405.

AN EIGHTH MEMOIR ON QUANTICS.

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THE present Memoir relates mainly to the binary quintic, continuing the investigations in relation to this form contained in my Second, Third, and Fifth Memoirs on Quantics, [141], [144], [156]; the investigations which it contains in relation to a quantic of any order are given with a view to their application to the quintic. All the invariants of a binary quintic (viz. those of the degrees 4, 8, 12, and 18) are given in the Memoirs above referred to, and also the covariants up to the degree 5; it was interesting to proceed one step further, viz. to the covariants of the degree 6; in fact, while for the degree 5 we obtain 3 covariants and a single syzygy, for the degree 6 we obtain only 2 covariants, but as many as 7 syzygies; one of these is, however, the syzygy of the degree 5 multiplied into the quintic itself, so that, excluding this derived syzygy, there remain $(7-1=)$ 6 syzygies of the degree 6. The determination of the two covariants (Tables 83 and 84 *post*) and of the syzygies of the degree 6, occupies the commencement of the present Memoir. [These covariants 83, 84 are the covariants M and N of the paper 143, "Tables of the covariants M to W of the binary quintic", and they are accordingly not here reproduced.]

The remainder of the Memoir is in a great measure a reproduction (with various additions and developments) of researches contained in Professor Sylvester's Trilogy, and in a recent memoir by M. Hermite[1]. In particular, I establish in a more general form (defining for that purpose the functions which I call "Auxiliars") the theory which is the basis of Professor Sylvester's criteria for the reality of the roots of a quintic equation, or, say, the theory of the determination of the character of an equation of any order. By way of illustration, I first apply this to the quartic equation; and

[1] Sylvester "On the Real and Imaginary Roots of Algebraical Equations; a Trilogy," *Phil. Trans.* vol. CLIV. (1864), pp. 579—666. Hermite, "Sur l'Équation du 5e degré," *Comptes Rendus*, t. LXI. (1866), and in a separate form, Paris, 1866.

I then apply it to the quintic equation, following Professor Sylvester's track, but so as to dispense altogether with his amphigenous surface, and making the investigation to depend solely on the discussion of the bicorn curve, which is a principal section of this surface. I explain the new form which M. Hermite has given to the Tschirnhausen transformation, leading to a transformed equation the coefficients whereof are all invariants; and, in the case of the quintic, I identify with my Tables his cubicovariants $\phi_1(x, y)$ and $\phi_2(x, y)$. And in the two new Tables, 85 and 86, I give the leading coefficients of the other two cubicovariants $\phi_3(x, y)$ and $\phi_4(x, y)$, [these are now also identified with my Tables]. In the transformed equation the second term (or that in z^4) vanishes, and the coefficient $\mathfrak{A}$ of z^3 is obtained as a quadric function of four indeterminates. The discussion of this form led to criteria for the character of a quintic equation, expressed like those of Professor Sylvester in terms of invariants, but of a different and less simple form; two such sets of criteria are obtained, and the identification of these, and of a third set resulting from a separate investigation, with the criteria of Professor Sylvester, is a point made out in the present memoir. The theory is also given of the canonical form which is the mechanism by which M. Hermite's investigations were carried on. The Memoir contains other investigations and formulæ in relation to the binary quintic; and as part of the foregoing theory of the determination of the character of an equation, I was led to consider the question of the imaginary linear transformations which give rise to a real equation: this is discussed in the concluding articles of the memoir, and in an Annex I have given a somewhat singular analytical theorem arising thereout.

The paragraphs and Tables are numbered consecutively with those of my former Memoirs on Quantics. I notice that in the Second Memoir, p. 126, we should have No. $26 = (\text{No. } 19)^2 - 128$ (No. 25), viz. the coefficient of the last term is 128 instead of 1152. [This correction is made in the present reprint, 141, where the equation is given in the form $Q' = G^2 - 128Q$.]

Article Nos. 251 to 254.—*The Binary Quintic, Covariants and Syzygies of the degree* 6.

251. The number of asyzygetic covariants of any degree is obtained as in my Second Memoir on Quantics, *Philosophical Transactions*, vol. CXLVI. (1856), pp. 101—126, [141], viz. by developing the function

$$\frac{1}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)},$$

as shown p. 114, and then subtracting from each coefficient that which immediately precedes it; or, what is the same thing, by developing the function

$$\frac{1-x}{(1-z)(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)},$$

which would lead directly to the second of the two Tables which are there given; the Table is there calculated only up to z^5, but I have since continued it up to z^{18}, so as to show the number of the asyzygetic covariants of every order in the variables up to the degree 18 in the coefficients, being the degree of the skew invariant, the highest of the irreducible invariants of the quintic. The Table is, for greater convenience, arranged in a different form, as follows:

Table No. 81.

