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ON THE TRANSFORMATION OF PLANE CURVES.

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1. THE expression a "double point," or, as I shall for shortness call it, a "dp," is to be throughout understood to include a cusp: thus, if a curve has δ nodes (or double points in the restricted sense of the expression) and κ cusps, it is here regarded as having $\delta+\kappa$ dps.

2. It was remarked by Cramer, in his "Théorie des Lignes Courbes" (1750), that a curve of the order n has at most $\frac{1}{2}(n-1)(n-2)$, $=\frac{1}{2}(n^2-3n)+1$, dps.

3. For several years past it has further been known that a curve such that the coordinates $(x:y:z)$ of any point thereof are as rational and integral functions of the order n of a variable parameter θ, is a curve of the order n having this maximum number $\frac{1}{2}(n-1)(n-2)$ of dps.

4. The converse theorem is also true, viz.: in a curve of the order n, with $\frac{1}{2}(n-1)(n-2)$ dps, the coordinates $(x:y:z)$ of any point are as rational and integral functions of the order n of a variable parameter θ—or, somewhat less precisely, the coordinates are expressible rationally in terms of a parameter θ.

5. The foregoing theorem, as a particular case of Riemann's general theorem, to be presently referred to, dates from the year 1857; but it was first explicitly stated only last year (1864) by Clebsch, in the Paper, "Ueber diejenigen ebenen Curven deren Coordinaten rationale Functionen eines Parameters sind," *Crelle*, t. LXIV. (1864), pp. 43—63.

6. The demonstration is, in fact, very simple; it depends merely on the remark that we may, through the $\frac{1}{2}(n-1)(n-2)$ dps, and through $2n-3$ other points on the given curve of the order n, together $\frac{1}{2}(n^2+n)-2$, $=\frac{1}{2}(n-1)(n+2)-1$, points, draw

a series of curves of the order $n-1$, given by an equation $U+\theta V=0$, containing an arbitrary parameter θ; any such curve intersects the given curve in the dps, each counting as two points, in the $2n-3$ points, and in *one* other point; hence, as there is only one variable point of intersection, the coordinates of this point, viz., the coordinates of an arbitrary point on the given curve, are expressible rationally in terms of the parameter θ. The demonstration may also be effected in a similar manner by means of curves of the order $n-2$.

7. Before going further, it will be convenient to introduce the term "Deficiency," viz., a curve of the order n with $\frac{1}{2}(n-1)(n-2)-D$ dps, is said to have a deficiency $=D$: the foregoing theorem is that for curves with a deficiency $=0$, the coordinates are expressible rationally in terms of a parameter θ. Since in such a curve the different points succeed each other in a certain definite order, viz., in the order obtained by giving to the parameter its different real values from $-\infty$ to ∞, the curve may be termed a *unicursal* curve.

8. Riemann's general theorem, as applied to plane curves, is stated, but not in its complete form, by Schwarz, in the Paper, "De superficiebus in planum explicabilibus primorum septem ordinum," *Crelle*, t. LXIV. (1864), pp. 1—17: to complete the enunciation it is necessary to refer to page 137 of Riemann's own Paper, "Theorie der Abelschen Functionen," *Crelle*, t. LIV. (1857), pp. 115—155, viz., the enunciation will be:

9. For a curve of any order with a given deficiency D, the coordinates may be expressed as follows:

$D=0$, rationally in terms of a parameter θ, or what comes to the same thing, rationally in terms of the parameters (ξ, η), connected by an equation of the form $(1, \xi)(1, \eta)=0$.

$D>0$, rationally in terms of the parameters (ξ, η) connected by an equation of a certain form, viz.:

$D=1$, the equation is $(1, \xi)^2(1, \eta)^2=0$, or (what comes to the same thing) η is the square root of a quartic function of ξ.

$D=2$, the equation is $(1, \xi)^3(1, \eta)^2=0$, or (what comes to the same thing) η is the square root of a sextic function of ξ.

$D>2$, viz.:

D odd, $=2\mu-3$, the equation is $(1, \xi)^\mu(1, \eta)^\mu=0$, and is besides such, that treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-2)^2$ dps.

D even, $=2\mu-2$, the equation is $(1, \xi)^\mu(1, \eta)^\mu=0$, and is besides such, that treating (ξ, η) as Cartesian coordinates, the curve thereby represented has $(\mu-1)(\mu-3)$ dps.

