos\beta$, $z+r\cos\gamma$ be the coordinates of any other point B of the same system, at the same moment; and let the components of the accelerating or retarding action of B on A be denoted by $m\cos\alpha f(r)$, $m\cos\beta f(r)$, $m\cos\gamma f(r)$, m being a constant factor which may be called the *mass* of the point B, & the function $f(r)$ being positive for the case of attraction and negative for the case of repulsion; while the distance r between the two points is considered as always positive: then the differential equations of the motion of the point A, if we attend only to the action of B upon it, are

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=m\cos\alpha f(r),\\ \frac{d^2y}{dt^2}&=m\cos\beta f(r),\\ \frac{d^2z}{dt^2}&=m\cos\gamma f(r);\end{aligned}\right\}\qquad(1.)$$

and if we take account of the actions of all the points B, C, &c., on the point A, the differential equations of the motion of A may be then denoted as follows,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=S.\,m\cos\alpha f(r),\\ \frac{d^2y}{dt^2}&=S.\,m\cos\beta f(r),\\ \frac{d^2z}{dt^2}&=S.\,m\cos\gamma f(r),\end{aligned}\right\}\qquad(2.)$$

* [See Cauchy, *Œuvres*, II[e] Série, Tome IX, pp. 390–450; I[e] Série, Tome II, pp. 111–177. Also Appendix, Note 11, p. 638. An abstract of this manuscript appears on page 450.]

the sign of summation S extending to all these other points B, C, &c., each of which has its own set of values of m, r, α, β, γ.

If the total number of attracting or repelling points A, B, C, &c. be n & if we attend only to their actions upon each other, there will be then $3n$ simultaneous differential equations of the 2nd order including and analogous to the 3 equations (2.); & the integrals of these $3n$ equations (if known) would express the $3n$ coordinates of the n points of the system as functions of the time t, involving also, besides the n masses, $6n$ arbitrary constants, which might be determined by the initial positions of the points and their initial components of velocities. And for every different supposition which we might make respecting these arbitrary initial circumstances, we should have (in general) a different result respecting the manner of motion of the system, though still consistent with the laws of mutual attraction or repulsion, as expressed by the differential equations.

2. Imagine then that while x, y, z and $x+r\cos\alpha$, $y+r\cos\beta$, $z+r\cos\gamma$ denote as before the coordinates of the points A & B at the time t in one possible state of motion (or rest) of the system, the coordinates of the same two points at the same moment in another possible state of motion of the same system are denoted by $x+\xi$, $y+\eta$, $z+\zeta$ and

$$x+r\cos\alpha+\xi+\Delta\xi,\quad y+r\cos\beta+\eta+\Delta\eta,\quad z+r\cos\gamma+\zeta+\Delta\zeta,$$

respectively; and let $r+\epsilon r$ denote the mutual distance of these two points in this new manner of motion, so that

$$r+\epsilon r=\sqrt{(r\cos\alpha+\Delta\xi)^2+(r\cos\beta+\Delta\eta)^2+(r\cos\gamma+\Delta\zeta)^2}. \tag{3.}$$

Then

$$\frac{r\cos\alpha+\Delta\xi}{r+\epsilon r},\ \frac{r\cos\beta+\Delta\eta}{r+\epsilon r},\ \frac{r\cos\gamma+\Delta\zeta}{r+\epsilon r},\ \text{that is,}\ \frac{\cos\alpha+\frac{\Delta\xi}{r}}{1+\epsilon},\ \frac{\cos\beta+\frac{\Delta\eta}{r}}{1+\epsilon},\ \frac{\cos\gamma+\frac{\Delta\zeta}{r}}{1+\epsilon},$$

will denote the 3 new cosines by which the new quantity $mf(r+\epsilon r)$ is to be multiplied, in order to get the three new components of accelerating or retarding action of B on A; & the differential equations (2.) of the old motion (or rest) of the system will be changed for the new motion, to the following

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{d^2\xi}{dt^2}&=S.m\left(\cos\alpha+\frac{\Delta\xi}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon},\\ \frac{d^2y}{dt^2}+\frac{d^2\eta}{dt^2}&=S.m\left(\cos\beta+\frac{\Delta\eta}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon},\\ \frac{d^2z}{dt^2}+\frac{d^2\zeta}{dt^2}&=S.m\left(\cos\gamma+\frac{\Delta\zeta}{r}\right)\frac{f(r+\epsilon r)}{1+\epsilon}.\end{aligned}\right\} \tag{4.}$$

And subtracting the equations (2.) of the old motion, from the equations (4.) of the new, we find the following equations:

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=S.m\cos\alpha\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\xi}{r}\frac{f(r+\epsilon r)}{1+\epsilon},\\ \frac{d^2\eta}{dt^2}&=S.m\cos\beta\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\eta}{r}\frac{f(r+\epsilon r)}{1+\epsilon},\\ \frac{d^2\zeta}{dt^2}&=S.m\cos\gamma\left(\frac{f(r+\epsilon r)}{1+\epsilon}-f(r)\right)+S.m\frac{\Delta\zeta}{r}\frac{f(r+\epsilon r)}{1+\epsilon},\end{aligned}\right\} \tag{5.}$$

which express, generally and rigorously, the laws of the differences between any two possible motions of any one system of attracting or repelling points: or, in other words, the laws of the dynamically possible deviations from any one dynamically possible manner of motion of an attracting or repelling system.

3. Let us now modify these general and rigorous expressions by introducing the particular & approximate supposition that the *length and direction of the varying line which connects any one moving point of the system with any other differ extremely little at any arbitrary moment t in the second manner of motion of the system from the length and direction of the same line at the same moment in the first manner of motion.* On this hypothesis the quantities $\frac{\Delta\xi}{r}, \frac{\Delta\eta}{r}, \frac{\Delta\zeta}{r}$ and ϵ will be extremely small, whether r be large or small, & we may neglect their squares and products; so that if we denote by $f'(r)$ the first differential coefficient of the function $f(r)$, the equations (5.) will take the simplified forms,

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= S.\, m\cos\alpha\,\{rf'(r) - f(r)\}\,\epsilon + S.\, mf(r)\,\frac{\Delta\xi}{r}, \\ \frac{d^2\eta}{dt^2} &= S.\, m\cos\beta\,\{rf'(r) - f(r)\}\,\epsilon + S.\, mf(r)\,\frac{\Delta\eta}{r}, \\ \frac{d^2\zeta}{dt^2} &= S.\, m\cos\gamma\,\{rf'(r) - f(r)\}\,\epsilon + S.\, mf(r)\,\frac{\Delta\zeta}{r}, \end{aligned}\right\} \tag{6.}$$

in which, by (3.),

$$\epsilon = \cos\alpha\,\frac{\Delta\xi}{r} + \cos\beta\,\frac{\Delta\eta}{r} + \cos\gamma\,\frac{\Delta\zeta}{r}. \tag{7.}$$

Such are the approximate laws of the dynamically possible deviations from one standard manner of motion of an attracting or repelling system, under the condition that the mutual distances of the several points of that system are altered extremely little by those deviations in length and in direction.

4. To illustrate by an example the foregoing general results, we may remark that they extend to the case of the solar system, considered as a system of points of which each attracts every other inversely as the square of the distance. In this case $f(r) = r^{-2}$, and the equations (2.) for the standard manner of motion, which may be supposed to be the actual motion of the system, become $\frac{d^2x}{dt^2} = S.\frac{m\cos\alpha}{r^2}$, $\frac{d^2y}{dt^2} = S.\frac{m\cos\beta}{r^2}$, $\frac{d^2z}{dt^2} = S.\frac{m\cos\gamma}{r^2}$. And if with this standard or actual motion of the solar system we compare any other motion which the same system might possibly have, (consistently with the present values of the masses & with the present law of attraction,) the deviations of the latter motion from the former are connected rigorously by equations of the form (5.)

$$\frac{d^2\xi}{dt^2} = S.\, m\cos\alpha\left(\frac{(r+\epsilon r)^{-2}}{1+\epsilon} - r^{-2}\right) + S.\, m\frac{\Delta\xi}{r}\,\frac{(r+\epsilon r)^{-2}}{1+\epsilon}, \text{ \&c.};$$

& if these deviations be imagined such as to alter very little the length and the direction of the distance of any one planet from any other or from the Sun, they may then be approximately expressed by equations of the form (6.), namely

$$\frac{d^2\xi}{dt^2} = -3S.\, m\cos\alpha\,\frac{\epsilon}{r^2} + S.\,\frac{m\Delta\xi}{r^3}, \text{ \&c.},$$

in which ϵ has the value (7.).

5. Let it next be supposed that *the law of mutual action is such that the system is capable of equilibrium* either exactly, or at least so very nearly that the difference shall produce no sensible effect in the final conclusions from this supposition: and let this possible state of equilibrium be taken as the *standard state*, to which the coordinates x, y, z refer & from which the deviations or displacements ξ, η, ζ are to be measured. Then the equations (6.) will determine approximately the motions of a displaced point of the system; and α, β, γ, r will become independent of the time and will satisfy the following equations of equilibrium,

$$0 = S.\, m \cos \alpha f(r), \quad 0 = S.\, m \cos \beta f(r), \quad 0 = S.\, m \cos \gamma f(r). \tag{8.}$$

More generally, we shall suppose that *the action of any one point of the system on any other is so small and extends sensibly to so small a distance, and that the distribution of these points in space is so symmetric*, as to satisfy, without sensible error, the conditions

$$\left.\begin{array}{l} S.\, m \cos \alpha^{i} \cos \beta^{i'} \cos \gamma^{i''} r^{h} f(r) = 0, \\ S.\, m \cos \alpha^{i} \cos \beta^{i'} \cos \gamma^{i''} r^{h} f'(r) = 0, \end{array}\right\} \tag{9.}$$

in which h, i, i', i'' may denote any positive integers, or zero, *provided that the sum of the exponents* i, i', i'' *of the cosines is an odd number*. That is, we shall suppose that in the state of equilibrium, for every point B with mass m & with coordinates $x + r\cos\alpha$, $y + r\cos\beta$, $z + r\cos\gamma$ which is near enough to influence sensibly the point A with coordinates x, y, z, there is another near point C with an equal mass m and having for coordinates

$$x - r\cos\alpha, \quad y - r\cos\beta, \quad z - r\cos\gamma;$$

or at least that the results deduced from this supposition are not liable to sensible error. It is worth while to observe that when the extent of sensible action is supposed small, the approximate equations of motion of a displaced point (6.) may be obtained from the rigorous equations (5.) by supposing only that *within this small extent of sensible action the mutual distance of any two points changes very little in length and in direction.*

[*Conditions for plane waves.*]

6. The foregoing suppositions being admitted, we shall next inquire *whether under any, and under what conditions the approximate equations of motion of a displaced point* (6.) *are compatible with the uniform propagation of an indefinite series of plane waves, or of displacements which succeed each other periodically in space and time & are the same for all the points situated at any one moment on any one plane parallel to a given plane*; in other words, we shall inquire whether & under what conditions the variable displacements ξ, η, ζ can be supposed to be periodical functions of any one linear function, such as $ux + vy + wz + st + \theta$, of the coordinates of equilibrium x, y, z & the time t, consistently with the approximate equations of motion (6.). But to simplify this inquiry, *which is not into the most general motion possible but only into the conditions of the possibility of a particular motion proposed*, we shall consider at present only the simplest kind of periodical motion; & shall therefore assume, for the laws of displacement to be examined, the following expressions,

$$\left.\begin{array}{l} \xi = \xi' \sin(ux + vy + wz + st + \theta), \\ \eta = \eta' \sin(ux + vy + wz + st + \theta), \\ \zeta = \zeta' \sin(ux + vy + wz + st + \theta). \end{array}\right\} \tag{10.}$$

So that the question now is, whether under *any*, & if so then under *what* conditions, the constants

$\xi', \eta', \zeta', u, v, w, s, \theta$ can be so chosen as to satisfy the differential equations of motion (6.) by the expressions of displacement (10.). If we can discover such conditions, we can then determine the *thickness* (λ) of the wave, & the *slowness* (μ) of its propagation; since these will be respectively equal to the square roots of the expressions $\frac{4\pi^2}{u^2+v^2+w^2}$ and $\frac{u^2+v^2+w^2}{s^2}$.

7. In this investigation, the operation Δ of differencing, by which we pass from the displacement of one to the displacement of another point of the system, affects only the coordinates of equilibrium x, y, z in the expressions of displacement (10.) & not the time t nor the constants $\xi', \eta', \zeta', u, v, w, s, \theta$, which are supposed to be the same for all the points of the system. We are therefore to substitute for $\Delta\xi$ in the equations (7.) and (6.) the value

$$\left.\begin{aligned}\Delta\xi = {} & \xi' \sin(ux+vy+wz+u\Delta x+v\Delta y+w\Delta z+st+\theta) \\ & -\xi' \sin(ux+vy+wz+st+\theta) \\ = {} & -2\xi' \sin(ux+vy+wz+st+\theta)\left(\sin\frac{u\Delta x+v\Delta y+w\Delta z}{2}\right)^2 \\ & +\xi' \cos(ux+vy+wz+st+\theta)\sin(u\Delta x+v\Delta y+w\Delta z) \\ = {} & -2\xi\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2 \\ & +\xi'\cos(ux+vy+wz+st+\theta)\sin(ur\cos\alpha+vr\cos\beta+wr\cos\gamma),\end{aligned}\right\} \quad (11.)$$

because

$$\xi = \xi' \sin(ux+vy+wz+st+\theta)$$

and

$$\Delta x = r\cos\alpha, \quad \Delta y = r\cos\beta, \quad \Delta z = r\cos\gamma; \quad (12.)$$

and similar substitutions are to be made for $\Delta\eta$ and $\Delta\zeta$. The sign of summation S refers to the various values of $m, r, \alpha, \beta, \gamma$ corresponding to the various other points B, C, &c. of the system which attract or repel the point A, whose coordinates when displaced are $x+\xi, y+\eta, z+\zeta$; so that if we attend to the suppositions made in the 5th paragraph and put, for abridgment,

$$\left.\begin{aligned}L &= S.\,2m\{\cos\alpha^2 f'(r)+\sin\alpha^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2, \\ M &= S.\,2m\{\cos\beta^2 f'(r)+\sin\beta^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2, \\ N &= S.\,2m\{\cos\gamma^2 f'(r)+\sin\gamma^2 r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\end{aligned}\right\} \quad (13.)$$

and

$$\left.\begin{aligned}P &= S.\,2m\cos\beta\cos\gamma\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2, \\ Q &= S.\,2m\cos\gamma\cos\alpha\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2, \\ R &= S.\,2m\cos\alpha\cos\beta\{f'(r)-r^{-1}f(r)\}\left(\sin\frac{ur\cos\alpha+vr\cos\beta+wr\cos\gamma}{2}\right)^2,\end{aligned}\right\} \quad (14.)$$

we shall have the three following conditions to be satisfied:

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=-(L\xi+R\eta+Q\zeta),\\ \frac{d^2\eta}{dt^2}&=-(M\eta+P\zeta+R\xi),\\ \frac{d^2\zeta}{dt^2}&=-(N\zeta+Q\xi+P\eta),\end{aligned}\right\}\qquad(15.)$$

or finally, by the supposed expressions of displacement,

$$\left.\begin{aligned}(L-s^2)\,\xi'+R\eta'+Q\zeta'&=0,\\ (M-s^2)\,\eta'+P\zeta'+R\xi'&=0,\\ (N-s^2)\,\zeta'+Q\xi'+P\eta'&=0.\end{aligned}\right\}\qquad(16.)$$

And these are the sought conditions for the possibility of the uniform propagation of a succession of plane waves with displacements of the forms (10.), in a system of attracting or repelling points distributed as supposed above. They do not at all restrict the constant θ but they give three relations between the constants u, v, w, s and those two other constants which express the ratios of ξ', η', ζ'.