Table for the number of the Asyzygetic Covariants of any order, to the degree 18.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	1		0		1		0		2		0		3		0		4		1
1		0		0		1		1		1		2		3		3		4	
2			1		0		2		1		4		2		6		4		9
3		0		1		1		2		3		4		5		7		8	
4			0		2		1		4		3		7		6		11		9
5		1		1		2		3		5		6		8		10		13	
6			1		1		4		3		7		7		12		11		18
7				1		2		4		5		8		10		13		16	
8			0		2		2		6		6		11		11		18		18
9				1		3		4		7		9		13		16		20	
10			1		1		4		5		10		10		17		18		26
11				1		2		5		7		11		14		19		23	
12					2		3		7		8		15		16		24		26
13				0		2		4		8		11		16		20		27	
14					1		4		5		11		13		21		23		33
15				1		2		5		8		13		17		24		29	
16					1		2		7		9		16		19		29		32
17						2		4		8		12		19		24		32	
18					0		3		5		11		14		23		27		39
19						1		4		7		13		18		26		33	
20					1		2		6		9		17		21		32		37
21						1		3		8		12		19		26		36	
22							2		4		10		14		24		29		42
23						0		3		6		12		18		27		35	
24							1		5		8		16		21		33		40
25						1		2		6		11		19		26		37	
26							1		3		9		13		23		30		44
27								2		5		11		17		27		36	
28							0		3		6		14		20		32		40
29								1		5		9		17		25		37	
30							1		2		7		12		22		29		44
31								1		3		9		15		25		35	
32									2		5		12		18		31		40
33								0		3		7		15		23		35	
34									1		5		9		19		27		42
35								1		2		7		13		23		33	
36									1		3		10		16		28		38
37										2		5		12		20		33	
38									0		3		7		16		24		39
39										1		5		10		20		30	
40									1		2		7		13		25		35
41										1		3		10		17		29	
42											2		5		13		21		36
43										0		3		7		16		26	
44											1		5		10		21		31
45										1		2		7		14		26	
46											1		3		10		17		31
47												2		5		13		22	
48											0		3		7		17		27
49												1		5		10		21	
50											1		2		7		14		27
51												1		3		10		18	
52													2		5		13		22
53												0		3		7		17	
54													1		5		10		22
55												1		2		7		14	
56													1		3		10		18
57														2		5		13	
58													0		3		7		17
59														1		5		10	
60													1		2		7		14
61														1		3		10	
62															2		5		13
63														0		3		7	
64															1		5		10
65														1		2		7	
66															1		3		10
67																2		5	
68															0		3		7
69																1		5	
70															1		2		7
71																1		3	
72																	2		5
73																0		3	
74																	1		5
75																1		2	
76																	1		3
77																		2	
78																	0		3
79																		1	
80																	1		2
81																		1	
82																			2
83																		0	
84																			1
85																		1	
86																			1
87																			
88																			0
89																			
90																			1

[In regard to this table No. 81 it is hardly necessary to notice that for any column with an even heading the numbers of the column correspond to the even outside numbers, while for any column with an odd heading the numbers of the column correspond to the odd outside numbers. The table is in fact a table of the differences of the numbers of the *af*-table, 142; thus in this table writing down cols. 5 and 6 and in each of them forming the differences by subtracting from each number the number immediately below it, we have cols. 5 and 6 of the table No. 81, viz.:

1	1	2	3	5	7	9	11	14	16	18	19	20	col. 5, 13—12 of *af*-table.
1	0	1	1	2	2	2	2	3	2	2	1	1	col. 5 of table No. 81.

1	1	2	3	5	7	10	12	16	19	23	25	29	30	32	32	col. 6, 15 of *af*-table.
1	0	1	1	2	2	3	2	4	3	4	2	4	1	2	0	col. 6 of table No. 81.]

252. The interpretation up to the degree 6 is as follows:

[In the following Table No. 82 as originally printed, the heading of the fourth column was "Constitution. Nos. in ( ) refer to Tables in former Memoirs except (83) and (84) which are given *post*," and the covariants were referred to by their Nos. accordingly.]

Table No. 82.