10. To see more clearly the meaning of this, write $\dfrac{\xi}{\eta}$, $\dfrac{\eta}{\zeta}$, in place of (ξ, η), so that the coordinates $(x : y : z)$ are expressible rationally and homogeneously in terms

of (ξ, η, ζ), connected by an equation of the form $(\zeta, \xi)^\mu (\zeta, \eta)^\mu = 0$. Such an equation, treating therein (ξ, η, ζ) as coordinates, belongs to a curve of the order 2μ, with a μ-tuple point at $(\xi = 0, \zeta = 0)$, a μ-tuple point at $(\eta = 0, \zeta = 0)$, and which has besides $(\mu - 2)^2$ or $(\mu - 1)(\mu - 3)$ dps, according as $D = 2\mu - 3$, or $2\mu - 2$. The coordinates $(x : y : z)$ of a point of the given curve are expressible rationally in terms of the coordinates $(\xi : \eta : \zeta)$ of a point on the new curve; and we may say that the original curve is by means of the equations which give $(x : y : z)$ in terms of $(\xi : \eta : \zeta)$ *transformed* into the new curve.

11. A curve of the order 2μ may have $\frac{1}{2}(2\mu - 1)(2\mu - 2)$, $= 2\mu^2 - 3\mu + 1$ dps; hence in the new curve, observing that the μ-tuple points each count for $\frac{1}{2}(\mu^2 - \mu)$ dps, we have

In the case $D = 2\mu - 3$,

$$\begin{aligned} \text{Deficiency} &= 2\mu^2 - 3\mu + 1 \\ &\quad - \mu^2 + \mu \\ &\quad - \mu^2 + 4\mu - 4 \\ &= 2\mu - 3, = D \end{aligned}$$

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Moreover for $D = 0$, the transformed curve is a conic, with 0 dps, and therefore with deficiency $= 0$; in the case $D = 1$, it is a quartic with 2 dps, and therefore deficiency $= 2$; in the case $D = 2$ it is a quintic with a triple point $= 3$, and a double point $= 1$, together 4 dps, and therefore deficiency $= 2$. Hence in every case the new curve has the same deficiency as the original curve.

12. The theorem thus is that the given curve of the order n, with deficiency D, may be rationally transformed into a curve of an order depending only on the deficiency, and having the same deficiency with the given curve, viz.: $D = 0$, the new curve is of the order $2\,(= D + 2)$; $D = 1$, it is of the order $4\,(= D + 3)$; $D = 2$, it is of the order $5\,(= D + 3)$; and $D > 2$, it is for D odd, of the order $D + 3$; and for D even, of the order $D + 2$. It will presently appear that these are not the lowest values which it is possible to give to the order of the new curve. Riemann's object was, not that the order of the transformed curve might be as low as possible, but that the equation in (ξ, η) might be in each of these parameters separately of the lowest possible order; and this he effected by giving to the transformed curve the two μ-tuple points.

13. It is to be noticed that the theorem that for any rational transformation of one curve into another the two curves have the same deficiency is in effect given (as a consequence of Riemann's theory) by Clebsch in the Paper, "Ueber die Singularitäten algebraischer Curven," *Crelle*, t. LXIV., pp. 98—100. I have, by the assistance of a formula communicated to me by Dr Salmon, obtained a direct analytical demonstration of this theorem.

14. I remark that (x, y, z) being connected by an equation, if $(x : y : z)$ are given rationally in terms of $(\xi : \eta : \zeta)$, then it follows that $(\xi : \eta : \zeta)$ are also

expressible rationally in terms of $(x : y : z)$: it is convenient to consider the transformation from this point of view, and I now proceed to the independent development of the theory, as follows:

15. We have a given curve $U = (x, y, z)^n = 0$, with deficiency D, which is by the transformation $\xi : \eta : \zeta = P : Q : R$ (where P, Q, R are given functions $(x, y, z)^k$ each of the same order k) transformed into the curve $\Upsilon = (\xi, \eta, \zeta)^\nu = 0$. The transformed curve has, as we know, the same deficiency D as the original curve.