[*Values of constants for an infinitesimally strained cubic lattice.*]

8. To simplify these conditions, we shall now introduce the additional supposition of an *equality between the quantities m, and nearly cubical arrangement of the attracting or repelling points* in their position of equilibrium, which differs from a cubical arrangement only by *three small and* (*in general*) *unequal dilatations or contractions in three rectangular directions*: in such a manner that in the expressions (13.), (14.) we may put

$$\left.\begin{aligned}r\cos\alpha&=(1+a_,)\,r_,\cos\alpha_,,\\ r\cos\beta&=(1+b_,)\,r_,\cos\beta_,,\\ r\cos\gamma&=(1+c_,)\,r_,\cos\gamma_,,\end{aligned}\right\}\qquad(17.)$$

$a_,, b_,, c_,$ being three small constant coefficients of dilatation or contraction & $r_,, \alpha_,, \beta_,, \gamma_,$ being the quantities analogous to r, α, β, γ in that exactly cubical arrangement of points from which the arrangement of the system, in the state of equilibrium considered lately, is now supposed to differ little. On this hypothesis, if we neglect all terms above the second dimension with respect to the small dilatations or contractions $a_,, b_,, c_,$, the accurate expression

$$r=r_,\sqrt{(1+a_,)^2\cos\alpha_,^2+(1+b_,)^2\cos\beta_,^2+(1+c_,)^2\cos\gamma_,^2}\qquad(18.)$$

deduced from the equations (17.) will give approximately

$$r=r_,\{1+a_,\cos\alpha_,^2+b_,\cos\beta_,^2+c_,\cos\gamma_,^2+\tfrac{1}{2}(a_,-b_,)^2\cos\alpha_,^2\cos\beta_,^2+\tfrac{1}{2}(b_,-c_,)^2\cos\beta_,^2\cos\gamma_,^2$$
$$+\tfrac{1}{2}(c_,-a_,)^2\cos\gamma_,^2\cos\alpha_,^2\};\qquad(19.)$$

or simply,

$$r=r_,(1+a_,\cos\alpha_,^2+b_,\cos\beta_,^2+c_,\cos\gamma_,^2),\qquad(20.)$$

if we neglect even the squares and products of $a_,, b_,, c_,$: and in this latter order of approximation, the equations (17.) will also give

$$\left.\begin{aligned}\cos\alpha&=\cos\alpha_,(1+a_,\sin\alpha_,^2-b_,\cos\beta_,^2-c_,\cos\gamma_,^2),\\ \cos\beta&=\cos\beta_,(1+b_,\sin\beta_,^2-c_,\cos\gamma_,^2-a_,\cos\alpha_,^2),\\ \cos\gamma&=\cos\gamma_,(1+c_,\sin\gamma_,^2-a_,\cos\alpha_,^2-b_,\cos\beta_,^2).\end{aligned}\right\}\qquad(21.)$$

On the same hypothesis, we may express the rectangular coordinates $r_, \cos \alpha_,$, $r_, \cos \beta_,$, $r_, \cos \gamma_,$ of the points of the cubical arrangement as follows:

$$r_, \cos \alpha_, = i_, h, \quad r_, \cos \beta_, = k_, h, \quad r_, \cos \gamma_, = l_, h, \tag{22.}$$

h denoting the *mean interval* of two adjacent points on the length of the side of any one of the elementary cubical spaces, and $i_,$, $k_,$, $l_,$ denoting any integer numbers positive or negative or null, which are however to be supposed subject to the conditions

$$i_,^2 + k_,^2 + l_,^2 > 0, \tag{23.}$$

in order not to include in the summation of the attractive or repulsive actions of the several points of the system on any one particular point the action of that point itself. And when, by substitution of the values (20.) and (21.), we have changed the 6 expressions (13.) & (14.) for the quantities L, M, N, P, Q, R to other expressions of the form

$$S. F(r_, \cos \alpha_,, r_, \cos \beta_,, r_, \cos \gamma_,, r_,) \tag{24.}$$

as it is possible in every case to do, we may then transform these latter expressions to others of the form

$$\Sigma_{(i_,, k_,, l_,) -\infty, -\infty, -\infty}^{\infty, \infty, \infty} F(i_, h, k_, h, l_, h, \sqrt{i_,^2 + k_,^2 + l_,^2}\, h) - F(0, 0, 0, 0), \tag{25.}$$

the sign of summation $\Sigma_{(i_,, k_,, l_,) -\infty, -\infty, -\infty}^{\infty, \infty, \infty}$ indicating a summation with respect to each of the three independent integers $i_,$, $k_,$, $l_,$ from $-\infty$ to ∞, and the term $-F(0, 0, 0, 0)$ being added in order to allow for the condition (23.). Besides, by the supposed smallness & equality of the mass constants m, & by the supposed uniformity of distribution of the attracting or repelling points & the supposed smallness of extent of their sensible influence on each other, the expression (25.) may be transformed without sensible error into the following triple sum

$$\Sigma_{(i_,, k_,, l_,) -\infty, -\infty, -\infty}^{\infty, \infty, \infty} \Phi(h\sqrt{i_,^2 + k_,^2 + l_,^2}) - \Phi(0) = S. \Phi(r_,), \tag{26.}$$

if we put for abridgment

$$\Phi(r_,) = \int_{-1}^{+1} \left\{ \frac{1}{4\pi} \int_0^{2\pi} F(r_, \kappa_,, r_, \sqrt{1 - \kappa_,^2} \cos \omega_,, r_, \sqrt{1 - \kappa_,^2} \sin \omega_,, r_,)\, d\omega_, \right\} d\kappa_,. \tag{27.}$$

It appears also to admit of proof that in virtue of the foregoing suppositions the triple sum (26.) may in general be very approximately expressed by the following single sum,

$$S. \Phi(r_,) = \Sigma_{(n)\,1}^{\infty} \Phi\left(h \sqrt[3]{\frac{3n}{4\pi}}\right) : \tag{28.}$$

but the investigations of Poisson* respecting similar cases of molecular action seem to show that we must distinguish this last sum Σ from the definite integral $\int_0^\infty \frac{4\pi}{h^3} \Phi(r_,)\, r_,^2 dr_,$, with which we might otherwise be tempted to confound it. Perhaps even the reduction of the triple sum (26.) to the single sum (28.) may in some cases sensibly err. So that it might be necessary, for some laws of molecular action, to employ the less simple expression

$$\begin{aligned} &S. F(r_, \cos \alpha_,, r_, \cos \beta_,, r_, \cos \gamma_,, r_,) \\ &\quad = \Sigma_{(i_,, k_,, l_,) -\infty, -\infty, -\infty}^{\infty, \infty, \infty} \Phi(h\sqrt{i_,^2 + k_,^2 + l_,^2}) - \Phi(0) = S. \Phi(r_,) \end{aligned} \tag{29.}$$

and not to admit the transformation

$$S. F(r_, \cos \alpha_,, r_, \cos \beta_,, r_, \cos \gamma_,, r_,) = \Sigma_{(n)\,1}^{\infty} \Phi\left(h \sqrt[3]{\frac{3n}{4\pi}}\right) : \tag{30.}$$

* [Poisson, *Nouveaux Mémoires de l'Académie des Sciences*, Tome VI.]

but it seems probable that such cases are rare, & that the simpler expression (30.) will usually be very exact.

9. The expressions $r^{-2}f'(r) - r^{-3}f(r)$ and $r^{-1}f(r)$ become, by the substitution (20.),

$$r_{,}^{-2}f'(r_{,}) - r_{,}^{-3}f(r_{,}) + \{r_{,}^{-1}f''(r_{,}) - 3r_{,}^{-2}f'(r_{,}) + 3r_{,}^{-3}f(r_{,})\}(a_{,}\cos\alpha_{,}^{2} + b_{,}\cos\beta_{,}^{2} + c_{,}\cos\gamma_{,}^{2}) \quad (31.)$$

and

$$r_{,}^{-1}f(r_{,}) + \{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}(a_{,}\cos\alpha_{,}^{2} + b_{,}\cos\beta_{,}^{2} + c_{,}\cos\gamma_{,}^{2}); \quad (32.)$$

$f''(r_{,})$ denoting the second differential coefficient of the function $f(r_{,})$. We have also, by (17.),

$$\left.\begin{aligned} r^2\cos\alpha^2 &= r_{,}^2\cos\alpha_{,}^2(1+2a_{,}),\\ r^2\cos\beta^2 &= r_{,}^2\cos\beta_{,}^2(1+2b_{,}),\\ r^2\cos\gamma^2 &= r_{,}^2\cos\gamma_{,}^2(1+2c_{,}),\end{aligned}\right\} \quad (33.)$$

and

$$\left.\begin{aligned} r^2\cos\beta\cos\gamma &= r_{,}^2\cos\beta_{,}\cos\gamma_{,}(1+b_{,}+c_{,}),\\ r^2\cos\gamma\cos\alpha &= r_{,}^2\cos\gamma_{,}\cos\alpha_{,}(1+c_{,}+a_{,}),\\ r^2\cos\alpha\cos\beta &= r_{,}^2\cos\alpha_{,}\cos\beta_{,}(1+a_{,}+b_{,});\end{aligned}\right\} \quad (34.)$$

and, by the same substitutions,

$$\left(\sin\frac{ur\cos\alpha + vr\cos\beta + wr\cos\gamma}{2}\right)^2 = \left(\sin\frac{ur_{,}\cos\alpha_{,} + vr_{,}\cos\beta_{,} + wr_{,}\cos\gamma_{,}}{2}\right)^2$$
$$+\left(\frac{ur_{,}a_{,}\cos\alpha_{,} + vr_{,}b_{,}\cos\beta_{,} + wr_{,}c_{,}\cos\gamma_{,}}{2}\right)\sin(ur_{,}\cos\alpha_{,} + vr_{,}\cos\beta_{,} + wr_{,}\cos\gamma_{,}). \quad (35.)$$

If then we denote by $L_{,}, M_{,}, N_{,}, P_{,}, Q_{,}, R_{,}$ the values of L, M, N, P, Q, R in the cubical arrangement of the points, and by

$$\frac{\delta L}{\delta a_{,}}, \quad \frac{\delta L}{\delta b_{,}}, \quad \frac{\delta L}{\delta c_{,}}, \quad \&c.$$

the coefficients of the first powers of the small dilatations or contractions $a_{,}, b_{,}, c_{,}$ in the general expressions of these six quantities L, &c., and if we put for abridgment

$$\iota_{,} = \frac{r_{,}}{2}(u\cos\alpha_{,} + v\cos\beta_{,} + w\cos\gamma_{,}), \quad (36.)$$

we shall have these 6 equations

$$\left.\begin{aligned} L_{,} &= S.\,2m\{\cos\alpha_{,}^2 f'(r_{,}) + \sin\alpha_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ M_{,} &= S.\,2m\{\cos\beta_{,}^2 f'(r_{,}) + \sin\beta_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ N_{,} &= S.\,2m\{\cos\gamma_{,}^2 f'(r_{,}) + \sin\gamma_{,}^2 r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\end{aligned}\right\} \quad (37.)$$

$$\left.\begin{aligned} P_{,} &= S.\,2m\cos\beta_{,}\cos\gamma_{,}\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2,\\ Q_{,} &= S.\,2m\cos\gamma_{,}\cos\alpha_{,}\{\qquad\qquad\}\sin\iota_{,}^2,\\ R_{,} &= S.\,2m\cos\alpha_{,}\cos\beta_{,}\{\qquad\qquad\}\sin\iota_{,}^2,\end{aligned}\right\} \quad (38.)$$

together with 18 others of which it is sufficient to write these 4,

$$\begin{aligned} \frac{\delta L}{\delta a_{,}} &= S.\,6m\cos\alpha_{,}^2\{f'(r_{,}) - r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2\\ &+ S.\,2m\cos\alpha_{,}^4\{r_{,}f''(r_{,}) - 3f'(r_{,}) + 3r_{,}^{-1}f(r_{,})\}\sin\iota_{,}^2\\ &+ S.\,mur_{,}\cos\alpha_{,}\{\cos\alpha_{,}^2 f'(r_{,}) + \sin\alpha_{,}^2 r_{,}^{-1}f(r_{,})\}\sin 2\iota_{,}; \end{aligned} \quad (39.)$$

$$\frac{\delta L}{\delta b_,} = S.\, 2m \cos\beta_,^2 \{f'(r_,) - r_,^{-1} f(r_,)\} \sin\iota_,^2$$
$$+ S.\, 2m \cos\alpha_,^2 \cos\beta_,^2 \{r_, f''(r_,) - 3f'(r_,) + 3r_,^{-1} f(r_,)\} \sin\iota_,^2$$
$$+ S.\, mvr_, \cos\beta_, \{\cos\alpha_,^2 f'(r_,) + \sin\alpha_,^2 r_,^{-1} f(r_,)\} \sin 2\iota_,; \qquad (40.)$$

$$\frac{\delta P}{\delta a_,} = S.\, 2m \cos\alpha_,^2 \cos\beta_, \cos\gamma_, \{r_, f''(r_,) - 3f'(r_,) + 3r_,^{-1} f(r_,)\} \sin\iota_,^2 \qquad (41.)$$
$$+ S.\, mur_, \cos\alpha_, \cos\beta_, \cos\gamma_, \{f'(r_,) - r_,^{-1} f(r_,)\} \sin 2\iota_,;$$

$$\frac{\delta P}{\delta b_,} = P_, + S.\, 2m \cos\beta_,^3 \cos\gamma_, \{r_, f''(r_,) - 3f'(r_,) + 3r_,^{-1} f(r_,)\} \sin\iota_,^2$$
$$+ S.\, mvr_, \cos\beta_,^2 \cos\gamma_, \{f'(r_,) - r_,^{-1} f(r_,)\} \sin 2\iota_,: \qquad (42.)$$

and the 6 quantities L, M, N, P, Q, R, in the formula of the 7th paragraph, will now be represented as follows,

$$L = L_, + a_, \frac{\delta L}{\delta a_,} + b_, \frac{\delta L}{\delta b_,} + c_, \frac{\delta L}{\delta c_,}, \text{ \&c.} \qquad (43.)$$

10. To attain a still greater simplicity, we shall suppose further that *the extent of sensible action of one attracting or repelling point upon another is small, yet not successively small, with respect to the thickness of a wave*, that is, with respect to the interval in space between two successive planes on which the attracting or repelling points of the system are similarly displaced from their positions of equilibrium, at any one moment of time. This interval or thickness λ is easily seen to be expressible by the formula

$$\lambda = \frac{2\pi}{k}, \qquad (44.)$$

if we put for abridgment
$$k = \sqrt{u^2 + v^2 + w^2}, \qquad (45.)$$
employing the symbols u, v, w to denote the same three constants of a wave as in the equations (10.). We shall, therefore, *suppose that the product*

$$kr = 2\pi . \frac{r}{\lambda}, \quad \text{or} \quad kr_, = 2\pi . \frac{r_,}{\lambda}, \qquad (46.)$$

is small, for all those distances r, or r for which the function $f(r)$, or $f(r_,)$, or even any one of its differential coefficients, has any sensible value; yet not so very small but that its square and perhaps higher powers may be sensible. And then, if we put

$$u = ka, \quad v = kb, \quad w = kc, \qquad (47.)$$

(so that a, b, c shall denote the cosines of the angles which one of the two outer seminormals to the surfaces of the wave makes with the three positive semiaxes of coordinates,) we shall be able to develope the square of the sine of the trinomial

$$\iota_, = \tfrac{1}{2}(ur_, \cos\alpha_, + vr_, \cos\beta_, + wr_, \cos\gamma_,)$$
$$= \tfrac{1}{2}kr_, (a\cos\alpha_, + b\cos\beta_, + c\cos\gamma_,), \qquad (48.)$$

according to the ascending powers of this small product $kr_,$, in a converging series of which the three first terms are

$$\left.\begin{aligned} \sin\iota_,^2 = {} & \frac{k^2 r_,^2}{4}(a\cos\alpha_, + b\cos\beta_, + c\cos\gamma_,)^2 \\ & - \frac{k^4 r_,^4}{48}(a\cos\alpha_, + b\cos\beta_, + c\cos\gamma_,)^4 \\ & + \frac{k^6 r_,^6}{1440}(a\cos\alpha_, + b\cos\beta_, + c\cos\gamma_,)^6: \end{aligned}\right\} \qquad (49.)$$

and in like manner we may develope sin 2ι, in a series of which the first terms are

$$\sin 2\iota_{,} = kr_{,}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,}) - \frac{k^3 r_{,}^3}{6}(a\cos\alpha_{,} + b\cos\beta_{,} + c\cos\gamma_{,})^3. \quad (50.)$$