Degree.	Order.	No.	Constitution. Notation is the alphabetic notation of 143, A the quintic itself, B, C quadricovariants, &c.	N=new covt. S=syzygy.
0	0	1	viz. the absolute constant unity.	
1	5	1	A	N.
2	10	1	A^2	
,,	6	1	C	N.
,,	2	1	B	N.
3	15	1	A^3	
,,	11	1	AC	
,,	9	1	F	N.
,,	7	1	AB	
,,	5	1	E	N.
,,	3	1	D	N.
4	20	1	A^4	
,,	16	1	A^2C	
,,	14	1	AF	
,,	12	2	C^2, A^2B	
,,	10	1	AE	
,,	8	2	AD, BC	
,,	6	1	I	N.
,,	4	2	H, B^2	N.
,,	0	1	G	N.
5	25	1	A^5	
,,	21	1	A^3C	
,,	19	1	A^2F	
,,	17	2	A^3B, AC^2	
,,	15	2	A^2E, CF	
,,	13	2	A^2D, ABC	
,,	11	2	$AI + BF - CE = 0$	S.
,,	9	3	AB^2, AH, CD	
,,	7	2	BE, L	N.
,,	5	2	AG, BD	
,,	3	1	K	N.
,,	1	1	J	N.
6	30	1	A^6	
,,	26	1	A^4C	
,,	24	1	A^3F	
,,	22	2	A^3C^2, A^4B	
,,	20	2	ACF, A^3E	
,,	18	3	$E^2 + 4C^3 + A^3D - A^2BC = 0$	S.
,,	16	2	$A\{AI + BF - CE\} = 0$	S'.
,,	14	4	$-6ACD - 1EF - 4BC^2 + A^2H = 0$, A^2B^2	S.
,,	12	3	$AL + 3DF - 2CI = 0$, ABE	S.
,,	10	4	$4B^2C + 12ABD - A^2G + E^2 = 0$, CH	S.
,,	8	2	$AK + 2BI - 3DE = 0$	S.
,,	6	4	$AJ + 2BH - B^3 - CG - 9D^2 = 0$	S.
,,	4	1	N	N.
,,	2	1	M	N.

253. For the explanation of this I remark that the Table No. 81 shows that we have for the degree 0 and order 0 one covariant; this is the absolute constant unity; for the degree 1 and order 5, 1 covariant, this is the quintic itself, A; for degree 2 and order 10, 1 covariant; this is the square of the quintic, A^2; for same degree and order 6, 1 covariant, which had accordingly to be calculated, viz. this is the covariant C; and similarly whenever the Table No. 81 indicates the existence of a covariant of any degree and order, and there does not exist a product of the covariants previously calculated, having the proper degree and order, then in each such case (shown in the last preceding Table by the letter N) a new covariant had to be calculated. On coming to degree 5, order 11, it appears that the number of asyzygetic invariants is only $=2$, whereas there exist of the right degree and order the 3 combinations AI, BF, CE; there is here a syzygy, or linear relation, between the combinations in question; which syzygy had to be calculated, and was found to be as shown, $AI+BF-CE=0$, a result given in the Second Memoir, p. 126. Any such case is indicated by the letter S. At the place degree 6, order 16, we find a syzygy between the combinations A^2I, A^2BF, ACE; as each term contains the factor A, this is only the last-mentioned syzygy multiplied by A, not a new syzygy, and I have written S' instead of S. The places degree 6, orders 18, 14, 12, 10, 8, 6 indicate each of them a syzygy, which syzygies, as being of the degree 6, were not given in the Second Memoir, and they were first calculated for the present Memoir. It is to be noticed that in some cases the combinations which might have entered into the syzygy do not all of them do so; thus degree 6, order 14, the syzygy is between the four combinations ACD, EF, BC^2. A^2H, and does not contain the remaining combination A^2B. The places degree 6, orders 4, 2, indicate each of them a new covariant, and these, as being of the degree 6, were not given in the Second Memoir, but had to be calculated for the present Memoir.

254. I notice the following results:

$$\begin{aligned}
&\text{Quadrinvt. } 6H && = 3G^2,\\
&\text{Cubinvt. } 6H && = -G^3+54GQ,\\
&\text{Disct. } (\alpha B+\beta M) && = (-G,\ Q,\ -3U\between \alpha,\ \beta)^2,\\
&\text{Jac. } (B,\ H) && = 6M,\\
&\text{Hess. } 3D && = N,
\end{aligned}$$

the last two of which indicate the formation of the covariants given in the new Tables $M=$ No. 83 and $N=$ No. 84: viz. if to avoid fractions we take 3 times the covariant D, being a cubic $(a, \ldots)^3(x, y)^3$, then the Hessian thereof is a covariant $(a, \ldots)^6(x, y)^2$, which is given in Table, M No. 83; and in like manner if we form the Jacobian of the Tables B and H which are respectively of the forms $(a, ..)^2(x, y)^2$, and $(a, ..)^6(x, y)^4$, this is a covariant $(a, ..)^6(x, y)^4$, and dividing it by 6 to obtain the coefficients in their lowest terms, we have the new Table, N No. 84. I have in these, for greater distinctness, written the numerical coefficients *after* instead of *before*, the literal terms to which they belong.