16. To find the order of the transformed curve, we must find the number of its intersections with an arbitrary line $a\xi + b\eta + c\zeta = 0$. Writing in this equation $\xi : \eta : \zeta = P : Q : R$, we obtain the equation $aP + bQ + cR = 0$, and combining therewith the equation $U = 0$, the two equations, being of the orders k and n respectively, give kn systems of values of $(x : y : z)$, and to each of these, in virtue of the equations $\xi : \eta : \zeta = P : Q : R$, there corresponds a single set of values of $(\xi : \eta : \zeta)$, and therefore a single point of intersection; hence the number of intersections, that is, the order of the transformed curve, is $= kn$.

17. If, however, the curves $P = 0$, $Q = 0$, $R = 0$, meet in an ordinary point of the curve $U = 0$, then it is easy to see there is a reduction $= 1$ in the foregoing value; and so if they meet in a dp of the curve $U = 0$, then there is a reduction $= 2$. More generally if the curves $P = 0$, $Q = 0$, $R = 0$ each pass through the same α dps and β ordinary points of the curve $U = 0$, then there is a reduction $= 2\alpha + \beta$. In fact the curve $aP + bQ + cR = 0$, meets the curve $U = 0$, in kn points; but among these are included the α dps, each counting as 2 intersections, and the β points; the number of the remaining intersections is $= kn - 2\alpha - \beta$, and the order of the transformed curve is equal to this number.

I assume that we have $k < n$:

18. A curve of the order k may be made to pass through $\frac{1}{2}k(k+3)$ points; it is moreover known that if any three curves, $P = 0$, $Q = 0$, $R = 0$, of the order k each pass through the same $\frac{1}{2}k(k+3) - 1$ points, then the three curves have all their intersections common, the equations being, in fact, connected by an identical relation of the form $\alpha P + \beta Q + \gamma R = 0$. To make the order of the transformed curve as low as possible, we must make the curves $P = 0$, $Q = 0$, $R = 0$, meet on the curve $U = 0$ in as many points as possible, and it appears from the remark just made, that the greatest possible number is $= \frac{1}{2}k(k+3) - 2$; in particular, for $k = n - 1$, $n - 2$, $n - 3$, the number of points on the curve $U = 0$ will be at most equal to $\frac{1}{2}(n^2 + n) - 3$, $\frac{1}{2}(n^2 - n) - 3$, $\frac{1}{2}(n^2 - 3n) - 2$, respectively.

19. Hence, considering the curve $U = 0$ with deficiency D, or with $\frac{1}{2}(n^2 - 3n) - D + 1$ dps, *first* if $k = n - 1$, we may assume that the transforming curves $P = 0$, $Q = 0$, $R = 0$ of the order $n - 1$, each pass

through the $\frac{1}{2}(n^2 - 3n) - D + 1$ dps,

and through $2n + D - 4$ other points,

together $\frac{1}{2}(n^2 + n) - 3$ points of the curve $U = 0$.

This being so, each of the three curves will meet the curve $U=0$

in the dps, counting as	$n^2-3n-2D+2$	points,
in the	$2n+D-4$	points,
and in	$D+2$	other points,
together	n^2-n	points;

whence the order of the transformed curve is $=D+2$.

20. In precisely the same manner, *secondly*, if $k=n-2$, then we may assume that the transforming curves $P=0$, $Q=0$, $R=0$, of the order $n-2$, each pass

through the	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$n+D-4$	other points,
together	$\frac{1}{2}(n^2-n)\quad -3$	points of the curve $U=0$;

and this being so, each of the three curves will meet the curve $U=0$

in the dps counting as	$n^2-3n-2D+2$	points,
in the	$n+D-4$	points,
and in	$D+2$	other points,
together	n^2-2n	points;

whence the order of the transformed curve is also in this case $=D+2$.

21. I was under the impression that the order of the transformed curve could not be reduced below $D+2$, but it was remarked to me by Dr Clebsch, that in the case $D>2$, the order might be reduced to $D+1$. In fact, considering, *thirdly*, the case $k=n-3$, we see that the transforming curves $P=0$, $Q=0$, $R=0$ of the order $n-3$ may be made to pass

through the	$\frac{1}{2}(n^2-3n)-D+1$	dps,
and through	$D-3$	other points,
together	$\frac{1}{2}(n^2-3n)\quad -2$	points of the curve $U=0$;

and this being so, each of the three curves meets the curve $U=0$,

in the dps counting as	$(n^2-3n)-2D+2$	points,
in the	$D-3$	points,
and in	$D+1$	other points,
together	n^2-3n	points;

whence the order of the transformed curve is in this case $=D+1$.