11. We shall now neglect the terms which are of the fourth or higher dimensions with respect to $kr_{,}$ in the 6 expressions $L_{,}, M_{,}, N_{,}, P_{,}, Q_{,}, R_{,}$; and also the terms which are of the third or higher dimensions with respect to the same small quantity $kr_{,}$ in the 6 small remaining parts

$$a_{,}\frac{\delta L}{\delta a_{,}} + b_{,}\frac{\delta L}{\delta b_{,}} + c_{,}\frac{\delta L}{\delta c_{,}}, \text{ \&c.,}$$

of the 6 expressions (43.). And thus we shall have

$$\left.\begin{aligned}\bar{L} &= \{(L+G)a^2 + (R+H)b^2 + (Q+I)c^2\}k^2,\\ \bar{M} &= \{(M+H)b^2 + (P+I)c^2 + (R+G)a^2\}k^2,\\ \bar{N} &= \{(N+I)c^2 + (Q+G)a^2 + (P+H)b^2\}k^2,\end{aligned}\right\} \quad (51.)$$

and

$$\bar{P} = 2Pbck^2, \quad \bar{Q} = 2Qcak^2, \quad \bar{R} = 2Rabk^2, \quad (52.)$$

if we put for abridgment

$$G = G_{,} + a_{,}\frac{\delta G}{\delta a_{,}} + b_{,}\frac{\delta G}{\delta b_{,}} + c_{,}\frac{\delta G}{\delta c_{,}}, \quad (53.)$$

with 8 other equations for H, I, L, M, N, P, Q, R, and if we also put

$$\left.\begin{aligned}G_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\alpha_{,}^2,\\ H_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\beta_{,}^2,\\ I_{,} &= S.\frac{mr_{,}}{2}f(r_{,})\cos\gamma_{,}^2,\end{aligned}\right\} \quad (54.)$$

$$\left.\begin{aligned}L_{,} &= S.\frac{mr_{,}}{2}\{r_{,}f'(r_{,}) - f(r_{,})\}\cos\alpha_{,}^4,\\ M_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\}\cos\beta_{,}^4,\\ N_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\}\cos\gamma_{,}^4,\end{aligned}\right\} \quad (55.)$$

$$\left.\begin{aligned}P_{,} &= S.\frac{mr_{,}}{2}\{r_{,}f'(r_{,}) - f(r_{,})\}\cos\beta_{,}^2\cos\gamma_{,}^2,\\ Q_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\}\cos\gamma_{,}^2\cos\alpha_{,}^2,\\ R_{,} &= S.\frac{mr_{,}}{2}\{\qquad\qquad\}\cos\alpha_{,}^2\cos\beta_{,}^2,\end{aligned}\right\} \quad (56.)$$

$$\left.\begin{aligned}&\frac{\delta G}{\delta a_{,}}=2G_{,}+L_{,}\,, \quad \frac{\delta G}{\delta b_{,}}=R_{,}\,, \quad \frac{\delta G}{\delta c_{,}}=Q_{,}\,,\\ &\frac{\delta H}{\delta b_{,}}=2H_{,}+M_{,}\,, \quad \frac{\delta H}{\delta c_{,}}=P_{,}\,, \quad \frac{\delta H}{\delta a_{,}}=R_{,}\,,\\ &\frac{\delta I}{\delta c_{,}}=2I_{,}+N_{,}\,, \quad \frac{\delta I}{\delta a_{,}}=Q_{,}\,, \quad \frac{\delta I}{\delta b_{,}}=P_{,}\,,\end{aligned}\right\} \qquad (57.)^*$$

$$\left.\begin{aligned}&\frac{\delta L}{\delta a_{,}}=4L_{,}+S.\phi(r_{,})\cos\alpha_{,}^{6}, \quad \frac{\delta L}{\delta b_{,}}=S.\phi(r_{,})\cos\alpha_{,}^{4}\cos\beta_{,}^{2}, \quad \frac{\delta L}{\delta c_{,}}=S.\phi(r_{,})\cos\alpha_{,}^{4}\cos\gamma_{,}^{2},\\ &\frac{\delta M}{\delta b_{,}}=4M_{,}+S.\phi(r_{,})\cos\beta_{,}^{6}, \quad \frac{\delta M}{\delta c_{,}}=S.\phi(r_{,})\cos\beta_{,}^{4}\cos\gamma_{,}^{2}, \quad \frac{\delta M}{\delta a_{,}}=S.\phi(r_{,})\cos\beta_{,}^{4}\cos\alpha_{,}^{2},\\ &\frac{\delta N}{\delta c_{,}}=4N_{,}+S.\phi(r_{,})\cos\gamma_{,}^{6}, \quad \frac{\delta N}{\delta a_{,}}=S.\phi(r_{,})\cos\gamma_{,}^{4}\cos\alpha_{,}^{2}, \quad \frac{\delta N}{\delta b_{,}}=S.\phi(r_{,})\cos\gamma_{,}^{4}\cos\beta_{,}^{2},\end{aligned}\right\} \qquad (58.)$$

and finally

$$\left.\begin{aligned}&\frac{\delta P}{\delta a_{,}}=\frac{\delta Q}{\delta b_{,}}=\frac{\delta R}{\delta c_{,}}=S.\phi(r_{,})\cos\alpha_{,}^{2}\cos\beta_{,}^{2}\cos\gamma_{,}^{2},\\ &\frac{\delta P}{\delta b_{,}}=2P_{,}+\frac{\delta M}{\delta c_{,}}, \quad \frac{\delta Q}{\delta c_{,}}=2Q_{,}+\frac{\delta N}{\delta a_{,}}, \quad \frac{\delta R}{\delta a_{,}}=2R_{,}+\frac{\delta L}{\delta b_{,}},\\ &\frac{\delta P}{\delta c_{,}}=2P_{,}+\frac{\delta N}{\delta b_{,}}, \quad \frac{\delta Q}{\delta a_{,}}=2Q_{,}+\frac{\delta L}{\delta c_{,}}, \quad \frac{\delta R}{\delta b_{,}}=2R_{,}+\frac{\delta M}{\delta a_{,}},\end{aligned}\right\} \qquad (59.)$$

in which

$$\phi(r_{,})=\frac{m}{2}r_{,}^{5}\frac{d}{dr_{,}}\left(\frac{1}{r_{,}}\frac{d}{dr_{,}}\frac{f(r_{,})}{r_{,}}\right)=\frac{m}{2}\{r_{,}^{3}f''(r_{,})-3r_{,}^{2}f'(r_{,})+3r_{,}f(r_{,})\}. \qquad (60.)$$

[*Wave velocities and directions of vibrations.*]

12. Again, by the 8th paragraph, we may transform every sum of the form

$$S.\psi(r_{,})\cos\alpha_{,}^{2i_{,,}}\cos\beta_{,}^{2k_{,,}}\cos\gamma_{,}^{2l_{,,}} \qquad (61.)$$

into an equivalent sum of the form

$$(i_{,,}, k_{,,}, l_{,,})\,S\psi(r_{,})=S.\psi(r_{,})\cos\alpha_{,}^{2i_{,,}}\cos\beta_{,}^{2k_{,,}}\cos\gamma_{,}^{2l_{,,}}, \qquad (62.)$$

in which $(i_{,,}, k_{,,}, l_{,,})$ is a fraction depending on the integer exponents $i_{,,}, k_{,,}, l_{,,}$ by the following rule

$$(i_{,,}, k_{,,}, l_{,,})=\frac{1}{\pi}\int_0^{\pi}\cos\omega_{,}^{2k_{,,}}\sin\omega_{,}^{2l_{,,}}d\omega_{,}\,.\int_0^1\kappa_{,}^{2i_{,,}}(1-\kappa_{,}^{2})^{k_{,,}+l_{,,}}d\kappa_{,}\,. \qquad (63.)$$

It is easily found to be a consequence of this rule that we may change the order of the 3 exponents $i_{,,}, k_{,,}, l_{,,}$ in any manner, without altering the value of the fraction $(i_{,,}, k_{,,}, l_{,,})$;† and therefore that it is sufficient to calculate the values of these 6 different combinations,

$$(1, 0, 0), \quad (2, 0, 0), \quad (1, 1, 0), \quad (3, 0, 0), \quad (2, 1, 0), \quad (1, 1, 1), \qquad (64.)$$

* [Equations (57.)–(60.) follow easily from results of the type

$$\frac{\partial}{\partial a_{,}}(r\cos\alpha)=r_{,}\cos\alpha_{,}, \quad \frac{\partial r}{\partial a_{,}}=r_{,}\cos^{2}\alpha_{,}, \quad \frac{\partial}{\partial b_{,}}(r\cos\beta)=0, \text{ etc.},$$

which are a direct consequence of (17.) and (19.).]

† $[(i_{,,}, k_{,,}, l_{,,})=\Gamma(i_{,,}+\frac{1}{2})\,\Gamma(k_{,,}+\frac{1}{2})\,\Gamma(l_{,,}+\frac{1}{2})/\{2\pi\Gamma(i_{,,}+k_{,,}+l_{,,}+\frac{3}{2})\}.]$

in order to be able to apply the transformation (62.) to all those expressions of the last paragraph which are of the form (61.). The 6 values (64.) are easily found to be:

$$\left.\begin{array}{lll}(1,0,0)=\frac{1}{3}; & (2,0,0)=\frac{1}{5}; & (1,1,0)=\frac{1}{15};\\ (3,0,0)=\frac{1}{7}; & (2,1,0)=\frac{1}{35}; & (1,1,1)=\frac{1}{105}.\end{array}\right\} \qquad (65.)$$

If then we put for abridgment

$$\left.\begin{aligned} R_0 &= S.\frac{m}{6} r_{,} f(r_{,}),\\ R_1 &= S.\frac{m}{30}\{r_{,}^2 f'(r_{,}) - r_{,} f(r_{,})\},\\ R_2 &= S.\frac{m}{210}\{r_{,}^3 f''(r_{,}) - 3r_{,}^2 f'(r_{,}) + 3r_{,} f(r_{,})\},\end{aligned}\right\} \qquad (66.)$$

we shall have the following results:

$$G_{,} = H_{,} = I_{,} = R_0, \qquad (67.)$$

$$L_{,} = M_{,} = N_{,} = 3R_1, \qquad (68.)$$

$$P_{,} = Q_{,} = R_{,} = R_1, \qquad (69.)$$

$$\frac{\delta G}{\delta a_{,}} = \frac{\delta H}{\delta b_{,}} = \frac{\delta I}{\delta c_{,}} = 2R_0 + 3R_1, \qquad (70.)$$

$$\frac{\delta G}{\delta b_{,}} = \frac{\delta H}{\delta c_{,}} = \frac{\delta I}{\delta a_{,}} = R_1, \qquad (71.)$$

$$\frac{\delta G}{\delta c_{,}} = \frac{\delta H}{\delta a_{,}} = \frac{\delta I}{\delta b_{,}} = R_1, \qquad (72.)$$

$$\frac{\delta L}{\delta a_{,}} = \frac{\delta M}{\delta b_{,}} = \frac{\delta N}{\delta c_{,}} = 12R_1 + 15R_2, \qquad (73.)$$

$$\frac{\delta L}{\delta b_{,}} = \frac{\delta M}{\delta c_{,}} = \frac{\delta N}{\delta a_{,}} = 3R_2, \qquad (74.)$$

$$\frac{\delta L}{\delta c_{,}} = \frac{\delta M}{\delta a_{,}} = \frac{\delta N}{\delta b_{,}} = 3R_2, \qquad (75.)$$

$$\frac{\delta P}{\delta a_{,}} = \frac{\delta Q}{\delta b_{,}} = \frac{\delta R}{\delta c_{,}} = R_2, \qquad (76.)$$

$$\frac{\delta P}{\delta b_{,}} = \frac{\delta Q}{\delta c_{,}} = \frac{\delta R}{\delta a_{,}} = 2R_1 + 3R_2, \qquad (77.)$$

$$\frac{\delta P}{\delta c_{,}} = \frac{\delta Q}{\delta a_{,}} = \frac{\delta R}{\delta b_{,}} = 2R_1 + 3R_2. \qquad (78.)$$

And substituting these values, in the expressions of the form (53.), we find

$$\left.\begin{aligned} G &= R_0(1 + 2a_{,}) + R_1(3a_{,} + b_{,} + c_{,}),\\ H &= R_0(1 + 2b_{,}) + R_1(3b_{,} + c_{,} + a_{,}),\\ I &= R_0(1 + 2c_{,}) + R_1(3c_{,} + a_{,} + b_{,}),\end{aligned}\right\} \qquad (79.)$$

$$\left.\begin{aligned}L&=3R_1(1+4a,)+3R_2(5a,+b,+c,),\\ M&=3R_1(1+4b,)+3R_2(5b,+c,+a,),\\ N&=3R_1(1+4c,)+3R_2(5c,+a,+b,),\end{aligned}\right\}\quad(80.)$$

$$\left.\begin{aligned}P&=R_1(1+2b,+2c,)+R_2(a,+3b,+3c,),\\ Q&=R_1(1+2c,+2a,)+R_2(b,+3c,+3a,),\\ R&=R_1(1+2a,+2b,)+R_2(c,+3a,+3b,);\end{aligned}\right\}\quad(81.)$$

and therefore

$$\left.\begin{aligned}L+G&=R_0(1+2a,)+R_1(3+15a,+b,+c,)+3R_2(5a,+b,+c,),\\ M+H&=R_0(1+2b,)+R_1(3+15b,+c,+a,)+3R_2(5b,+c,+a,),\\ N+I&=R_0(1+2c,)+R_1(3+15c,+a,+b,)+3R_2(5c,+a,+b,),\end{aligned}\right\}\quad(82.)$$

$$\left.\begin{aligned}R+H&=R_0(1+2b,)+R_1(1+3a,+5b,+c,)+R_2(3a,+3b,+c,),\\ P+I&=R_0(1+2c,)+R_1(1+3b,+5c,+a,)+R_2(3b,+3c,+a,),\\ Q+G&=R_0(1+2a,)+R_1(1+3c,+5a,+b,)+R_2(3c,+3a,+b,),\end{aligned}\right\}\quad(83.)$$

$$\left.\begin{aligned}Q+I&=R_0(1+2c,)+R_1(1+3a,+b,+5c,)+R_2(3a,+b,+3c,),\\ R+G&=R_0(1+2a,)+R_1(1+3b,+c,+5a,)+R_2(3b,+c,+3a,),\\ P+H&=R_0(1+2b,)+R_1(1+3c,+a,+5b,)+R_2(3c,+a,+3b,).\end{aligned}\right\}\quad(84.)$$

Hence, by (51.) and (52.),

$$\begin{aligned}k^{-2}\overline{L}=R_0+R_1+2R_1a^2+2R_0(a,a^2+b,b^2+c,c^2)+R_1(3a,+b,+c,)+3R_2(a,+b,+c,)\\ +4R_1(3a,a^2+b,b^2+c,c^2)+2R_2(6a,a^2-c,b^2-b,c^2);\end{aligned}\quad(85.)$$

$$\begin{aligned}k^{-2}\overline{M}=R_0+R_1+2R_1b^2+2R_0(b,b^2+c,c^2+a,a^2)+R_1(3b,+c,+a,)+3R_2(a,+b,+c,)\\ +4R_1(3b,b^2+c,c^2+a,a^2)+2R_2(6b,b^2-a,c^2-c,a^2);\end{aligned}\quad(86.)$$

$$\begin{aligned}k^{-2}\overline{N}=R_0+R_1+2R_1c^2+2R_0(c,c^2+a,a^2+b,b^2)+R_1(3c,+a,+b,)+3R_2(a,+b,+c,)\\ +4R_1(3c,c^2+a,a^2+b,b^2)+2R_2(6c,c^2-b,a^2-a,b^2);\end{aligned}\quad(87.)$$

$$\left.\begin{aligned}k^{-2}\overline{P}&=2R_1(1+2b,+2c,)\,bc+2R_2(a,+3b,+3c,)\,bc,\\ k^{-2}\overline{Q}&=2R_1(1+2c,+2a,)\,ca+2R_2(b,+3c,+3a,)\,ca,\\ k^{-2}\overline{R}&=2R_1(1+2a,+2b,)\,ab+2R_2(c,+3a,+3b,)\,ab.\end{aligned}\right\}\quad(88.)$$

And hence, finally, if we can put for abridgment

$$-s^2k^{-2}+R_0+R_1+(2R_0+4R_1)(a,a^2+b,b^2+c,c^2)+(R_1+3R_2)(a,+b,+c,)=2\sigma,\quad(89.)$$

the equations (16.) become

$$\left.\begin{aligned}&\{\sigma+R_1(a^2+a,+4a,a^2)+R_2(6a,a^2-c,b^2-b,c^2)\}\xi'\\ &\quad+\{R_1(1+2a,+2b,)+R_2(3a,+3b,+c,)\}ab\eta'\\ &\quad+\{R_1(1+2a,+2c,)+R_2(3a,+3c,+b,)\}ac\zeta'=0,\end{aligned}\right\}\quad(90.)$$

$$\left.\begin{aligned}&\{\sigma+R_1(b^2+b,+4b,b^2)+R_2(6b,b^2-a,c^2-c,a^2)\}\eta'\\ &\quad+\{R_1(1+2b,+2c,)+R_2(3b,+3c,+a,)\}bc\zeta'\\ &\quad+\{R_1(1+2b,+2a,)+R_2(3b,+3a,+c,)\}ba\xi'=0,\end{aligned}\right\}\quad(91.)$$

$$\left.\begin{aligned}&\{\sigma+R_1(c^2+c,+4c,c^2)+R_2(6c,c^2-b,a^2-a,b^2)\}\zeta'\\ &\quad+\{R_1(1+2c,+2a,)+R_2(3c,+3a,+b,)\}ca\xi'\\ &\quad+\{R_1(1+2c,+2b,)+R_2(3c,+3b,+a,)\}cb\eta'=0.\end{aligned}\right\}\quad(92.)$$

It is to be remembered that in these 4 last equations the quantities R_0, R_1, R_2 and $a_{,}, b_{,}, c_{,}$ are constant for any one medium depending only on the distribution of the attracting or repelling points of the system in their positions of equilibrium and on their law of action; ξ', η', ζ' are also constants for any one succession of plane waves and are proportional to the three coordinate displacements of any one vibrating point from its position of equilibrium; a, b, c are the cosines of the angles of direction of a normal to such a wave; and s^2k^{-2}, with which the auxiliary quantity σ is connected by the equation (89.), is the square of the velocity wherewith that wave is propagated.