The two new Tables are:

Table No. 83. $M = (*\between x, y)^2$. See 143.

Table No. 84. $N = (*\between x, y)^4$. See 143.

Article No. 255.—*Formulæ for the canonical form* $ax^5 + by^5 + cz^5 = 0$, *where* $x + y + z = 0$.

255. The quintic $(a, b, c, d, e, f\between x, y)^5$ may be expressed in the form

$$ru^5 + sv^5 + tw^5,$$

where u, v, w are linear functions of (x, y) such that $u + v + w = 0$. Or, what is the same thing, the quintic may be represented in the canonical form

$$ax^5 + by^5 + cz^5,$$

where $x + y + z = 0$; this is $= (a - c, \ -c, \ -c, \ -c, \ -c, \ b - c\between x, y)^5$, and the different covariants and invariants of the quintic may hence be expressed in terms of these coefficients (a, b, c).

For the invariants we have

$$\begin{aligned} G &= J = b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a + b + c), \\ Q &= K = a^2b^2c^2\,(bc + ca + ab), \\ -U &= L = a^4b^4c^4, \\ W &= I = 4a^5b^5c^5\,(b - c)(c - a)(a - b). \end{aligned}$$

[Observe that throughout the present Memoir, the invariants, instead of being called G, Q, $-U$, W are called I, J, K, L, viz. the I, J, K, L in all that follows denote the invariants, and not the covariants denoted by these letters in 142, 143. Moreover D is used to denote the invariant Q', which is in fact the discriminant of the quintic.]

Hence, writing for a moment

$$\begin{aligned} a + b + c &= p, \text{ and therefore } & J &= q^2 - 4pr, \\ bc + ca + ab &= q & K &= r^2q, \\ abc &= r & L &= r^4, \end{aligned}$$

we have

$$(a - b)^2(b - c)^2(c - a)^2 = p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2,$$

and thence

$$I^2 = 16r^{10}\,(p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2),$$

and

$$\begin{aligned} &J(K^2 - JL)^2 + 8K^3L - 72JKL^2 - 432L^3 \\ &\quad = r^{10}\,\{(q^2 - 4pr)\,16p^2 + 8q^3 - (q^2 - 4pr)\,72q - 432r^2\}, \\ &\quad = 8r^{10}\,\{(q^2 - 4pr)(2p^2 - 9q) + q^3 - 54r^2\}, \\ &\quad = 16r^{10}\,\{p^2q^2 - 4q^3 - 4p^3r + 18pqr - 27r^2\}, \end{aligned}$$

that is,

$$I^2 = J(K^2 - JL)^2 + 8K^3L - 72JKL^2 - 432L^3,$$

which is the simplest mode of obtaining the expression for the square of the 18-thic or skew invariant I in terms of the invariants J, K, L of the degrees 4, 8, 12 respectively.

If instead of the invariant K of the degree 8 we consider the invariant D [$=Q'$ as before-mentioned] of the same degree, this is

$$Q' = D = \{b^2c^2 + c^2a^2 + a^2b^2 - 2abc\,(a+b+c)\}^2 - 128a^2b^2c^2\,(bc+ca+ab),$$

$$= q^4 - 8q^2pr - 128qr^2 + 16p^2r^2,$$

$$D = \text{Norm}\ \left((bc)^{\frac{1}{2}} + (ca)^{\frac{1}{2}} + (ab)^{\frac{1}{2}}\right);$$

and we have also the following covariants:

$$B = (-ac,\ ab - ac - bc,\ -bc\between x,\ y)^2,$$

$$= bcyz + cazx + abxy.$$

$$C = (-ac,\ -3ac,\ -3ac,\ ab - ac - bc,\ -3bc,\ -3bc,\ -bc\between x,\ y)^6$$

$$= bcy^3z^3 + caz^3x^3 + abx^3y^3.$$

$$D = (0,\ -abc,\ -abc,\ 0\between x,\ y)^3 = abc\,xyz.$$

$$\begin{aligned} E = {} & (\ a^2b - ac^2 + bc^2 - a^2c - 2abc)\,x^5 \\ & + (\ -5ac^2 + 5bc^2 - 5abc)\,x^4y \\ & + (\ -10ac^2 + 10bc^2 - 2abc)\,x^3y^2 \\ & + (\ -10ac^2 + 10bc^2 + 2abc)\,x^2y^3 \\ & + (\ -5ac^2 + 5bc^2 + 5abc)\,xy^4 \\ & + (-ab^2 - ac^2 + bc^2 + b^2c + 2abc)\,y^5 \\ = {} & (b-c)\,a^2x^5 + (c-a)\,b^2y^5 + (a-b)\,c^2z^5 \\ & - abc\,(y-z)(z-x)(x-y)(yz+zx+xy). \end{aligned}$$

