391.

SOLUTION OF A PROBLEM OF ELIMINATION.

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It is required to eliminate x, y from the equations

$$\left\| \begin{matrix} x^4, & x^3y, & x^2y^2, & xy^3, & y^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{matrix} \right\| = 0.$$

This system may be written

$$\begin{aligned} x^4 &= \Sigma\lambda a, \\ x^3y &= \Sigma\lambda b, \\ x^2y^2 &= \Sigma\lambda c, \\ xy^3 &= \Sigma\lambda d, \\ y^4 &= \Sigma\lambda e; \end{aligned}$$

if for shortness

$$\Sigma\lambda a = \lambda a + \lambda' a' + \lambda'' a'', \text{ \&c.}$$

Or putting

$$\frac{x}{y} = -k,$$

we have

$$\begin{aligned} \Sigma\lambda\,(a+kb) &= 0, \\ \Sigma\lambda\,(b+kc) &= 0, \\ \Sigma\lambda\,(c+kd) &= 0, \\ \Sigma\lambda\,(d+ke) &= 0\,; \end{aligned}$$

or, what is the same thing,

$$\lambda(a+kb)+\lambda'(a'+kb')+\lambda''(a''+kb'')=0,$$
$$\lambda(b+kc)+\lambda'(b'+kc')+\lambda''(b''+kc'')=0,$$
$$\lambda(c+kd)+\lambda'(c'+kd')+\lambda''(c''+kd'')=0,$$
$$\lambda(d+ke)+\lambda'(d'+ke')+\lambda''(d''+ke'')=0;$$

and representing the columns

$$\begin{matrix} a & b, & a' & b', & a'' & b'', \\ b & c, & b' & c', & b'' & c'', \\ c & d, & c' & d', & c'' & d'', \\ d & e, & d' & e', & d'' & e'', \end{matrix}$$

by

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6,$$

each equation is of the type

$$\lambda(1+k2)+\lambda'(3+k4)+\lambda''(5+k6)=0.$$

Multiplying the several equations by the minors of 135, each with its proper sign, and adding, the terms independent of k disappear, the equation divides by k, and we find

$$\lambda\,2135+\lambda'\,4135+\lambda''\,6135=0;$$

operating in a similar manner with the minors of 246, the terms in k disappear, and we find

$$\lambda\,1246+\lambda'\,3246+\lambda''\,5246=0;$$

again, operating with the minors of $(146+236+245+k246)$, we find

$$\begin{aligned} \lambda\ &\{1236+1245+k(2146+1246)\} \\ +\lambda'\ &\{3146+3245+k(4236+3246)\} \\ +\lambda''\ &\{5146+5236+k(6245+5246)\}=0, \end{aligned}$$

where the terms in k disappear, and this is

$$\lambda(1236+1245)+\lambda'(3146+3245)+\lambda''(5146+5236)=0.$$

We have thus three linear equations, which written in a slightly different form are

$$\begin{aligned} &\lambda\,1235 &&+\lambda'\,3451 &&+\lambda''\,5613 &&=0, \\ &\lambda(1236+1245) &&+\lambda'(3452+3461) &&+\lambda''(5614+5623) &&=0, \\ &\lambda\,1246 &&+\lambda'\,3462 &&+\lambda''\,5624 &&=0, \end{aligned}$$

and thence eliminating $\lambda, \lambda', \lambda''$, we have

$$\begin{vmatrix} 1235, & 1236+1245, & 1246 \\ 3451, & 3452+3461, & 3462 \\ 5613, & 5614+5623, & 5624 \end{vmatrix}=0,$$

which is the required result. It may be remarked that the second and third column are obtained from the first by operating on it with Δ, $\frac{1}{2}\Delta^2$, if $\Delta = 2\delta_1 + 4\delta_3 + 6\delta_5$. Or say the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2)\left|\begin{matrix} 1235 \\ 3451 \\ 5613 \end{matrix}\right| = 0.$$

In like manner for the system

$$\left\|\begin{matrix} x^5, & x^4y, & x^3y^2, & x^2y^3, & xy^4, & y^5 \\ a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \\ a''', & b''', & c''', & d''', & e''', & f''' \end{matrix}\right\| = 0,$$

if the columns are

$$\begin{matrix} a\,b, & a'\,b', & a''\,b'', & a'''\,b''', \\ b\ c, & b'\,c', & b''\,c'', & b'''\,c''', \\ c\,d, & c'\,d', & c''\,d'', & c'''\,d''', \\ d\ e, & d'\,e', & d''\,e'', & d'''\,e''', \\ e\ f, & e'\,f', & e''\,f'', & e'''\,f''', \\ =1,\ 2, & 3,\ 4, & 5,\ 6, & 7,\ 8; \end{matrix}$$

then the result is

$$(1,\ \Delta,\ \tfrac{1}{2}\Delta^2,\ \tfrac{1}{6}\Delta^3)\left|\begin{matrix} 12357 \\ 34571 \\ 56713 \\ 78135 \end{matrix}\right| = 0,$$

where

$$\Delta = 2\delta_1 + 4\delta_3 + 6\delta_5 + 8\lambda_7.$$