13. In the particular case when

$$b_{,} = a_{,}, \tag{93.}$$

that is, when the system of points is supposed to be, in its position of equilibrium, dilated equally or contracted equally in the 2 rectangular directions of x and y from the standard cubical arrangement, the three equations (90.), (91.), (92.) become

$$\begin{aligned} &\{\sigma + R_1(a^2 + a_{,} + 4a_{,}a^2) + R_2(6a_{,}a^2 - a_{,}c^2 - c_{,}b^2)\}\xi' \\ &+ \{R_1(1 + 4a_{,}) + R_2(6a_{,} + c_{,})\}ab\eta' \\ &+ \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}ac\zeta' = 0; \end{aligned} \tag{94.}$$

$$\begin{aligned} &\{\sigma + R_1(b^2 + a_{,} + 4a_{,}b^2) + R_2(6a_{,}b^2 - a_{,}c^2 - c_{,}a^2)\}\eta' \\ &+ \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}bc\zeta' \\ &+ \{R_1(1 + 4a_{,}) + R_2(6a_{,} + c_{,})\}ba\xi' = 0; \end{aligned} \tag{95.}$$

$$\begin{aligned} &\{\sigma + R_1(c^2 + c_{,} + 4c_{,}c^2) + R_2(6c_{,}c^2 - a_{,}a^2 - a_{,}b^2)\}\zeta' \\ &+ \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}ca\xi' \\ &+ \{R_1(1 + 2a_{,} + 2c_{,}) + R_2(4a_{,} + 3c_{,})\}cb\eta' = 0. \end{aligned} \tag{96.}$$

And if we further suppose that these two equal dilatations or contractions in the directions of x & y to vanish, so that

$$a_{,} = 0, \quad b_{,} = 0, \tag{97.}$$

(which comes to taking for the standard cubical arrangement $x_{,}, y_{,}, z$, that which differs from the position of equilibrium x, y, z only by a dilatation or contraction in the direction of the axis of z,) we shall have, more simply, the 3 equations following:

$$\left.\begin{aligned} &(\sigma + R_1a^2 - R_2c_{,}b^2)\xi' + (R_1 + R_2c_{,})ab\eta' + (R_1 + 2R_1c_{,} + 3R_2c_{,})ac\zeta' = 0, \\ &(\sigma + R_1b^2 - R_2c_{,}a^2)\eta' + (R_1 + R_2c_{,})ab\xi' + (R_1 + 2R_1c_{,} + 3R_2c_{,})bc\zeta' = 0, \\ &\{\sigma + R_1c^2 + R_1c_{,}(1 + 4c^2) + 6R_2c_{,}c^2\}\zeta' + (R_1 + 2R_1c_{,} + 3R_2c_{,})c(a\xi' + b\eta') = 0; \end{aligned}\right\} \tag{98.}$$

which give, by elimination of the ratios of ξ', η', ζ', the following cubic equation

$$\begin{aligned} 0 = {}&\sigma^3 + \sigma^2R_1 + \sigma^2c_{,}\{R_1 - R_2 + (4R_1 + 7R_2)c^2\} \\ &+ \sigma c_{,}(1 - c^2)R_1(R_1 - R_2) - \sigma c_{,}^2(1 - c^2)\{4R_1^2c^2 + R_1R_2(1 + 16c^2) + 15R_2^2c^2\} \\ &- c_{,}^2(1 - c^2)^2R_1^2R_2 + c_{,}^3(1 - c^2)^2c^2(2R_1 + 3R_2)^2R_2 \\ = {}&\{\sigma - c_{,}(1 - c^2)R_2\}\{\sigma + c_{,}(1 - c^2)R_1\}\{\sigma + R_1 + c_{,}c^2(5R_1 + 6R_2)\} \\ &- 9\{\sigma - c_{,}(1 - c^2)R_2\}c_{,}^2(1 - c^2)c^2(R_1 + R_2)^2: \end{aligned} \tag{99.}$$

and the three roots of this cubic equation will give three corresponding values of the square s^2k^{-2} of the velocity of propagation of a plane wave by the formula

$$s^2k^{-2} = R_0 + R_1 + c,(R_1 + 3R_2) + 2c,c^2(R_0 + 2R_1) - 2\sigma, \tag{100.}$$

to which the formula (89.) reduces itself by the supposition (97.). With respect to the three directions of displacement of a vibrating point corresponding to these three velocities of propagation of a wave, they may be deduced from the equations (98.) by determining, for each value of the velocity or of the auxiliary quantity σ, the ratios of $\xi_\backprime$, $\eta_\backprime$, $\zeta_\backprime$.

14. One root of the cubic equation (99.) is evidently

$$\sigma_1 = c,(1 - c^2) R_2; \tag{101.}$$

which is easily seen to correspond, by the equations (98.), to the following relations between $\xi_\backprime, \eta_\backprime, \zeta_\backprime$,

$$a\xi_{\backprime 1} + b\eta_{\backprime 1} = 0, \quad \zeta_{\backprime 1} = 0, \tag{102.}$$

that is, to a direction of displacement or vibration which is contained upon the surface of the wave & is perpendicular to the axis of z, that is, to the direction of dilatation or contraction of the system of points when in equilibrium. And the square of the velocity of a wave when propagated with such vibrations is, by (100.) and (101.),

$$s_1^2k^{-2} = R_0 + R_1 + c,(R_1 + R_2) + 2c,(R_0 + 2R_1 + R_2)c^2; \tag{103.}$$

so that it consists of a constant part, namely,

$$R_0 + R_1 + c,(R_1 + R_2), \tag{104.}$$

which is independent of the direction of the wave, and a variable part, namely,

$$2c,(R_0 + 2R_1 + R_2)c^2, \tag{105.}$$

which is proportional to the square c^2 of the cosine of the angle between the normal to the surface of the wave and the direction of dilatation or contraction.

Another root of the same cubic (99.) is, very nearly,

$$\sigma_2 = -c,(1 - c^2) R_1, \tag{106.}$$

the error being of the order of the square of the dilatation or contraction, which we have already supposed that we may neglect. This root corresponds to the following value of the square of the velocity of the wave,

$$s_2^2k^{-2} = R_0 + R_1 + 3c,(R_1 + R_2) + 2c,(R_0 + R_1)c^2; \tag{107.}$$

which consists like the former value (103.) of a constant part, in this case

$$R_0 + R_1 + 3c,(R_1 + R_2), \tag{108.}$$

and a part proportional to the square of the cosine of the angle between the direction of dilatation or contraction & the normal to the surface of the wave, namely the part

$$2c,(R_0 + R_1)c^2. \tag{109.}$$

The vibrations, for this velocity of propagation of the wave, have their directions determined very nearly by the following equations, deduced from (98.) and (106.),

$$\frac{\xi_{\backprime 2}}{a} = \frac{\eta_{\backprime 2}}{b} = \frac{\zeta_{\backprime 2}}{c - \dfrac{1}{c}} \left\{1 + \frac{3c,}{R_1}(R_1 + R_2)\right\}. \tag{110.}$$

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They are therefore very nearly tangential to the surface of the wave, & are perpendicular to the vibrations which correspond to the former velocity.

Finally, the remaining root of the cubic equation (99.) is very nearly

$$\sigma_3 = -R_1 - c_{,}c^2(5R_1 + 6R_2); \qquad (111.)$$

it corresponds to vibrations which are perpendicular to both the two former directions of vibration and very nearly normal to the wave, being determined very approximately by the equations

$$\frac{\xi'_3}{a} = \frac{\eta'_3}{b} = \frac{\zeta'_3}{c}\left\{1 - \frac{3c_{,}}{R_1}(R_1 + R_2)\right\}; \qquad (112.)$$

and for these last vibrations the square of the velocity of propagation of the wave is very nearly

$$s_3^2 k^{-2} = R_0 + 3R_1 + c_{,}(R_1 + 3R_2) + 2c_{,}(R_0 + 7R_1 + 6R_2)c^2, \qquad (113.)$$

so that it is composed of a constant and a variable part like the squares of the two former velocities. With respect to the small angular deviation of the third set of vibrations from the normal, or of the second set of vibrations from the surface of the wave, it varies very nearly as the sine of twice the angle between the normal and the direction of dilatation or of contraction.

15. Results of the same kind would have followed if we had made in (89.), (90.), (91.) and (92.)

$$b_{,} = a_{,}, \quad c_{,} = 0; \qquad (114.)$$

that is, if we had supposed the arrangement of the attracting or repelling points of the system, when in equilibrium, to differ from the standard cubical arrangement by two equal dilatations or equal contractions in the two rectangular directions of x and y. We should then have had this formula for the square of the velocity of the wave,

$$s^2 k^{-2} = R_0 + R_1 + 2a_{,}(R_1 + 3R_2) + 2a_{,}(R_0 + 2R_1)(1 - c^2) - 2\sigma; \qquad (115.)$$

the auxiliary quantity σ, and the ratios of ξ', η', ζ', which determine the direction of vibration, being found from the following equations:

$$\left.\begin{aligned}
0 &= \{\sigma + R_1 a^2 + a_{,}R_1(1 + 4a^2) + a_{,}R_2(6a^2 - c^2)\}\xi' \\
&\quad + \{R_1 + 2a_{,}(2R_1 + 3R_2)\}ab\eta' + \{R_1 + 2a_{,}(R_1 + 2R_2)\}ac\zeta', \\
0 &= \{\sigma + R_1 b^2 + a_{,}R_1(1 + 4b^2) + a_{,}R_2(6b^2 - c^2)\}\eta' \\
&\quad + \{R_1 + 2a_{,}(2R_1 + 3R_2)\}ab\xi' + \{R_1 + 2a_{,}(R_1 + 2R_2)\}bc\zeta', \\
0 &= \{\sigma + R_1 c^2 - a_{,}R_2(1 - c^2)\}\zeta' + \{R_1 + 2a_{,}(R_1 + 2R_2)\}c(a\xi' + b\eta'),
\end{aligned}\right\} \qquad (116.)$$

which give, by elimination,

$$\begin{aligned}
0 = \{\sigma + a_{,}(R_1 - R_2c^2)\}\{\sigma + a_{,}(R_1c^2 - R_2)\}\{\sigma + R_1 + a_{,}(1 - c^2)(5R_1 + 6R_2)\} \\
- 9\{\sigma + a_{,}(R_1 - R_2c^2)\}a_{,}^2(1 - c^2)c^2(R_1 + R_2)^2. \qquad (117.)
\end{aligned}$$

One root of this cubic is

$$\sigma_1 = -a_{,}(R_1 - R_2c^2) \qquad (118.)$$

& corresponds to the following expressions for the velocity & the vibrations,

$$\left.\begin{aligned}
s_1^2 k^{-2} &= R_0 + R_1 + 2a_{,}(R_0 + 4R_1 + 3R_2) - 2a_{,}(R_0 + 2R_1 + R_2)c^2, \\
& a\xi'_1 + b\eta'_1 = 0, \quad \zeta'_1 = 0;
\end{aligned}\right\} \qquad (119.)$$

another root is, nearly,

$$\sigma_2 = -a_{,}(R_1c^2 - R_2) \qquad (120.)$$

& corresponds to the expressions

$$\left.\begin{aligned}
s_2^2 k^{-2} &= R_0 + R_1 + 2a_{,}(R_0 + 3R_1 + 2R_2) - 2a_{,}(R_0 + R_1)c^2, \\
\frac{\xi'_2}{a} &= \frac{\eta'_2}{b} = \frac{\zeta'_2}{c - \frac{1}{c}}\left\{1 - 3a_{,}\frac{R_1 + R_2}{R_1}\right\};
\end{aligned}\right\} \qquad (121.)$$

and the third root is, nearly,

$$\sigma_3 = -R_1 - a_{,}(1-c^2)(5R_1+6R_2), \tag{122.}$$

which gives the expressions

$$\left.\begin{aligned} s_3^2 k^{-2} &= R_0 + 3R_1 + 2a_{,}(R_0+8R_1+9R_2) - 2a_{,}(R_0+7R_1+6R_2)c^2, \\ \frac{\xi'_3}{a} &= \frac{\eta'_3}{b} = \frac{\zeta'_3}{c}\left\{1+3a_{,}\frac{R_1+R_2}{R_1}\right\}. \end{aligned}\right\} \tag{123.}$$

And all these results are evidently similar in kind to those obtained in the foregoing paragraph and lead to analogous conclusions.

16. In general, if we put for abridgment

$$\left.\begin{aligned} c\xi' - a\zeta' &= \xi'', \\ c\eta' - b\zeta' &= \eta'', \\ a\xi' + b\eta' + c\zeta' &= \zeta'', \end{aligned}\right\} \tag{124.}$$

we shall get the expressions

$$\left.\begin{aligned} \xi' &= \frac{b^2+c^2}{c}\xi'' - \frac{ab}{c}\eta'' + a\zeta'', \\ \eta' &= -\frac{ab}{c}\xi'' + \frac{a^2+c^2}{c}\eta'' + b\zeta'', \\ \zeta' &= -a\xi'' - b\eta'' + c\zeta'', \end{aligned}\right\} \tag{125.}$$

by means of which the system of the 3 equations (90.), (91.), (92.) may easily be transformed into the following:

$$0 = \xi''\{\sigma + a_{,}R_1(b^2+c^2) - b_{,}R_2(a^2+c^2) + c_{,}(R_1a^2 - R_2b^2)\} \\ + a(c_{,}-a_{,})(R_1+R_2)(b\eta'' - 3c\zeta''); \tag{126.}$$

$$0 = \eta''\{\sigma + b_{,}R_1(a^2+c^2) - a_{,}R_2(b^2+c^2) + c_{,}(R_1b^2 - R_2a^2)\} \\ + b(c_{,}-b_{,})(R_1+R_2)(a\xi'' - 3c\zeta''); \tag{127.}$$

$$0 = c\zeta''\{\sigma + R_1 + (5R_1+6R_2)(a_{,}a^2+b_{,}b^2+c_{,}c^2)\} \\ + 3(R_1+R_2)a\xi''\{a_{,} - (a_{,}a^2+b_{,}b^2+c_{,}c^2)\} \\ + 3(R_1+R_2)b\eta''\{b_{,} - (a_{,}a^2+b_{,}b^2+c_{,}c^2)\}. \tag{128.*}$$

It is easy hence to perceive that either the two quantities σ and ζ'', or else the three quantities $\sigma + R_1$, ξ'' and η'', must be considered as small of the order of the dilatations or contractions $a_{,}, b_{,}, c_{,}$; of which two cases, the first corresponds to nearly tangential & the second case to nearly normal vibrations. In the first case we may suppress the products of ζ'', σ, $a_{,}$, $b_{,}$, $c_{,}$ and then the equations (126.), (127.), (128.) reduce themselves to the three following:

$$0 = \xi''\{\sigma + a_{,}R_1(b^2+c^2) - b_{,}R_2(a^2+c^2) + c_{,}(R_1a^2 - R_2b^2)\} + \eta''(c_{,}-a_{,})(R_1+R_2)ab; \tag{129.}$$

$$0 = \eta''\{\sigma + b_{,}R_1(a^2+c^2) - a_{,}R_2(b^2+c^2) + c_{,}(R_1b^2 - R_2a^2)\} + \xi''(c_{,}-b_{,})(R_1+R_2)ab; \tag{130.}$$

$$0 = c\zeta''R_1 + 3(R_1+R_2)\{aa_{,}\xi'' + bb_{,}\eta'' - (a\xi''+b\eta'')(a_{,}a^2+b_{,}b^2+c_{,}c^2)\}: \tag{131.}$$

* [The terms in (90.), (91.), (92.) not involving $a_{,}, b_{,}, c_{,}$ relate to a system of revolution, the axis of revolution being the direction a, b, c. This is the reason for adopting the transformation (124.), which is the simplest of its kind. (126.), (127.), (128.) do not follow directly from (90.), (91.), (92.). A transformation of the type of (124.) is first employed, e.g. (90.) multiplied by c, (91.) multiplied by a, etc.]

and of these the third determines a plane of vibration coinciding nearly with the surface of the wave, while the two first determine in general two distinct directions of vibration upon that plane, to each of which corresponds a certain velocity of propagation determined by the same two equations through the help of the two values assigned by them for the auxiliary quantity σ. In the second case the equation (128.) reduces itself to the form

$$\sigma = -R_1 - (5R_1 + 6R_2)(a, a^2 + b, b^2 + c, c^2), \tag{132.}$$

which determines this auxiliary quantity σ, and therefore also, by (89.), the velocity of propagation of the wave when the vibrations are nearly normal to its surface, having the direction determined by the equations (126.), (127.), which now reduce themselves to the following:

$$\left.\begin{aligned} \xi'' &= 3ac\frac{R_1 + R_2}{R_1}(a, -c,)\zeta'', \\ \eta'' &= 2bc\frac{R_1 + R_2}{R_1}(b, -c,)\zeta''. \end{aligned}\right\} \tag{133.}$$