Article No. 256.—*Expression of the 18-thic Invariant in terms of the roots.*

256. It was remarked by Dr Salmon, that for a quintic $(a, b, c, d, e, f\between x, y)^5$ which is linearly transformable into the form $(a, 0, c, 0, e, 0\between x, y)^5$, the invariant I is $=0$. Now putting for convenience $y=1$, and considering for a moment the equation

$$x(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon) = 0,$$

then writing herein $\dfrac{x}{mx+n}$ for x, the transformed equation is

$$x(x-\beta')(x-\gamma')(x-\delta')(x-\epsilon') = 0,$$

where

$$\beta' = \frac{n\beta}{1-m\beta},\ \gamma' = \frac{n\gamma}{1-m\gamma},\ \&c.;$$

hence m may be so determined that $\beta'+\gamma'$ may be $=0$; viz. this will be the case if $\beta+\gamma = 2m\beta\gamma$, or $m = \dfrac{\beta+\gamma}{2\beta\gamma}$. In order that $\delta'+\epsilon'$ may be $=0$, we must of course have $m = \dfrac{\delta+\epsilon}{2\delta\epsilon}$, and hence the condition that simultaneously $\beta'+\gamma'=0$ and $\delta'+\epsilon'=0$ is $\dfrac{\beta+\gamma}{2\beta\gamma} = \dfrac{\delta+\epsilon}{2\delta\epsilon}$; that is, $(\beta+\gamma)\,\delta\epsilon - \beta\gamma(\delta+\epsilon) = 0$. Or putting $x-\alpha$ for x and $\beta-\alpha$, $\gamma-\alpha$, &c. for β, γ, &c., we have the equation

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon) = 0,$$

which is by the transformation $x-\alpha$ into $\dfrac{x-\alpha}{m(x-\alpha)+n}$ changed into

$$(x-\alpha')(x-\beta')(x-\gamma')(x-\delta')(x-\epsilon') = 0$$

(where $\alpha'=\alpha$), and the condition in order that in the new equation it may be possible to have simultaneously $\beta'+\gamma'-2\alpha'=0$, $\delta'+\epsilon'-2\alpha'=0$, is

$$(\beta+\gamma-2\alpha)(\delta-\alpha)(\epsilon-\alpha) - (\delta+\epsilon-2\alpha)(\beta-\alpha)(\gamma-\alpha) = 0,$$

or, as this may be written,

$$\begin{vmatrix} 1, & 2\alpha\ \ , & \alpha^2 \\ 1, & \beta+\gamma, & \beta\gamma \\ 1, & \delta+\epsilon, & \delta\epsilon \end{vmatrix} = 0.$$

Hence writing $x+\alpha'$ for x, the last-mentioned equation is the condition in order that the equation

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon) = 0$$

may be transformable into

$$x(x-\beta')(x-\gamma')(x-\delta')(x-\epsilon') = 0,$$

where $\beta'+\gamma'=0$, $\delta'+\epsilon'=0$, that is, into the form $x(x^2-\beta'^2)(x^2-\delta'^2) = 0$. Or replacing y, if we have

$$(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5 = a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\epsilon y),$$

then the equation in question is the condition in order that this may be transformable into the form $(a',\ 0,\ c',\ 0,\ e',\ 0\between x,\ y)^5$; that is, in order that the 18-thic invariant I may vanish. Hence observing that there are 15 determinants of the form in question, and that any root, for instance α, enters as α^2 in 3 of them and in the simple power α in the remaining 12, we see that the product

$$a^{18}\,\Pi \begin{vmatrix} 1, & 2\alpha\ \ , & \alpha^2 \\ 1, & \beta+\gamma, & \beta\gamma \\ 1, & \delta+\epsilon, & \delta\epsilon \end{vmatrix}$$

contains each root in the power 18, and is consequently a rational and integral function of the coefficients of the degree 18, viz. save as to a numerical factor it is equal to the invariant I. And considering the equation $(a, \ldots \between x, y)^5 = 0$ as representing a range of points, the signification of the equation $I = 0$ is that, the pairs (β, γ) and (δ, ϵ) being properly selected, the fifth point α is a focus or sibiconjugate point of the involution formed by the pairs (β, γ) and (δ, ϵ).