22. The general theorem thus is that a curve of the order n with deficiency D, can be, by a transformation of the order $n-1$ or $n-2$, transformed into a curve of

the order $D+2$; and if $D>2$, then the given curve can be by a transformation of the order $n-3$ transformed to a curve of the order $D+1$: the transformed curve having in each case the same deficiency D as the original curve.

23. In particular, if $D=1$, a curve of the order n with deficiency 1, or with $\frac{1}{2}(n^2-3n)$ dps, can be transformed into a cubic curve with the same deficiency, that is with 0 dps; or the given curve can be transformed into a cubic. This case is discussed by Clebsch in the Memoir "Ueber diejenigen Curven deren Coordinaten elliptische Functionen eines Parameters sind," *Crelle*, t. LXIV., pp. 210—271. And he has there given in relation to it a theorem which I establish as follows:

24. Using the transformation of the order $n-1$, if besides the $2n+D-4\,(=2n-3)$ points on the given curve $U=0$, we consider another point O on the curve, then we may, through the $\frac{1}{2}(n^2-3n)$ dps, the $2n-3$ points and the point O, draw a series of curves of the order $n-1$, viz., if P_0, Q_0, R_0, are what the functions P, Q, R, become on substituting therein for (x, y, z), the coordinates (x_0, y_0, z_0) of the given point O, then the equation of any such curve will be $aP+bQ+cR=0$, with the relation $aP_0+bQ_0+cR_0$ between the parameters a, b, c; or (what is the same thing) eliminating c, the equation will be $a(PR_0-P_0R)+b(QR_0-Q_0R)=0$, which contains the single arbitrary parameter $a:b$. In the cubic which is the transformation of the given curve we have a point O' corresponding to O and if (ξ_0, η_0, ζ_0) be the coordinates of this point, then corresponding to the series of curves of the order $n-1$, we have a series of lines through the point O' of the cubic, viz., the lines $a\xi+b\eta+c\zeta=0$ with the relation $a\xi_0+b\eta_0+c\zeta_0=0$ between the parameters; or, what is the same thing, we have the series of lines $a(\xi\zeta_0-\zeta\xi_0)+b(\eta\zeta_0-\zeta\eta_0)=0$, containing the same single parameter $a:b$. By determining this parameter, the curves of the order $n-1$, will be the curves of this order through the dps, the $2n-3$ points, and the point O, which touch the given curve $U=0$; and the lines will be the tangents to the cubic from the point O'; as the number of tangents to a cubic from a point on the cubic is $=4$, it is clear that the values of the parameter $a:b$ will be determined by a certain quartic equation; and there will of course be 4 tangent curves of the order $n-1$ corresponding to the 4 tangents to the cubic. Now by Dr Salmon's anharmonic property of the tangents of a cubic, if on the cubic we vary the position of the point O', the absolute invariant $I^3 \div J^2$ of the quartic in $(a:b)$ remains unaltered; that is the absolute invariant $I^3 \div J^2$ of the quartic which determines the 4 tangent curves of the order $n-1$ is independent of the position of the point O on the given curve $U=0$, and since the tangent curves in question have the same relation to each of the $2n-3$ points and to the point O, it follows that the invariant is also independent of the position of each of the $2n-3$ points; that is, we have the following theorem, viz.:

25. Considering a curve of the order n with deficiency $=1$; we may, through the $\frac{1}{2}(n^2-3n)$ dps, and through any $2n-2$ points on the curve, draw so as to touch the curve, four curves of the order $n-1$; viz., these are given by an equation $aP'+bQ'=0$, where the ratio $a:b$ is determined by a certain quartic equation $(*\between a, b)^4=0$; then *theorem*, the absolute invariant $I^3 \div J^2$ of the quartic function, is independent of the

positions of the $2n-2$ points on the curve $U=0$, and it is consequently a function of only the coefficients of the curve $U=0$, being, as is obvious, an absolute invariant of the curve $U=0$.

26. And, moreover, if the curve $U=0$ is by a transformation of the order $n-1$, by means of $2n-3$ points on the curve as above, transformed into a cubic, then the absolute invariant $I^3 \div J^2$ of the quartic equation which determines the tangents to the cubic from any point O' on the cubic (or, what is the same thing, the absolute invariant $S^3 \div T^2$ of the cubic, taken with a proper numerical multiplier) is independent of the positions of the $2n-3$ points on the curve $U=0$, being in fact equal to the above-mentioned absolute invariant of the curve $U=0$. The like results apply to the transformation of the order $n-2$.