It is easy to prove that this line of nearly normal vibration is perpendicular to the plane (131.) of the two nearly tangential vibrations, of which the equation is, when referred to the rectangular coordinates ξ', η', ζ',

$$\begin{aligned} 0 = R_1(a\xi' + b\eta' + c\zeta') + 3(R_1 + R_2)\{ab(a, -b,)(b\xi' - a\eta') \\ + bc(b, -c,)(c\eta' - b\zeta') + ca(c, -a,)(a\zeta' - c\xi')\}. \end{aligned} \tag{134.}$$

And in this plane the two nearly tangential vibrations themselves, of which the directions are determined by the following quadratic equation, deduced from (129.) & (130.) by elimination of the auxiliary quantity σ,

$$0 = \eta''^2(c, -a,)ab - \xi''^2(c, -b,)ab + \xi''\eta''\{(c, -b,)(a^2 + c^2) - (c, -a,)(b^2 + c^2)\}, \tag{135.}$$

are easily seen to be perpendicular to each other as well as to the direction (133.) of nearly normal vibration;* for if we consider them as represented separately by the 2 equations

$$\eta''_1 = g_1\xi''_1, \quad \eta''_2 = g_2\xi''_2, \tag{136.}$$

in which, by (135.),

$$g_1 + g_2 = \frac{(c, -a,)(b^2 + c^2) - (c, -b,)(a^2 + c^2)}{(c, -a,)ab}, \quad g_1g_2 = -\frac{c, -b,}{c, -a,}, \tag{137.}$$

then the two normal planes to the wave, containing the two directions (135.) or (136.) & making with each other an angle which in the present order of approximation must be considered as equal to the angle between those two directions themselves, have for their two equations in the rectangular coordinates ξ', η', ζ',

$$\left.\begin{aligned} c\eta'_1 - b\zeta'_1 = g_1(c\xi'_1 - a\zeta'_1), \\ c\eta'_2 - b\zeta'_2 = g_2(c\eta'_2 - a\zeta'_2), \end{aligned}\right\} \tag{138.}$$

and are therefore perpendicular to each other, because, by (137.),

$$c^2(1 + g_1g_2) + (b - ag_1)(b - ag_2) = 0. \tag{139.}$$

With respect to the 2 velocities of propagation of the wave corresponding to these two rectangular & nearly tangential vibrations (135.) or (136.), they may be determined (as was above

* [Three directions perpendicular to one another are given by (90.), (91.), (92.), if the cubic in σ has unequal roots.]

observed) by the formula (89.) through the help of the two values of the auxiliary quantity σ by these 2 linear equations

$$\left.\begin{aligned}\sigma_1 &= -a_{,}R_1(b^2+c^2)+b_{,}R_2(a^2+c^2)-c_{,}(R_1a^2-R_2b^2)+g_1(a_{,}-c_{,})(R_1+R_2)ab,\\ \sigma_2 &= -a_{,}R_1(b^2+c^2)+b_{,}R_2(a^2+c^2)-c_{,}(R_1a^2-R_2b^2)+g_2(a_{,}-c_{,})(R_1+R_2)ab;\end{aligned}\right\} \quad (140.)$$

or, jointly, by this one quadratic,

$$\begin{aligned}0 &= \sigma^2+\sigma(R_1-R_2)\{a_{,}(b^2+c^2)+b_{,}(c^2+a^2)+c_{,}(a^2+b^2)\}\\ &\quad +(R_1^2+R_2^2)(a_{,}b_{,}c^2+b_{,}c_{,}a^2+c_{,}a_{,}b^2)\\ &\quad -R_1R_2\{a_{,}^2(b^2+c^2)^2+b_{,}^2(c^2+a^2)^2+c_{,}^2(a^2+b^2)^2+2a_{,}b_{,}a^2b^2+2b_{,}c_{,}b^2c^2+2c_{,}a_{,}c^2a^2\}, \quad (141.)\end{aligned}$$

which may be put under the form

$$\begin{aligned}&[2\sigma+(R_1-R_2)\{a_{,}(b^2+c^2)+b_{,}(c^2+a^2)+c_{,}(a^2+b^2)\}]^2\\ &= (R_1+R_2)^2\{a_{,}^2(b^2+c^2)^2+b_{,}^2(c^2+a^2)^2+c_{,}^2(a^2+b^2)^2+2a_{,}b_{,}(a^2b^2-c^2)\\ &\quad +2b_{,}c_{,}(b^2c^2-a^2)+2c_{,}a_{,}(c^2a^2-b^2)\}\\ &= (R_1+R_2)^2\{(c_{,}-a_{,})^2(b^2+c^2)^2+(c_{,}-b_{,})^2(a^2+c^2)^2+2(c_{,}-a_{,})(c_{,}-b_{,})(a^2b^2-c^2)\}\\ &= (R_1+R_2)^2\{(a_{,}-b_{,})^2(c^2+a^2)^2+(a_{,}-c_{,})^2(b^2+a^2)^2+2(a_{,}-b_{,})(a_{,}-c_{,})(b^2c^2-a^2)\}. \quad (142.)\end{aligned}$$

17. When any two of the three quantities $a_{,}, b_{,}, c_{,}$ are equal to each other, the last expression (142.) becomes an exact square, & the quadratic equation (141.) resolves itself into two linear factors. For example, in the case (93.) when

$$b_{,}=a_{,};$$

the roots of the quadratic (141.) may be thus separately and rationally expressed,

$$\left.\begin{aligned}\sigma_1 &= -a_{,}(R_1-R_2)+(c_{,}-a_{,})R_2(1-c^2),\\ \sigma_2 &= -a_{,}(R_1-R_2)+(a_{,}-c_{,})R_1(1-c^2),\end{aligned}\right\} \quad (143.)$$

and are connected by (140.) with the following separate & rational values of g_1, g_2,

$$g_1=-\frac{a}{b}, \quad g_2=\frac{b}{a}, \quad (144.)$$

which are to be substituted in the equations (136.), while the equation (131.) of the plane of nearly tangential vibrations becomes

$$0=R_1\grave{\grave{\zeta}}+3(R_1+R_2)(a_{,}-c_{,})c(a\grave{\grave{\xi}}+b\grave{\grave{\eta}}), \quad (145.)$$

& the formula (89.) for the velocity of propagation of the wave becomes

$$s^2k^{-2}=R_0+R_1+2a_{,}(R_0+3R_1+3R_2)+c_{,}(R_1+3R_2)+2(c_{,}-a_{,})c^2(R_0+2R_1)-2\sigma; \quad (146.)$$

so that in this case the two directions of nearly tangential vibration, with the two corresponding velocities of propagation of the wave, are expressed in the following manner,*

$$\left.\begin{aligned}&a\grave{\xi}_1+b\grave{\eta}_1=0, \quad \grave{\zeta}_1=0,\\ &s_1^2k^{-2}=R_0+R_1+2a_{,}(R_0+4R_1+3R_2)+c_{,}(R_1+R_2)+2(c_{,}-a_{,})(R_0+2R_1+R_2)c^2,\end{aligned}\right\} \quad (147.)$$

and

$$\left.\begin{aligned}&\frac{\grave{\xi}_2}{a}=\frac{\grave{\eta}_2}{b}=\frac{\grave{\grave{\zeta}}_2}{c-\dfrac{1}{c}}\left\{1+3(c_{,}-a_{,})\frac{R_1+R_2}{R_1}\right\},\\ &s_2^2k^{-2}=R_0+R_1+2a_{,}(R_0+3R_1+2R_2)+3c_{,}(R_1+R_2)+2(c_{,}-a_{,})(R_0+R_1)c^2.\end{aligned}\right\} \quad (148.)$$

* [In the equations determining the directions of the vibrations the square of $(c_{,}-a_{,})$ is neglected.]

In the same case (93.) the direction of nearly normal vibration & the corresponding velocity of the wave are determined by the following formula, deduced from equations (133.) & (132.) combined with (124.) & (89.):

$$\left.\begin{aligned}\frac{\xi_3'}{a}=\frac{\eta_3'}{b}=\frac{\zeta_3'}{c}\left\{1+3(a,-c,)\frac{R_1+R_2}{R_1}\right\},\\ s_3^2k^{-2}=R_0+3R_1+2a,(R_0+8R_1+9R_2)+c,(R_1+3R_2)\\ +2(c,-a,)(R_0+7R_1+6R_2)c^2.\end{aligned}\right\} \quad (149.)$$

These results agree with and include those of the 14th & 15th paragraphs; and give the following expression for the difference of the squares of the two velocities of propagation of a wave, corresponding to the two directions of tangential or nearly tangential vibration,

$$s_1^2k^{-2}-s_2^2k^{-2}=2(a,-c,)(R_1+R_2)(1-c^2). \quad (150.)$$

[*Fresnel's wave surface.*]

[**18.**] $0=\sigma^2+\sigma(R_1-R_2)\{a,(b^2+c^2)+b,(c^2+a^2)+c,(a^2+b^2)\}$
$+(R_1^2+R_2^2)(a,b,c^2+b,c,a^2+c,a,b^2)(a^2+b^2+c^2)$
$-R_1R_2\{a,^2(b^2+c^2)^2+b,^2(c^2+a^2)^2+2a,b,a^2b^2+2b,c,b^2c^2+2c,a,c^2a^2\};$

if $c,=0$, then

$$0=\sigma^2+\sigma(R_1-R_2)\{a,(b^2+c^2)+b,(c^2+a^2)\}$$
$$+(R_1^2+R_2^2)a,b,c^2(a^2+b^2+c^2)-R_1R_2\{a,^2(b^2+c^2)^2+2a,b,a^2b^2+b,^2(c^2+a^2)^2\};$$

also

$$2\sigma=\{-s^2k^{-2}+R_0+R_1+(R_1+3R_2)(a,+b,)\}(a^2+b^2+c^2)+2(R_0+2R_1)(a,a^2+b,b^2);$$

when $a=0,\ b=0,\ c=1$, then $\sigma=R_2a,-R_1b,$ or $\sigma=R_2b,-R_1a,$;

when $b=0,\ c=0,\ a=1$, then $\sigma=-R_1b,$ or $\sigma=R_2b,$;

when $c=0,\ a=0,\ b=1$, then $\sigma=-R_1a,$ or $\sigma=R_2a,$.

In the 1st case

$$s^2k^{-2}-(R_0+R_1)=(a,+3b,)(R_1+R_2) \quad \text{or} \quad =(3a,+b,)(R_1+R_2);$$

in the 2nd case

$$=2R_0a,+R_1(5a,+3b,)+3R_2(a,+b,) \quad \text{or} \quad =2R_0a,+R_1(5a,+b,)+R_2(3a,+b,);$$

in the 3rd case

$$=2R_0b,+R_1(3a,+5b,)+3R_2(a,+b,) \quad \text{or} \quad =2R_0b,+R_1(a,+5b,)+R_2(a,+3b,).$$

If these 6 values of s^2k^{-2} be called (1), (2), (3), (4), (5), (6), then

$$(3)-(1)=2a,(R_0+2R_1+R_2);\quad (5)-(2)=2b,(R_0+2R_1+R_2);$$
$$(6)-(4)=2(b,-a,)(R_0+2R_1+R_2).$$

If, then, we suppose $R_0+2R_1+R_2=0$, we shall have

$$(1)=(3)=(R_0+R_1)(1-a,-3b,),$$
$$(2)=(5)=(R_0+R_1)(1-3a,-b,),$$
$$(4)=(6)=(R_0+R_1)(1-a,-b,).$$

Is then

$$\frac{a^2}{s^2k^{-2}-(2)}+\frac{b^2}{s^2k^{-2}-(1)}+\frac{c^2}{s^2k^{-2}-(4)}=0,$$

that is,

$$0=s^4k^{-4}-s^2k^{-2}\{a^2\,\overline{(1)+(4)}+b^2\,\overline{(2)+(4)}+c^2\,\overline{(1)+(2)}\}+a^2(1)(4)+b^2(2)(4)+c^2(1)(2)?$$

$$\frac{a^2(1)(4)}{(R_1+R_2)^2}=a^2(1-2a_{,}-4b_{,}+a_{,}^2+4a_{,}b_{,}+3b_{,}^2),$$

$$\frac{b^2(2)(4)}{(R_1+R_2)^2}=b^2(1-4a_{,}-2b_{,}+3a_{,}^2+4a_{,}b_{,}+b_{,}^2),$$

$$\frac{c^2(1)(2)}{(R_1+R_2)^2}=c^2(1-4a_{,}-4b_{,}+3a_{,}^2+10a_{,}b_{,}+3b_{,}^2);$$

$$\text{Sum}=1-4a_{,}-4b_{,}+3a_{,}^2+4a_{,}b_{,}+3b_{,}^2+2(a_{,}a^2+b_{,}b^2)-2(a_{,}^2a^2-3a_{,}b_{,}c^2+b_{,}^2b^2);$$

$$-a^2\frac{(1)+(4)}{R_1+R_2}=2(1-a_{,}-2b_{,})a^2,$$

$$-b^2\frac{(2)+(4)}{R_1+R_2}=2(1-2a_{,}-b_{,})b^2,$$

$$-c^2\frac{(1)+(2)}{R_1+R_2}=2(1-2a_{,}-2b_{,})c^2;$$

$$\text{Sum}=2(1-2a_{,}-2b_{,})+2(a_{,}a^2+b_{,}b^2).$$

Putting

$$R_1=-R_2+(R_1+R_2),\quad R_2=\rho^{\backprime}(R_1+R_2),\quad \sigma=\sigma^{\backprime}(R_1+R_2),$$

then

$$\frac{s^2k^{-2}}{R_1+R_2}=-1+(2\rho^{\backprime}+1)(a_{,}+b_{,})-2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-2\sigma^{\backprime}.$$

$$\begin{aligned}\because\quad 0=&\,1-4a_{,}-4b_{,}+3a_{,}^2+4a_{,}b_{,}+3b_{,}^2+2(a_{,}a^2+b_{,}b^2)-2(a_{,}^2a^2-3a_{,}b_{,}c^2+b_{,}^2b^2)\\&-2(1-2a_{,}-2b_{,}+a_{,}a^2+b_{,}b^2)\{1+2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-(2\rho^{\backprime}+1)(a_{,}+b_{,})+2\sigma^{\backprime}\}\\&+\{1+2\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-(2\rho^{\backprime}+1)(a_{,}+b_{,})+2\sigma^{\backprime}\}^2=\Sigma_0^{\backprime}+\Sigma_1^{\backprime}+\Sigma_2^{\backprime};^*\end{aligned}$$

where

$$\Sigma_0^{\backprime}=1-2+1=0,$$

$$\begin{aligned}\Sigma_1^{\backprime}=&-4a_{,}-4b_{,}+2(a_{,}a^2+b_{,}b^2)+4(a_{,}+b_{,})-2(a_{,}a^2+b_{,}b^2)-4\rho^{\backprime}(a_{,}a^2+b_{,}b^2)\\&+2(2\rho^{\backprime}+1)(a_{,}+b_{,})-4\sigma^{\backprime}+4\rho^{\backprime}(a_{,}a^2+b_{,}b^2)-2(2\rho^{\backprime}+1)(a_{,}+b_{,})+4\sigma^{\backprime}=0,\end{aligned}$$

$$\tfrac{1}{4}\Sigma_2^{\backprime}=\sigma^{\backprime 2}+\sigma^{\backprime}(1-2\rho^{\backprime})(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)+a_{,}b_{,}c^2-\rho^{\backprime}(1-\rho^{\backprime})(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)^2;$$

$$\begin{aligned}\because\quad 0=&\,\sigma^2+\sigma(R_1-R_2)(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)+a_{,}b_{,}c^2(R_1+R_2)^2\\&-R_1R_2(a_{,}+b_{,}-a_{,}a^2-b_{,}b^2)^2;\end{aligned}$$

$$\begin{aligned}0=&\,\sigma^2+\sigma(R_1-R_2)\{a_{,}(b^2+c^2)+b_{,}(c^2+a^2)\}+(R_1^2+R_2^2)a_{,}b_{,}c^2(a^2+b^2+c^2)\\&-R_1R_2\{a_{,}^2(b^2+c^2)^2+2a_{,}b_{,}a^2b^2+b_{,}^2(a^2+c^2)\};\end{aligned}$$

& this quadratic equation does in fact agree with that lately found for σ. *Thus Fresnel's law for the square of the normal velocity of a wave in a biaxal crystal would in fact result from the supposition of a rectangular and nearly cubical arrangement of attracting and repelling points,*

* [I.e. absolute term, terms of 1st degree, terms of 2nd degree in $a_{,}$, $b_{,}$, $\sigma^{\backprime}$.]

if we suppose the law of force & the mean interval to be such that $R_0 + 2R_1 + R_2 = 0$*: & the vibrations would then be perpendicular to the plane of polarisation, at least for uniaxal and ∵ probably for biaxal crystals.*

But if, instead, we suppose R_0 and R_1 to be so small with respect to R_2 that when multiplied by $a,$ or $b,$ they may be neglected, then the quadratic in σ becomes

$$0 = \sigma^2 - \sigma R_2 (a, + b, - a, a^2 - b, b^2) + R_2^2 a, b, c^2;$$

& the question now is, whether this quadratic can be deduced from the developement of the formula

$$0 = \frac{a^2}{\sigma - R_2 a,} + \frac{b^2}{\sigma - R_2 b,} + \frac{c^2}{\sigma};$$

which gives, accordingly,

$$0 = \sigma^2 - \sigma R_2 \{a, (b^2 + c^2) + b, (c^2 + a^2)\} + R_2^2 a, b, c^2.$$

Thus we should equally deduce Fresnel's law of velocity of a wave in a biaxal crystal, by supposing R_0 & R_1 very small with respect to R_2; but we should now have the vibrations *parallel* to the plane of polarisation, at least for uniaxal crystals and probably for biaxal also.