Article Nos. 257 to 267.—*Theory of the determination of the Character of an Equation; Auxiliars; Facultative and Non-facultative space.*

257. The equation $(a, b, c \ldots \between x, y)^n = 0$ is a *real* equation if the ratios $a : b : c, \ldots$ of the coefficients are all real. In considering a given real equation, there is no loss of generality in considering the coefficients $(a, b, c \ldots)$ as being themselves real, or in taking the coefficient a to be $= 1$; and it is also for the most part convenient to write $y = 1$, and thus to consider the equation under the form $(1, b, c \ldots \between x, 1)^n = 0$. It will therefore (unless the contrary is expressed) be throughout assumed that the coefficients (including the coefficient a when it is not put $= 1$) are all of them real; and, in speaking of any functions of the coefficients, it is assumed that these are rational and integral real functions, and that any values attributed to these functions are also real.

258. The equation $(1, b, c \ldots \between x, 1)^n = 0$, with α real roots and 2β imaginary roots, is said to have the character $\alpha r + 2\beta i$; thus a quintic equation will have the character $5r$, $3r + 2i$, or $r + 4i$, according as its roots are all real, or as it has a single pair, or two pairs, of imaginary roots.

259. Consider any m functions $(A, B, \ldots K)$ of the coefficients, ($m =$ or $< n$). For given values of $(A, B, \ldots K)$, *non constat* that there is any corresponding equation (that is, the corresponding values of the coefficients $(b, c, \ldots)$ may be of necessity imaginary), but attending only to those values of $(A, B, \ldots K)$ which have a corresponding equation or corresponding equations, let it be assumed that the equations which correspond to a given set of values of $(A, B, \ldots K)$ have a determinate character (one and the same for all such equations): this assumption is of course a condition imposed on the form of the functions $(A, B, \ldots K)$; and any functions satisfying the condition are said to be "auxiliars." It may be remarked that the n coefficients $(b, c, \ldots)$ are themselves auxiliars; in fact for given values of the coefficients there is only a single equation, which equation has of course a determinate character. To fix the ideas we may consider the auxiliars $(A, B, \ldots K)$ as the coordinates of a point in m-dimensional space, or say in m-space.

260. Any given point in the m-space is either "facultative," that is, we have corresponding thereto an equation or equations (and if more than one equation then by what precedes these equations have all of them the same character), or else it is "non-facultative," that is, the point has no corresponding equation.

261. The entire system of facultative points forms a region or regions, and the entire system of non-facultative points a region or regions; and the m-space is thus divided into facultative and non-facultative regions. The surface which divides the

facultative and non-facultative regions may be spoken of simply as the bounding surface, whether the same be analytically a single surface, or consist of portions of more than one surface.

262. Consider the discriminant D, and to fix the ideas let the sign be determined in such wise that D is $+$ or $-$ according as the number of imaginary roots is $\equiv 0$ (mod. 4), or is $\equiv 2$ (mod. 4); then expressing the equation $D=0$ in terms of the auxiliars $(A, B, \ldots K)$, we have a surface, say the discriminatrix, dividing the m-space into regions for which D is $+$, and for which D is $-$, or, say, into positive and negative regions.

263. A given facultative or non-facultative region may be wholly positive or wholly negative, or it may be intersected by the discriminatrix and thus divided into positive and negative regions. Hence taking account of the division by the discriminatrix, but attending only to the facultative regions, we have positive facultative regions and negative facultative regions. Now using the simple term region to denote indifferently a positive facultative region or a negative facultative region, it appears from the very notion of a region as above explained that we may pass from any point in a given region to any other point in the same region without traversing either the bounding surface or the discriminatrix; and it follows that the equations which correspond to the several points of the same region have each of them one and the same character; that is, to a given region there correspond equations of a given character.

264. It is proper to remark that there may very well be two or more regions which have corresponding to them equations with the same character; any such regions may be associated together and considered as forming a kingdom; the number of kingdoms is of course equal to the number of characters, viz. it is $=\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ according as n is even or odd; and this being so, the general conclusion from the preceding considerations is that the whole of facultative space will be divided into kingdoms, such that to a given kingdom there correspond equations having a given character; and conversely, that the equations with a given character correspond to a given kingdom. Hence (the characters for the several kingdoms being ascertained) knowing in what kingdom is situate a point $(A, B, \ldots K)$, we know also the character of the corresponding equations.