27. Suppose now that we have $D>2$, and consider a curve of the order n with the deficiency D, that is with $\frac{1}{2}(n^2-3n)-D+1$ dps, transformed by a transformation of the order $n-3$ into a curve of the order $D+1$ with deficiency D; then, assuming the truth of the subsidiary theorem to be presently mentioned, it may be shown by very similar reasoning to that above employed, that the absolute invariants of the transformed curve of the order $D+1$ (the number of which is $=4D-6$), will be independent of the positions of the $D-3$ points used in the transformation, and will be equal to absolute invariants([1]) of the given curve $U=0$.

28. The subsidiary theorem is as follows: consider a curve of the order $D+1$, with deficiency D, that is, with $\frac{1}{2}D(D-1)-D=\frac{1}{2}(D^2-3D)$ dps; the number of tangents to the curve from any point O' on the curve is $=(D+1)D-(D^2-3D)-2$, $=4D-2$, (this assumes however, that the dps are proper dps, not *cusps*,) the pencil of tangents has $4D-5$ absolute invariants, and of these all but one, that is, $4D-6$, absolute invariants of the pencil are independent of the position of the point O' on the curve, and are respectively equal to absolute invariants of the curve.

29. To establish it, I observe that a curve of the order $D+1$ with deficiency D, or with $\frac{1}{2}(D^2-3D)$ dps, contains $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)$, $=4D+2$ arbitrary constants, and it may therefore be made to satisfy $4D+2$ conditions. Now imagine a given pencil of $4D-2$ lines, and let a curve of the form in question be determined so as to pass through the centre of the pencil, and touch each of the $4D-2$ lines; the curve thus satisfies $4D-1$ conditions, and its equation will contain $4D+2-(4D-1)$, $=3$ arbitrary constants. But if we have any particular curve satisfying the $4D-1$ conditions, then by transforming the whole figure homologously, taking the centre of the pencil as pole and any arbitrary line as axis of homology, so as to leave the pencil of lines unaltered (analytically if at the centre of the pencil $x=0$, $y=0$, then by writing $\alpha x+\beta y+\gamma z$ in place of z) the transformed curve still satisfies the $4D-1$ conditions, and we have by the homologous transformation introduced into its equation 3 arbitrary constants, that is, we have obtained the most general curve which satisfies the conditions in question. The absolute invariants of the general curve are independent of the

[1] It is right to notice that the absolute invariants spoken of here, and in what follows, are not in general rational ones.

3 arbitrary constants introduced by the homologous transformation; and they are consequently functions of only the coefficients of the given pencil of $4D-2$ lines; this being so, it is obvious that they will be respectively equal to absolute invariants of the pencil of $4D-2$ lines. The number of the absolute invariants of the general curve of the order $D+1$ is $=\frac{1}{2}(D+1)(D+4)+1-9$, but there is a reduction $=1$, for each of the dps, hence in the present case the number is $\frac{1}{2}(D+1)(D+4)-\frac{1}{2}(D^2-3D)-8$, $=4D-6$; and there are thus $4D-6$ absolute invariants of the curve, each of them equal to an absolute invariant of the pencil; that is, of the $4D-5$ absolute invariants of the pencil, there are $4D-6$, each of them equal to an absolute invariant of the curve, and consequently independent of the position of the point O' on the curve; which is the theorem which was to be proved. I believe the reasoning is quite correct, but there are some points in it which require further examination, it is therefore given subject to any correction which may hereafter appear to be necessary.

30. The general subject may be illustrated by considerations belonging to solid geometry. If we imagine the original curve and the transformed curve as situate in different planes, then joining each point of the original curve with the corresponding point on the transformed curve, we have a series of lines forming a scroll (skew surface): if the two curves are of the orders n, n' respectively, then the complete section by the plane of the original curve is made up of this curve of the order n, and of n' generating lines; and similarly the complete section by the plane of the transformed curve is made up of this curve of the order n', and of n generating lines. Conversely, given a scroll of the order $n+n'$, any two sections of this scroll, being in general curves of the same order $n+n'$, are rational transformations the one of the other; but for the *general* scroll of the order $n+n'$, it is not possible to find sections breaking up as above.