Supposing $R_0 = -2R_1 - R_2$, we shall deduce, very nearly, Fresnel's law of polarisation in a biaxal crystal if we can prove that ξ', η', ζ' are very nearly proportional to

$$\frac{a}{\sigma + R_2 (a, a^2 + b, b^2) + R_1 a, - R_2 b,}, \quad \frac{b}{\sigma + R_2 (a, a^2 + b, b^2) + R_1 b, - R_2 a,}, \quad \frac{c}{\sigma + R_2 (a, a^2 + b, b^2) - R_2 (a, + b,)}.$$

That is,

$$0 = \{\sigma + R_2 (a, a^2 + b, b^2 - b,)\} (c\xi' - a\zeta') + a, (R_1 c\xi' + R_2 a\zeta'),$$

$$0 = \{\sigma + R_2 (a, a^2 + b, b^2 - a,)\} (c\eta' - a\zeta') + b, (R_1 c\eta' + R_2 b\zeta');$$

that is [(125.)], since $a\xi' + b\eta' + c\zeta' = 0$ very nearly,

$$0 = \xi'' \{\sigma + R_2 (b, b^2 - b,) + a, R_1 (b^2 + c^2)\} - ab\eta'' (R_1 + R_2) a,,$$

$$0 = \eta'' \{\sigma + R_2 (a, a^2 - a,) + b, R_1 (a^2 + c^2)\} - ab\xi'' (R_1 + R_2) b,;$$

& these do in fact agree with the equations (129.), (130.).

But if R_0 and R_1 be very small compared with R_2, these last equations give

$$\frac{\xi''}{\eta''} = \frac{aba, R_2}{\sigma + R_2 (b^2 - 1) b,} = \frac{\sigma + R_2 (a^2 - 1) a,}{abb, R_2};$$

also

$$\frac{\xi'}{\eta'} = \frac{(a^2 - 1)\xi'' + ab\eta''}{(b^2 - 1)\eta'' + ab\xi''}, \quad \frac{\zeta'}{\eta'} = \frac{ac\xi'' + bc\eta''}{(b^2 - 1)\eta'' + ab\xi''};$$

is $\dfrac{a\xi'}{\sigma - R_2 a,} + \dfrac{b\eta'}{\sigma - R_2 b,} + \dfrac{c\zeta'}{\sigma} = 0$? is $\dfrac{a (c\xi' - a\zeta')}{\sigma - R_2 a,} + \dfrac{b (c\eta' - b\zeta')}{\sigma - R_2 b,} = 0$?

is $\dfrac{a\xi''}{\sigma - R_2 a,} + \dfrac{b\eta''}{\sigma - R_2 b,} = 0$?

is $(\sigma - R_2 b,) (\sigma - R_2 a, + R_2 a^2 a,) + b^2 b, R_2 (\sigma - R_2 a,) = 0$?

is $\sigma^2 - \sigma R_2 \{a, (b^2 + c^2) + b, (a^2 + c^2)\} + R_2^2 a, b, c^2 = 0$? Yes.

Reflexion & Refraction.

(June 27, 1838.) (See the series of sheets begun at Kilmore near Nenagh, Sept. 27th, 1836.)*

[**19.**] Components of attractive action of $x+\Delta x, y+\Delta y, z+\Delta z$ on $x, y, z \ldots \frac{\Delta x}{r}f(r), \frac{\Delta y}{r}f(r), \frac{\Delta z}{r}f(r)$. Suppose $S.f(r)=0$ and $S.rf'(r)=0$, in order to account for the equilibrium of a point placed at the separating surface of two ordinary media.

To account for the change of velocity which corresponds to ordinary refraction & yet to leave the density almost unaltered, suppose

$$\frac{S.r\{r^2f'(r)+4rf(r)\}'}{S.\{r^2f'(r)+4rf(r)\}} \text{ very great;}$$

but

$$\frac{S.r[r\{r^2f'(r)+4rf(r)\}']'}{S.r\{r^2f'(r)+4rf(r)\}'}$$

not very great, at least for the gases; because taking the velocity in vacuo for unity, we are to have

$$\frac{30}{\mu^2}=S.\{(r+\alpha r)^2f'(r+\alpha r)+4(r+\alpha r)f(r+\alpha r)\}, \quad 30=S.\{r^2f'(r)+4rf(r)\},$$

$$\frac{\mu^2-1}{\mu^2}=A_1\alpha+A_2\alpha^2+\ldots, \quad A_1=\frac{S.\{r^3f''(r)+6r^2f'(r)+4rf(r)\}}{S.\{r^2f'(r)+4rf(r)\}}, \quad A_2=\ldots.$$

However, for the gases, the extreme smallness of dilatation of the ether may be sufficient to account for the (probably) near proportion of this to the change of velocity.

To account for the independence of velocity on colour in vacuo, suppose

$$S.\{r^4f'(r)+6r^3f(r)\}=0;$$

or at least

$$\frac{S.r\{r^4f'(r)+6r^3f(r)\}'}{S.\{r^4f'(r)+6r^3f(r)\}} \text{ very great.}$$

We are then to suppose in like manner

$$\frac{S.r\{r^6f'(r)+8r^5f(r)\}'}{S.\{r^6f'(r)+8r^5f(r)\}}=0, \text{ \&c.}$$

We have ∵ been led by *both refraction and dispersion* to suppose the quotient

$$\frac{S.r\{r^{2i}f'(r)+\overline{2i+2}\,r^{2i-1}f(r)\}'}{S.\{r^{2i}f'(r)+\overline{2i+2}\,r^{2i-1}f(r)\}} \text{ great;}$$

* [These sheets seem to have been lost. If we take equations (13.), p. 417, and replace $mf(r)$ by $\frac{f(r)}{r}$ and expand, we get, on taking average values for Δx^2, Δx^4, etc., i.e. $\frac{1}{3}r^2$, $\frac{1}{5}r^4$, etc.,

$$(\text{velocity})^2=\frac{1}{30}S.\{r^2f'(r)+4rf(r)\}-\frac{\pi^2}{105\lambda^2}S.\{r^4f'(r)+6r^3f(r)\}+\text{etc.}$$

In the other medium every distance r is supposed to be increased to $r+\alpha r$.]

& it is remarkable that this quotient becomes, when the sums for all space are changed to sums along one radius vector & these again to single definite integrals,

$$\frac{\int_0^\infty r^3\{r^{2i}f'(r)+\overline{2i+2}\,r^{2i-1}f(r)\}'\,dr}{\int_0^\infty \{r^{2i+2}f'(r)+\overline{2i+2}\,r^{2i+1}f(r)\}\,dr},$$

in which denominator $=\Delta_0^\infty\{r^{2i+2}f(r)\}$ & numerator $=\Delta_0^\infty\{r^{2i+3}f'(r)+(2i-1)\,r^{2i+2}f(r)\}$; so that $\Delta_0^\infty.\,r^3\{r^{2i}f(r)\}'$ ought on this hypothesis to be greater than $\Delta_0^\infty.\,r^2\{r^{2i}f(r)\}$.

(June 28th.)

Let refracting surface be plane of xy, z being negligible for incident and reflected rays, but positive for refracted. Also let all the vibrations be parallel to y, so that $\xi=0$, $\zeta=0$. Let incident vibration be

$$\eta_1=\eta_1'\cos\left(\frac{2\pi}{\lambda}\overline{z-t}\right),\quad \text{refracted } \eta_2=\eta_2'\cos\left(\frac{2\pi}{\lambda}\overline{\mu z-t}\right),\quad \text{reflected } \eta_3=\eta_3'\cos\left(\frac{2\pi}{\lambda}\overline{z+t}\right),$$

the incidence being perpendicular. When $z<0$, $\eta=\eta_1+\eta_3$; when $z>0$, $\eta=\eta_2$; when $z=0$, both expressions hold, $\eta_2'=\eta_1'+\eta_3'$. When $z<0$, x, y, z are integers i, k, l, l being negative; when $z>0$, x, y, z are integers i, k, l multiplied by $1+\alpha$, α being a *very* small fraction positive or negative & l being positive. The components of attraction on that point, which when at rest was 0, 0, 0 & when in motion 0, $\eta_2'\cos\frac{2\pi t}{\lambda}$, 0, are, in the directions $+x$, $+y$, $+z$,

$$S.\frac{x}{r}\frac{f(r+\epsilon r)}{1+\epsilon}=S.\frac{x\epsilon}{r}\{rf'(r)-f(r)\}=S.\frac{xy}{r^3}\{rf'(r)-f(r)\}\left(\eta-\eta_2'\cos\frac{2\pi t}{\lambda}\right),^*$$

$$S.\frac{y}{r}\frac{f(r+\epsilon r)}{1+\epsilon}+S.\left(\eta-\eta_2'\cos\frac{2\pi t}{\lambda}\right)\frac{f(r)}{r}=S.\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\}\left(\eta-\eta_2'\cos\frac{2\pi t}{\lambda}\right),$$

$$S.\frac{z}{r}\frac{f(r+\epsilon r)}{1+\epsilon}=S.\frac{z\epsilon}{r}\{rf'(r)-f(r)\}=S.\frac{zy}{r^3}\{rf'(r)-f(r)\}\left(\eta-\eta_2'\cos\frac{2\pi t}{\lambda}\right).$$

The 1st and 3rd vanish; the 2nd ought to be $=-\left(\frac{2\pi}{\lambda}\right)^2\eta_2'\cos\frac{2\pi t}{\lambda}$. We ought ∵ to have

$$\left(\frac{2\pi}{\lambda}\right)^2=\underset{z>0}{S}.\operatorname{vers}\frac{2\pi\mu z}{\lambda}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\}+\underset{z<0}{S}.\operatorname{vers}\frac{2\pi z}{\lambda}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\},$$

$$0=(\eta_1'+\eta_3')\underset{z>0}{S}.\sin\frac{2\pi\mu z}{\lambda}\left\{\frac{y^2}{r^3}\{\quad\}+\frac{f(r)}{r}\right\}+(\eta_1'-\eta_3')\underset{z<0}{S}.\sin\frac{2\pi z}{\lambda}\left\{\frac{y^2}{r^3}\{\quad\}+\frac{f(r)}{r}\right\};$$

that is,

$$1=\left(\mu^2\underset{z>0}{S}+\underset{z<0}{S}\right).\frac{z^2}{2}\left\{\frac{y^2}{r^3}\{rf'(r)-f(r)\}+\frac{f(r)}{r}\right\};$$

$$0=\left\{(\eta_1'+\eta_3')\,\mu\underset{z>0}{S}+(\eta_1'-\eta_3')\underset{z<0}{S}\right\}\left\{\frac{zy^2}{r^3}\{rf'(r)-f(r)\}+\frac{zf(r)}{r}\right\}.$$

* [ϵr is the increase in r when y is changed to $y+\eta-\eta_2\cos\frac{2\pi t}{\lambda}$.]

The 1st is satisfied because*

$$\mu^2 \underset{z>0}{S}(\&c.) = \underset{z<0}{S}(\&c.) = \tfrac{1}{2};$$

the 2nd becomes

$$0=\left(\mu^2 \underset{z>0}{S} + \underset{z<0}{S}\right)\left\{\frac{zy^2}{r^3}\{rf'(r)-f(r)\}+\frac{zf(r)}{r}\right\},$$

when we introduce the lately established laws of reflection and refraction [i.e. $\eta_1' + \eta_3' = \mu(\eta_1' - \eta_3')$]. Now

$$\underset{z>0}{S} F(x, y, z, r) = \underset{l>0}{S} F(i+\alpha i, k+\alpha k, l+\alpha l, n+\alpha n);$$

$$\underset{z>0}{S}\left\{\frac{zy^2}{r^3}\{rf'(r)-f(r)\}+\frac{zf(r)}{r}\right\} = \underset{l>0}{S}\left\{\frac{lk^2}{n}\left(\frac{f(n)}{n}\right)' + \frac{lf(n)}{n}\right\}$$
$$+\alpha \underset{l>0}{S}\left\{lk^2\left(\frac{f(n)}{n}\right)'' + \frac{lf(n)}{n} + \frac{l}{n}(2k^2+n^2)\left(\frac{f(n)}{n}\right)'\right\} = \alpha \underset{l>0}{S}.\, lk^2\frac{f''(n)}{n},$$

because $Sf(n)=0$, $S.\,nf'(n)=0$; & the same reason which led us to establish these 2 last equations seems sufficient to lead us to establish also the equation $S.\,n^2f''(n)=0$; whereby the condition ending the last sentence is satisfied, even without supposing, according to the laws of reflected and refracted vibrations $\eta_1' + \eta_3' = \mu(\eta_1' - \eta_3')$. Thus the case of *perpendicular incidence* appears to be satisfactorily explained even without introducing the law of the *vis viva*, provided that we do employ the principle of *equivalent vibrations*.

[*Extension of above to dispersion in a biaxal crystal.*]

(July 9th, 1838.)

[20.]

$$\frac{d^2x}{dt^2} = S.\,\Delta x\frac{f(r)}{r}; \quad \frac{d^2\delta x}{dt^2} = S.\left\{\Delta\delta x\frac{f(r)}{r} + \Delta x\left(\frac{f(r)}{r}\right)'\delta r\right\},$$

$$\delta r = \frac{\Delta x}{r}\Delta\delta x + \frac{\Delta y}{r}\Delta\delta y + \frac{\Delta z}{r}\Delta\delta z; \quad \delta x = \text{const.} \times \cos\frac{2\pi}{\lambda}(t-V), \quad V = \sigma x + \tau y + v z;$$

$$\Delta\delta x = -2\delta x\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2 + \text{useless terms [i.e. terms which vanish on summation]};$$

$$\Delta V = \sigma\Delta x + \tau\Delta y + v\Delta z; \quad \sigma^2+\tau^2+v^2 = \omega^{-2};$$

let

$$L = S.\left\{\frac{f(r)}{r} + \frac{\Delta x^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega^2}{2\pi^2}\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2, \quad M = \quad, \quad N = \quad,$$

$$P = S.\frac{\Delta y\Delta z}{r}\left(\frac{f(r)}{r}\right)'\frac{\lambda^2\omega^2}{2\pi^2}\left(\sin\frac{\pi\Delta V}{\lambda}\right)^2, \quad Q = \quad, \quad R = \quad,$$

then

$$0 = (L-\omega^2)\delta x + R\delta y + Q\delta z = (M-\omega^2)\delta y + P\delta z + R\delta x = (N-\omega^2)\delta z + Q\delta x + P\delta y;$$

let plane of polarisation be xy, then we have 2 different velocities ω_1, ω_2, determined as follows:

1st... $\Delta V = \sigma_1\Delta x = \omega_1^{-1}\Delta x, \quad \delta x = 0, \quad \delta z = 0, \quad P = 0, \quad Q = 0, \quad R = 0,$

$$\omega_1^2 = M = S.\left\{\frac{f(r)}{r} + \frac{\Delta y^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega_1^2}{2\pi^2}\left(\sin\frac{\pi\Delta x}{\lambda\omega_1}\right)^2$$

$$= \tfrac{1}{2}S\left\{\frac{f(r)}{r}\Delta x^2 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\} - \frac{\pi^2}{6\lambda^2\omega_1^2}S\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\} + \&c.;$$

* [This result follows from the equations in the footnote of page 435 when we use the fact that the average values over a hemisphere of y^2z^2 and z^2 are $r^4/30$ and $r^2/6$ respectively.]

$$\text{II}^{\text{nd}}\ldots\ \omega_2^2 = \tfrac{1}{2}S\left\{\frac{f(r)}{r}\Delta y^2 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\} - \frac{\pi^2}{6\lambda^2\omega_2^2}S\left\{\frac{f(r)}{r}\Delta y^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta y^4\Delta x^2}{r}\right\} + \&\text{c.}$$

Is $\omega_1^2 = \omega_2^2$ sensibly?