265. Any conditions which determine in what kingdom is situate the point $(A, B, \ldots K)$ which belongs to a given equation $(1, b, c \ldots \between x, 1)^n = 0$, determine therefore the character of the equation. It is very important to notice that the form of these conditions is to a certain extent indeterminate; for if to a given kingdom we attach any portion or portions of non-facultative space, then any condition or conditions which confine the point $(A, B, \ldots K)$ to the resulting aggregate portion of space, in effect confine it to the kingdom in question; for of the points within the aggregate portion of space it is only those within the kingdom which have corresponding to them an equation, and therefore, if the coefficients $(b, c, \ldots)$ of the given equation are such as to give to the auxiliars $(A, B, \ldots K)$ values which correspond to a point situate within the above-mentioned aggregate portion of space, such point will of necessity be within the kingdom.

266. In the case where the auxiliars are the coefficients $(b, c, \ldots)$, to any given values of the auxiliars there corresponds an equation, that is, all space is facultative space. And the division into regions or kingdoms is effected by means of the discriminatrix, or surface $D=0$, alone. Thus in the case of the quadric equation $(1, x, y \between \theta, 1)^2=0$ the m-space is the plane. We have $D=x^2-y$, and the discriminatrix is thus the parabola $x^2-y=0$. There are two kingdoms, each consisting of a single region, viz. the positive kingdom or region $(x^2-y=+)$ outside the parabola, and the negative kingdom or region $(x^2-y=-)$ inside the parabola, which have the characters $2r$ and $2i$, or correspond to the cases of two real roots and two imaginary roots, respectively. And the like as regards the cubic $(1, x, y, z \between \theta, 1)^3=0$; the m-space is here ordinary space, $D=-4x^3z+3x^2y^2+6xyz-4y^3-z^2$, and the division into kingdoms is effected by means of the surface $D=0$; but as in this case there are only the two characters $3r$ and $r+2i$, there can be only the two kingdoms $D=+$ and $D=-$ having these characters $3r$ and $r+2i$ respectively, and the determination of the character of the cubic equation is thus effected without its being necessary to proceed further, or inquire as to the form or number of the regions determined by the surface $D=0$: I believe that there are only two regions, so that in this case also each kingdom consists of a single region. But proceeding in the same manner, that is, with the coefficients themselves as auxiliars, to the case of a quartic equation, the m-space is here a 4-dimensional space, so that we cannot by an actual geometrical discussion show how the 4-space is by the discriminatrix or hypersurface $D=0$ divided into kingdoms having the characters $4r$, $2r+2i$, $4i$ respectively. The employment therefore of the coefficients themselves as auxiliars, although theoretically applicable to an equation of any order whatever, can in practice be applied only to the cases for which a geometrical illustration is in fact unnecessary.

267. I will consider in a different manner the case of the quartic, chiefly as an instance of the actual employment of a surface in the discussion of the character of an equation; for in the case of a quintic the auxiliars are in the sequel selected in such manner that the surface breaks up into a plane and cylinder, and the discussion is in fact almost independent of the surface, being conducted by means of the curve (Professor Sylvester's Bicorn) which is the intersection of the plane and cylinder.

Article Nos. 268—273.—*Application to the Quartic equation.*

268. Considering then the quartic equation $(a, b, c, d, e \between \theta, 1)^4=0$ (I retain for symmetry the coefficient a, but suppose it to be $=1$, or at all events positive), then if I, J signify as usual, and if for a moment

$$\vartheta = a^2d-3abc+2b^3,$$

$$X=3aJ+2\,(b^2-ac)\,I,$$

we have identically

$$\tfrac{9}{4}\,(3a^2J^2+X^2)\,\vartheta^2=9\,(b^2-ac)^3\,X^2-a^2\,(b^2-ac)^3\,(I^3-27J^2)-a^2X^3$$

(see my paper, "A discussion of the Sturmian Constants for Cubic and Quartic Equations," *Quart. Math. Journ.*, vol. IV. (1861) pp. 7—12), [290]. And I write