$$\Delta x = \Delta x_{,}(1+a_{,}),\quad \Delta y = \Delta y_{,}(1+b_{,}),\quad \Delta z = \Delta z_{,}(1+c_{,}),\quad r = r_{,} + \frac{a_{,}\Delta x_{,}^2 + b_{,}\Delta y_{,}^2 + c_{,}\Delta z_{,}^2}{r_{,}};$$

$a_{,}, b_{,}, c_{,}$ are small dilatations or contractions & the lower accents refer to a cubical system;

$$\tfrac{1}{2}S.\frac{f(r)}{r}\Delta x^2 = \frac{1+2a_{,}}{2}S.\frac{f(r_{,})}{r_{,}}\Delta x_{,}^2 + \tfrac{1}{2}S.\left(\frac{f(r_{,})}{r_{,}}\right)'\frac{\Delta x_{,}^2}{r_{,}}(a_{,}\Delta x_{,}^2 + b_{,}\Delta y_{,}^2 + c_{,}\Delta z_{,}^2)$$

$$= \frac{1+2a_{,}}{6}S.r_{,}f(r_{,}) + \tfrac{1}{30}(3a_{,}+b_{,}+c_{,})\,S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)';$$

$$\tfrac{1}{2}S.\left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r} = \frac{1+2a_{,}+2b_{,}}{2}S.\left(\frac{f(r_{,})}{r_{,}}\right)'\frac{\Delta x_{,}^2\Delta y_{,}^2}{r_{,}} + \tfrac{1}{2}S.\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)'\frac{\Delta x_{,}^2\Delta y_{,}^2}{r_{,}}(a_{,}\Delta x_{,}^2 + \quad + \quad)$$

$$= \frac{1+2a_{,}+2b_{,}}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)' + \frac{3a_{,}+3b_{,}+c_{,}}{210}S.r_{,}^5\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)';$$

$$\because\ (\text{if } \lambda=\infty)\quad \omega_1^2 = (1+2a_{,})\left\{\tfrac{1}{6}S.r_{,}f(r_{,}) + \tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'\right\}$$

$$+(3a_{,}+3b_{,}+c_{,})\left\{\tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)' + \tfrac{1}{210}S.r_{,}^5\left(\frac{1}{r_{,}}\left(\frac{f(r_{,})}{r_{,}}\right)'\right)'\right\};$$

$$\because\qquad \omega_1^2 - \omega_2^2 = 2(a_{,}-b_{,})\left\{\tfrac{1}{6}S.r_{,}f(r_{,}) + \tfrac{1}{30}S.r_{,}^3\left(\frac{f(r_{,})}{r_{,}}\right)'\right\} = 2(a_{,}-b_{,})$$

& $\because$ insensible if the value of ω_1 for the cubical system be unity.

But we must show also that

$$S.\left\{\frac{f(r)}{r}(\Delta x^2 + \Delta y^2) + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta y^2}{r}\right\}(\Delta x^2 - \Delta y^2)$$

is much less than
$$S.\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\},$$

in order to account for the existence of Fresnel's law of propagation in dispersive crystals.

$$S.\frac{f(r)}{r}\Delta x^4 = \frac{1+4a_{,}}{5}S_{,}r^3f(r) + \frac{5a_{,}+b_{,}+c_{,}}{35}S_{,}r^5\left(\frac{f(r)}{r}\right)',^{*}$$

$$S.\left(\frac{f(r)}{r}\right)'\Delta x^4\Delta y^2 = \frac{1+4a_{,}+2b_{,}}{35}S_{,}r^5\left(\frac{f(r)}{r}\right)' + \frac{5a_{,}+3b_{,}+c_{,}}{315}S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)',$$

$$\because\ \frac{\text{diff}^{\text{ce}}}{\text{sum}}\text{ of } S\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta y^2}{r}\right\}\text{ and } S\left\{\frac{f(r)}{r}\Delta y^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta y^4\Delta x^2}{r}\right\}$$

$$= (a_{,}-b_{,})$$

$$\times\frac{126S_{,}r^3f(r) + 27S_{,}r^5\left(\frac{f(r)}{r}\right)' + S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)'}{63(1+2a_{,}+2b_{,})S_{,}r^3f(r) + 9(1+6a_{,}+6b_{,}+c_{,})S_{,}r^5\left(\frac{f(r)}{r}\right)' + (4a_{,}+4b_{,}+c_{,})S_{,}r^7\left(\frac{1}{r}\left(\frac{f(r)}{r}\right)'\right)'}$$

$$= \frac{a_{,}-b_{,}}{9}\,\frac{18 + S_{,}\left\{9r^5\left(\frac{f}{r}\right)' + r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\} \div S_{,}\left\{7r^3f + r^3\left(\frac{f}{r}\right)'\right\}}{1+2a_{,}+2b_{,}+\frac{1}{9}(4a_{,}+4b_{,}+c_{,})\,S_{,}\div S_{,}\ (\text{as in num}^{\text{r}})}$$

$$= \frac{a_{,}-b_{,}}{4a_{,}+4b_{,}+c_{,}}\text{ nearly;}$$

* [$S_{,}r^3f(r)$ means $S.r_{,}^3f(r_{,})$, etc.]

which may easily be supposed to be very small. Yet this result seems to indicate that Fresnel's law of propagation will hold less perfectly for the more refrangible rays.

Consider III[rd] that velocity ω_3 which corresponds to the same plane wave as in I[st] case, but to vibrations parallel to z & $\because$ perpendicular to axis of crystal; that is,

$$\text{III}^{\text{rd}}\ldots \qquad \Delta V = \sigma_3 \Delta x = \omega_3^{-1}\Delta x, \quad \delta x = 0, \quad \delta y = 0,$$

$$\omega_3^2 = N = S.\left\{\frac{f(r)}{r} + \frac{\Delta z^2}{r}\left(\frac{f(r)}{r}\right)'\right\}\frac{\lambda^2\omega_3^2}{2\pi^2}\left(\sin\frac{\pi\Delta x}{\lambda\omega_3}\right)^2$$

$$= \tfrac{1}{2}S\left\{\frac{f(r)}{r}\Delta x^2 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^2\Delta z^2}{r}\right\} - \frac{\pi^2}{6\lambda^2\omega_3^2}S\left\{\frac{f(r)}{r}\Delta x^4 + \left(\frac{f(r)}{r}\right)'\frac{\Delta x^4\Delta z^2}{r}\right\} + \&\text{c}.$$

$$= (1+2a_,)\,S_,\left\{\tfrac{1}{6}rf + \tfrac{1}{30}r^3\left(\frac{f}{r}\right)'\right\} + (3a_, + b_, + 3c_,)\,S_,\left\{\tfrac{1}{30}r^3\left(\frac{f}{r}\right)' + \tfrac{1}{210}r^5\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}$$

$$- \frac{\pi^2}{1890\lambda^2\omega_3^2}\left\{63\,(1+4a_,)\,S_, r^3 f + 9\,(1+9a_, + b_, + 3c_,)\,S_, r^5\left(\frac{f}{r}\right)'\right.$$

$$\left. + (5a_, + b_, + 3c_,)\,S_, r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\};$$

$$\omega_1^2 - \omega_3^2 = 2\,(b_, - c_,)\left[S_,\left\{\tfrac{1}{30}r^3\left(\frac{f}{r}\right)' + \tfrac{1}{210}r^5\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\} - \frac{\pi^2}{1890\lambda^2\omega_1^2}S_,\left\{9r^5\left(\frac{f}{r}\right)' + r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}\right];$$

which is the effect of a double refraction on the square of velocity of the wave and is sensible on account of the greatness of the factor

$$S_,\left\{\tfrac{1}{30}r^3\left(\frac{f}{r}\right)' + \frac{r^5}{210}\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\}.$$

(July 9[th], 1838.)

I think that [the] ordinary [ray of] violet light should have a slightly but sensibly different velocity in Iceland Spar, according as its course is parallel or perpendicular to the axis & I will ask Prof. Lloyd to try this experimentally.*

(July 10[th], 1838.)

$$\omega_1^2 = 1 + \frac{(3a_, + 3b_, + c_,)}{210}S_,\left\{7r^3\left(\frac{f}{r}\right)' + r^5\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\} - \frac{\pi^2(5a_, + 3b_, + c_,)}{1890\lambda^2\omega_1^2}S_,\left\{9r^5\left(\frac{f}{r}\right)' + r^7\left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)'\right\};$$

$$\omega_2^2 = 1 + \frac{(3a_, + 3b_, + c_,)}{210}S_,\left\{\qquad\right\} - \frac{\pi^2(3a_, + 5b_, + c_,)}{1890\lambda^2\omega_2^2}S_,\left\{\qquad\right\};$$

$$\omega_3^2 = 1 + \frac{(3a_, + b_, + 3c_,)}{210}S_,\left\{\qquad\right\} - \frac{\pi^2(5a_, + b_, + 3c_,)}{1890\lambda^2\omega_3^2}S_,\left\{\qquad\right\}.$$

Let

$$\omega_1^2 = \Omega_1^{(1)} - \frac{\Omega_1^{(2)}}{\lambda^2\omega_1^2}, \quad \omega_2^2 = \Omega_2^{(1)} - \frac{\Omega_2^{(2)}}{\lambda^2\omega_2^2}, \quad \omega_3^2 = \Omega_3^{(1)} - \frac{\Omega_3^{(2)}}{\lambda^2\omega_3^2};$$

hence

$$\Omega_2^{(2)} + \Omega_3^{(2)} = 2\Omega_1^{(2)}, \quad \text{if } c_, = a_,.$$

Using Rudberg's indices as given by Powell in *Phil. Trans.* 1836, Part I, page 18, I find for Iceland Spar by calculations of this date... $(\omega_H^{-1})_1 = 1{,}6806$; $(\omega_H^{-1})_2 = 1{,}6860$. These then I con-

* I did ask him in presence of Poggendorf and Lord Adare in Observatory (S.E.L. room) on July 10[th].

jecture or rather conclude from the foregoing theory (if *it* be right) to be the two indices of refraction of an *ordinary* ray H according as it passes through Iceland Spar in one or other of two different directions, namely perpendicular or parallel to axis [the axis of y].

(July 11th, 1838.)

But I believe that it will be more accurate to consider the observed quantity 0,00014478... as being $\Omega_1^{(2)}$ and not as $\dfrac{\Omega_1^{(2)}+\Omega_2^{(2)}}{2}$. [Then] $(\omega_H^{-1})_2 = 1{,}69152$. I am led to think it likely that the value of the same index for light *parallel to axis* would be found to be 1,6915: the difference being about the 200th part of either. Malus's differences between observation and calculation seem to have been about as great as this, sometimes.

(July 12th, 1838.)

The comparison of those parts of the squares of the ordinary and extraordinary velocities which are *independent of dispersion* give $b_, = \frac{8}{5}a_,$ & the comparison of those parts which *depend on dispersion* give $b_, = 2a_,$; so that, by each method, the *ether in Iceland Spar* ought to be about twice as much dilated or compressed (I suspect dilated) in the direction of the axis of the crystal as in a direction perpendicular thereto.

Again, supposing $1-\Omega_1^{(1)} : 1-\Omega_3^{(1)} = 4a_, + 3b_, : 6a_, + b_,$, we see that the second changes faster than the first when $a_,$ recedes from 0 and $b_,$ approaches 0, if this recession and this approach be nearly equal. Suppose that $db_, = -\frac{4}{3}da_,$, $a_, > 0$, $da_, > 0$, then $d\Omega_1^{(1)} = 0$, $\dfrac{d(1-\Omega_3^{(1)})}{1-\Omega_3^{(1)}} = \dfrac{\frac{14}{3}da_,}{6a_, + b_,}$ $\because > 0$; & thus we shall explain the phenomenon observed by Mitscherlich of the extraordinary index changing much faster than the ordinary with *heat & increasing* while the crystal tended to become cubical....

[21.] How far would my theory agree with the experiments of Malus as cited by Biot, *Traité de Physique*, Vol. III, the plane of incidence containing the axis of the crystal?

Let $$\Delta V = \sigma \Delta x + \tau \Delta y, \quad v = 0.$$

[Introducing new notation

$$\sigma = \omega^{-1}\omega_x, \quad \tau = \omega^{-1}\omega_y, \quad \omega_x^2 + \omega_y^2 = 1,$$

$$R_0' = \frac{\pi^2}{30} S_, r^3 f(r), \quad R_1' = \frac{\pi^2}{210} S_, r^5 \left(\frac{f}{r}\right)', \quad R_2' = \frac{\pi^2}{1890} S_, r^7 \left(\frac{1}{r}\left(\frac{f}{r}\right)'\right)',$$

expressions for L, M, N, R are obtained: $P=0$, $Q=0$. It will be sufficient to quote L and R.]

$$\begin{aligned} L = {} & \omega_x^2\{(1+2a_,)R_0 + (3a_, + b_, + c_,)R_1 + 3(1+4a_,)R_1 + 3(5a_, + b_, + c_,)R_2\} \\ & + \omega_y^2\{(1+2b_,)R_0 + (a_, + 3b_, + c_,)R_1 + (1+2a_, + 2b_,)R_1 + (3a_, + 3b_, + c_,)R_2\} \\ & - \frac{\omega_x^4}{\lambda^2\omega^2}\{(1+4a_,)R_0' + (5a_, + b_, + c_,)R_1' + 5(1+6a_,)R_1' + 5(7a_, + b_, + c_,)R_2'\} \\ & - \frac{2\omega_x^2\omega_y^2}{\lambda^2\omega^2}\{(1+2a_, + 2b_,)R_0' + (3a_, + 3b_, + c_,)R_1' + 3(1+4a_, + 2b_,)R_1' + 3(5a_, + 3b_, + c_,)R_2'\} \\ & - \frac{\omega_y^4}{\lambda^2\omega^2}\{(1+4b_,)R_0' + (a_, + 5b_, + c_,)R_1' + (1+2a_, + 4b_,)R_1' + (3a_, + 5b_, + c_,)R_2'\}; \end{aligned}$$

$$R = 2\omega_x\omega_y\{(1+2a_,+2b_,)R_1+(3a_,+3b_,+c_,)R_2\}$$
$$-\frac{4\omega_x^3\omega_y}{\lambda^2\omega^2}\{(1+4a_,+2b_,)R_1'+(5a_,+3b_,+c_,)R_2'\}-\frac{4\omega_x\omega_y^3}{\lambda^2\omega^2}\{(1+2a_,+4b_,)R_1'$$
$$+(3a_,+5b_,+c_,)R_2'\};$$
$$0=(L-\omega^2)\delta x+R\delta y=(M-\omega^2)\delta y+R\delta x=(N-\omega^2)\delta z;$$

for one set of vibrations $\delta x=0$, $\delta y=0$, $\omega^2=N$; these vibrations are perpendicular to the plane of xy & are tangential to wave & correspond to the extraordinary ray if the axis of the crystal be (as before) the axis of y; for the other set of vibrations, namely those which give the ordinary ray, we have

$$\delta z=0,\quad (\omega^2-L)(\omega^2-M)=R^2,\quad R(\delta y^2-\delta x^2)=(M-L)\,\delta x\,\delta y.$$

[ω^2 is then found, neglecting squares and higher powers of $\lambda^{-2}\omega^{-2}$.]

$$\omega^2=(1+a_,+b_,)(R_0+R_1)+(3a_,+3b_,+c_,)(R_1+R_2)$$
$$-\lambda^{-2}\omega^{-2}\{(1+2a_,+2b_,)(R_0'+R_1')+(4a_,+4b_,+c_,)(R_1'+R_2')\}$$
$$+(a_,-b_,)(\omega_x^2-\omega_y^2)\{R_0+R_1-\lambda^{-2}\omega^{-2}(2R_0'+3R_1'+R_2')\};$$

and this expression may, with sufficient accuracy, be written

$$\omega^2=R_0+R_1+(3a_,+3b_,+c_,)(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')\{(4a_,+4b_,+c_,)+(a_,-b_,)(\omega_x^2-\omega_y^2)\}.$$

This includes the 2 expressions

$$\omega_1^2=R_0+R_1+(3a_,+3b_,+c_,)(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')(5a_,+3b_,+c_,),$$

and
$$\omega_2^2=R_0+R_1+(3a_,+3b_,+c_,)(R_1+R_2)-\lambda^{-2}\omega^{-2}(R_1'+R_2')(3a_,+5b_,+c_,);$$

which correspond (as we have seen) to the cases of an ordinary ray perpendicular and parallel to the axis.

In general the square of the *ordinary* velocity ought, by this theory, for Iceland Spar, to *increase* as the ray recedes from the axis, the increase being proportional to the *square of the sine of the inclination* just as for the extraordinary ray, except that this increase is much less rapid for the ordinary ray. And the double wave surface for Iceland Spar, and for light of any one colour, ought to be *a system of 2 oblate spheroids of revolution touching at their common poles*.

For the extraordinary light

$$\omega^2=N=\omega_x^2\{(1+2a_,)(R_0+R_1)+(3a_,+b_,+3c_,)(R_1+R_2)\}$$
$$+\omega_y^2\{(1+2b_,)(R_0+R_1)+(a_,+3b_,+3c_,)(R_1+R_2)\}$$
$$-\omega_x^2\lambda^{-2}\omega^{-2}\{(1+4a_,)(R_0'+R_1')+(5a_,+b_,+3c_,)(R_1'+R_2')\}$$
$$-\omega_y^2\lambda^{-2}\omega^{-2}\{(1+4b_,)(R_0'+R_1')+(a_,+5b_,+3c_,)(R_1'+R_2')\}$$
$$=R_0+R_1+(2a_,+2b_,+3c_,)(R_1+R_2)-3(a_,+b_,+c_,)\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$+(a_,-b_,)(\omega_x^2-\omega_y^2)\{R_1+R_2-2\lambda^{-2}\omega^{-2}(R_1'+R_2')\}.$$

Or, referring to axis of crystal, by making $\omega_x^2-\omega_y^2=2\omega_x^2-1$ & observing that $c_,=a_,$, we have for ordinary light ...

$$\omega_o^2=R_0+R_1+(4a_,+3b_,)(R_1+R_2)-(4a_,+5b_,)\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$-2(a_,-b_,)\lambda^{-2}\omega^{-2}(R_1'+R_2')\,\omega_x^2;$$

for extraordinary ...

$$\omega_e^2=R_0+R_1+(4a_,+3b_,)(R_1+R_2)-(4a_,+5b_,)\lambda^{-2}\omega^{-2}(R_1'+R_2')$$
$$+2(a_,-b_,)\,\omega_x^2\{R_1+R_2-2\lambda^{-2}\omega^{-2}(R_1'+R_2')\}.$$

Hence $$\omega_e^2 - \omega_o^2 = 2(a, -b,)\,\omega_x^2\{R_1 + R_2 - \lambda^{-2}\omega^{-2}(R_1' + R_2')\},$$

ω_x^2 being square of sine of inclination to axis.*

In the notation lately used

$$\Omega_1^{(2)} = (R_1' + R_2')(6a, +3b,); \quad \Omega_2^{(2)} = (R_1' + R_2')(4a, +5b,); \quad \Omega_3^{(2)} = (R_1' + R_2')(8a, +b,);$$

& $$\Omega_1^{(2)} = \frac{\Omega_2^{(2)} + \Omega_3^{(2)}}{2}.$$

The common difference of the increasing arithmetical progression $\Omega_3^{(2)}, \Omega_1^{(2)}, \Omega_2^{(2)}$ is

$$2(b, -a,)(R_1' + R_2').$$

Making $R_0 + R_1 = 1$ & introducing a and b with the same meanings as Biot, we have

$$a^2 = 1 + (6a, +b,)(R_1 + R_2) - (8a, +b,)(R_1' + R_2')\lambda^{-2}\omega^{-2};$$

$$b^2 = 1 + (4a, +3b,)(R_1 + R_2) - (6a, +3b,)(R_1' + R_2')\lambda^{-2}\omega^{-2};$$

$$\omega_o^2 = b^2 - \frac{\Omega\omega_y^2}{\lambda^2\omega^2}; \quad \omega_e^2 = a^2 - \left(\Psi + \frac{2\Omega}{\lambda^2\omega^2}\right)\omega_y^2;$$

$$\Omega = 2(b, -a,)(R_1' + R_2'); \quad \Psi = 2(a, -b,)(R_1 + R_2);$$

$$a^2 - b^2 = \Psi + \frac{\Omega}{\lambda^2 b^2}; \quad \Psi = a^2 - b^2 - \frac{\Omega}{\lambda^2 b^2};$$

$$\omega_e^2 = a^2\omega_x^2 + \left(b^2 - \frac{\Omega}{\lambda^2 b^2}\right)\omega_y^2.$$

Thus the received constructions and rules for determining the course of the ordinary and extraordinary rays through Iceland Spar are to be modified by supposing that a and b are the two equatorial semidiameters of my two oblate spheroids & that $\sqrt{b^2 - \frac{\Omega^2}{\lambda^2 b^2}}$ is the common polar semidiameter of both. [Hamilton finds from numerical calculations that "Malus' experiments as quoted by Biot are quite inadequate to decide a question which turns on such small quantities."]

(July 13th, 1838.)

The recent calculations, having been carried on nearly to the conclusion without supposing $c, = a,$, apply so far to a biaxal crystal also, and show that the section of the wave surface in such a crystal made by any one of the 3 principal planes is composed of 2 concentric and coaxal ellipses, though one of them is *very nearly circular*, its ellipticity being proportional to the product of the doubly refractive & dispersive powers or to the *difference of the* 2 *dispersions*.

To examine other sections of the biaxal wave surface & in particular to examine *whether it has really two optic axes* now supposed and indeed known to exist, let

$$L = L^{(1)} - \lambda^{-2}\omega^{-2}L^{(2)}, \quad M = \quad, \quad N = \quad, \quad P = \quad, \quad Q = \quad, \quad R = \quad;$$

& let $$\Delta V = \sigma\Delta x + \tau\Delta y + \upsilon\Delta z;$$

* [Various terms in the expressions for ω_o^2 and ω_e^2 have been neglected. They would however have disappeared in the expression for $\omega_e^2 - \omega_o^2$.]

then $L^{(1)}$, $M^{(1)}$, $N^{(1)}$ have all one common part, namely

$$\omega_x^2\{(1+2a,)R_0+(3a,+b,+c,)R_1\}+\omega_y^2\{(1+2b,)R_0+(a,+3b,+c,)R_1\}$$
$$+\omega_z^2\{(1+2c,)R_0+(a,+b,+3c,)R_1\}$$
$$=(R_0+R_1)\{1+2(a,\omega_x^2+b,\omega_y^2+c,\omega_z^2)\}-R_1(1-a,-b,-c,);$$

while the other parts are

$$\omega_x^2\{3(1+4a,)R_1+3(5a,+b,+c,)R_2\}+\omega_y^2\{(1+2a,+2b,)R_1+(3a,+3b,+c,)R_2\}$$
$$+\omega_z^2\{(1+2a,+2c,)R_1+(3a,+b,+3c,)R_2\},$$
$$\omega_y^2\{3(1+4b,)R_1+3(a,+5b,+c,)R_2\}+\omega_z^2\{(1+2b,+2c,)R_1+(a,+3b,+3c,)R_2\}$$
$$+\omega_x^2\{(1+2a,+2b,)R_1+(3a,+3b,+c,)R_2\},$$
$$\omega_z^2\{3(1+4c,)R_1+3(a,+b,+5c,)R_2\}+\omega_x^2\{(1+2a,+2c,)R_1+(3a,+b,+3c,)R_2\}$$
$$+\omega_y^2\{(1+2b,+2c,)R_1+(a,+3b,+3c,)R_2\}.$$

Of these the first is

$$=(a,+b,+c,)(R_1+R_2)+2(R_1+R_2)(a,\omega_x^2+b,\omega_y^2+c,\omega_z^2)+(1-a,-b,-c,)R_1$$
$$+2a,(R_1+R_2)(1+4\omega_x^2)+2\{(a,+b,+c,)(R_1+R_2)+(1-a,-b,-c,)R_1\}\omega_x^2,$$

the 1st line being common to all 3, but the 2nd line being peculiar to $L^{(1)}$. As to $R^{(1)}$, it has been assigned.

Our fundamental *equations of vibration* are $0=(L-\omega^2)\delta x+R\delta y+Q\delta z \;=\; =$. These give

$$\omega^2\xi^{``}=\omega_z(L\delta x+R\delta y+Q\delta z)-\omega_x(Q\delta x+P\delta y+N\delta z),$$
$$\omega^2\eta^{``}=\omega_z(R\delta x+M\delta y+P\delta z)-\omega_y(Q\delta x+P\delta y+N\delta z),$$
$$\omega^2\zeta^{``}=\omega_x(L\delta x+R\delta y+Q\delta z)+\omega_y(R\delta x+M\delta y+P\delta z)+\omega_z(Q\delta x+P\delta y+N\delta z),$$

if $$\xi^{``}=\omega_z\delta x-\omega_x\delta z,\quad \eta^{``}=\omega_z\delta y-\omega_y\delta z,\quad \zeta^{``}=\omega_x\delta x+\omega_y\delta y+\omega_z\delta z.$$

These last expressions give

$$\omega_z\delta x=\xi^{``}-\omega_x(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}),$$
$$\omega_z\delta y=\eta^{``}-\omega_y(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}),$$
$$\omega_z\delta z=\quad-\omega_z(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}).$$

$$\because\quad \omega^2\omega_z\xi^{``}=\omega_z(L\xi^{``}+R\eta^{``})-\omega_x(Q\xi^{``}+P\eta^{``})+\{-\omega_z(\omega_xL+\omega_yR+\omega_zQ)$$
$$+\omega_x(\omega_xQ+\omega_yP+\omega_zN)\}(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}),$$
$$\omega^2\omega_z\eta^{``}=\omega_z(R\xi^{``}+M\eta^{``})-\omega_y(Q\xi^{``}+P\eta^{``})+\{-\omega_z(\omega_xR+\omega_yM+\omega_zP)$$
$$+\omega_y(\omega_xQ+\omega_yP+\omega_zN)\}(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}),$$
$$\omega^2\omega_z\zeta^{``}=\omega_x(L\xi^{``}+R\eta^{``})+\omega_y(R\xi^{``}+M\eta^{``})+\omega_z(Q\xi^{``}+P\eta^{``})$$
$$-\{\omega_x(\omega_xL+\omega_yR+\omega_zQ)+\omega_y(\omega_xR+\omega_yM+\omega_zP)+\omega_z(\omega_xQ+\omega_yP+\omega_zN)\}$$
$$\times(\omega_x\xi^{``}+\omega_y\eta^{``}-\omega_z\zeta^{``}).$$

Put $$R=R^{`}\omega_x\omega_y,\quad P=P^{`}\omega_y\omega_z,\quad Q=Q^{`}\omega_x\omega_z;$$

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then, omitting the small terms containing ζ'' as a factor [i.e. the vibrations are assumed to be nearly tangential],

$$0=\xi''\{-\omega^2+L-Q'\omega_x^2+\omega_x^2(N-L+Q'(\omega_x^2-\omega_z^2)+(P'-R')\omega_y^2)\}$$
$$+\omega_x\omega_y\eta''\{R'-P'+N-L+Q'(\omega_x^2-\omega_z^2)+(P'-R')\omega_y^2\},$$
$$0=\eta''\{-\omega^2+M-P'\omega_y^2+\omega_y^2(N-M+P'(\omega_y^2-\omega_z^2)+(Q'-R')\omega_x^2)\}$$
$$+\omega_x\omega_y\xi''\{R'-Q'+N-M+P'(\omega_y^2-\omega_z^2)+(Q'-R')\omega_x^2\}.$$

But by an investigation made lately [pp. 432–434] these formulae will give *Fresnel's laws of velocity & polarisation* to semiaxes $a_{\prime\prime}$, $b_{\prime\prime}$, $c_{\prime\prime}$, provided we can have the 4 following relations:

$$c_{\prime\prime}^2+(1-\omega_y^2)(b_{\prime\prime}^2-c_{\prime\prime}^2)=(L-Q'\omega_x^2)(1-\omega_x^2)+\omega_x^2(N-Q'\omega_z^2)+\omega_x^2\omega_y^2(P'-R'),$$
$$c_{\prime\prime}^2+(1-\omega_x^2)(a_{\prime\prime}^2-c_{\prime\prime}^2)=(M-P'\omega_y^2)(1-\omega_y^2)+\omega_y^2(N-P'\omega_z^2)+\omega_x^2\omega_y^2(Q'-R'),$$
$$a_{\prime\prime}^2-c_{\prime\prime}^2=R'-P'+N-Q'\omega_z^2-(L-Q'\omega_x^2)+\omega_y^2(P'-R'),$$
$$b_{\prime\prime}^2-c_{\prime\prime}^2=R'-Q'+N-P'\omega_z^2-(M-P'\omega_y^2)+\omega_x^2(Q'-R').$$

Of these 4 equations, it is remarkable that *any one is a consequence of the 3 others*, independently of L, M, N, P', Q', R', ω_x, ω_y and in virtue of the relation $\omega_z^2=1-\omega_x^2-\omega_y^2$. They give

$$c_{\prime\prime}^2=L+M+\omega_z^2(P'+Q'-R')-(L\omega_x^2+M\omega_y^2+N\omega_z^2+2R'\omega_x^2\omega_y^2+2P'\omega_y^2\omega_z^2+2Q'\omega_z^2\omega_x^2),$$
$$a_{\prime\prime}^2=M+N+\omega_x^2(Q'+R'-P')-(L\omega_x^2+M\omega_y^2+N\omega_z^2+2R'\omega_x^2\omega_y^2+2P'\omega_y^2\omega_z^2+2Q'\omega_z^2\omega_x^2),$$
$$b_{\prime\prime}^2=N+L+\omega_y^2(R'+P'-Q')-(L\omega_x^2+M\omega_y^2+N\omega_z^2+2R'\omega_x^2\omega_y^2+2P'\omega_y^2\omega_z^2+2Q'\omega_z^2\omega_x^2);$$

so that it only remains to examine *how nearly these* 3 *quantities are constant.* For this examination it will be convenient to calculate

$$L\omega_x+R\omega_y+Q\omega_z,\quad R\omega_x+M\omega_y+N\omega_z,\quad Q\omega_x+P\omega_y+N\omega_z;$$

or rather

$$L+R'\omega_y^2+Q'\omega_z^2,\quad M+P'\omega_z^2+R'\omega_x^2,\quad N+Q'\omega_x^2+P'\omega_y^2.$$

Any one of these will give the 2 others by transformation of the letters in the cycles a, b, c and x, y, z.

Confining ourselves at first to the terms independent of dispersion, we have

$$c_{\prime\prime}^2=R_0+R_1+(R_1+R_2)(3a_\prime+3b_\prime+c_\prime);\quad a_{\prime\prime}^2=R_0+R_1+(R_1+R_2)(a_\prime+3b_\prime+3c_\prime);$$
$$b_{\prime\prime}^2=R_0+R_1+(R_1+R_2)(3a_\prime+b_\prime+3c_\prime).$$

How nearly would this same constancy hold when we take account of dispersion?...

..."neglecting $R_0'+R_1'$ on account of there being no dispersion in a vacuum"....

We may then determine the velocity & polarisation of light in a biaxal crystal by Fresnel's rules, (the vibrations however being parallel to the plane of polarisation & at least *very nearly* tangential to the wave) if we employ *variable semi-axes* $a_{\prime\prime}$, $b_{\prime\prime}$, $c_{\prime\prime}$ to construct the auxiliary ellipsoid or the wave surface; these variable semi-axes depending on the colour & on the direction, according to the formulae

$$a_{\prime\prime}^2=R_0+R_1+\left(R_1+R_2-\frac{R_1'+R_2'}{\lambda^2\omega^2}\right)(a_\prime+3b_\prime+3c_\prime)-2\frac{R_1'+R_2'}{\lambda^2\omega^2}(a_\prime\omega_x^2+b_\prime\omega_y^2+c_\prime\omega_z^2),$$
$$b_{\prime\prime}^2=R_0+R_1+\left(\qquad\qquad\right)(3a_\prime+b_\prime+3c_\prime)-2\frac{R_1'+R_2'}{\lambda^2\omega^2}(\qquad\qquad),$$
$$c_{\prime\prime}^2=R_0+R_1+\left(\qquad\qquad\right)(3a_\prime+3b_\prime+c_\prime)-2\frac{R_1'+R_2'}{\lambda^2\omega^2}(\qquad\qquad).$$

It is remarkable that the two differences

$$a_{\prime\prime}^2 - b_{\prime\prime}^2 = 2\left(R_1 + R_2 - \frac{R_1' + R_2'}{\lambda^2\omega^2}\right)(b_{\prime} - a_{\prime}), \quad b_{\prime\prime}^2 - c_{\prime\prime}^2 = 2\left(R_1 + R_2 - \frac{R_1' + R_2'}{\lambda^2\omega^2}\right)(c_{\prime} - b_{\prime}),$$

are independent of the direction of the ray or wave....

But it is very important to observe that although we have only been led to add a common part to each of Fresnel's 2 values of the square of the velocity, so that *the difference of those* 2 *squares is still proportional to the product of the sines of the inclinations of the wave-normal to* 2 *fixed axes* A & B and thus the phenomena of the rings appears to be satisfied, yet on the other hand these axes seem in the present theory to be *independent of the colour*; which is *not true for light*. And this would seem to be a fatal objection, unless it can be removed by some modification of my hypothesis....

[Here follows an attempt at reconciling the values of $a_{\prime\prime}$, $b_{\prime\prime}$, $c_{\prime\prime}$ for topaz with the above results.]