$$x = b^2 - ac,$$

$$y = 3aJ + 2(b^2 - ac) I,$$

$$z = I^3 - 27J^2 (= D).$$

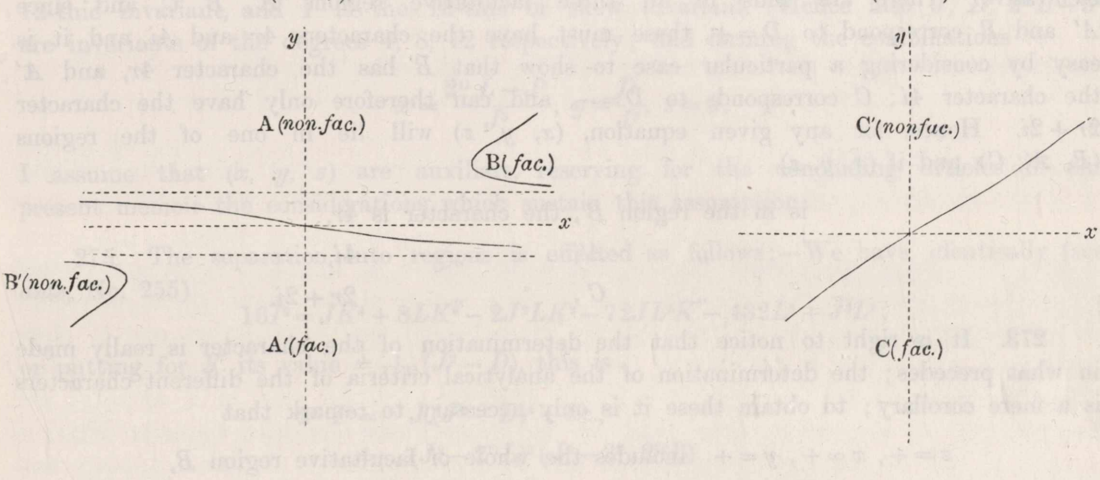
269. I borrow from Sturm's theorem the conclusion (but nothing else than this conclusion) that (x, y, z) possess the fundamental property of auxiliars (that is, that the quartic equations (if any) corresponding to a given system of values of (x, y, z) have one and the same character). The foregoing equation gives $9x^3y^2 - x^3z - y^3 =$ a square function, and therefore positive; that is, the facultative portion of space is that for which $9x^3y^2 - x^3z - y^3$ is $= +$. And the equation

$$x^3 (9y^2 - z) - y^3 = 0$$

is that of the bounding surface, dividing the facultative and non-facultative portions of space.

270. To explain the form of the surface we may imagine the plane of xy to be that of the paper, and the positive direction of the axis of z to be in front of the paper. Taking z constant, or considering the sections by planes parallel to that of xy,

$z = 0$, gives $y^2 (9x^3 - y) = 0$, viz. the section is the line $y = 0$, or axis of x twice, and the cubical parabola $y = x^3$.



$z = +$, the curve $x^3 = \dfrac{y^3}{9y^2 - z}$ has two asymptotes $y = \pm \frac{1}{3}\sqrt{z}$, parallel to and equidistant from the axis of x, and consists of a branch included between the two parallel asymptotes, and two other portions branches outside the asymptotes, as shown in the figure $(z = +)$.

$z=-$, the curve $x^3=\dfrac{y^3}{9y^2-z}$ has no real asymptote, and consists of a single branch, resembling in its appearance the cubical parabola as shown in the figure $(z=-)$.

Taking x as constant, or considering the sections by planes parallel to that of zy, the equation of the section is $z=9y^2-\dfrac{y^3}{x^3}$, which is a cubical parabola, meeting the plane of xy in a point on the cubical parabola $y=9x^3$, and also in a twofold point on the axis of x, that is, touching the plane of xy at the last-mentioned point.

271. The surface consists of a single sheet extending to infinity, the form of which is most easily understood by considering the sections by a system of spheres having the origin of coordinates for their common centre. These sections have all of them the same general form; and one of them is shown fig. 1 of the Plate at the end of the present Memoir, the projection being orthogonal on the plane of xy or plane of the paper, and the spherical curve being shown, the portion of it above the plane of the paper by a continuous line, that below it by a dotted line (the double point in the figure is thus of course only an apparent one): the same figure shows also the sections by planes parallel to that of xy previously shown in the figures $(z=+)$ and $(z=-)$.

272. Now considering the discriminatrix $D=0$, in this case the plane $z=0$, it appears that the bounding surface and this plane divide space into six regions, viz. above the plane of the paper we have the four regions, A non-facultative, B facultative, A' facultative, B' non-facultative, and below it the two regions, C facultative, C' non-facultative. There are thus in all three facultative regions A', B, C, and since A' and B correspond to $D=+$, these must have the characters $4r$ and $4i$, and it is easy by considering a particular case to show that B has the character $4r$, and A' the character $4i$; C corresponds to $D=-$, and can therefore only have the character $2r+2i$. Hence, for any given equation, $(x,\ y,\ z)$ will lie in one of the regions $(B,\ A',\ C)$, and if $(x,\ y,\ z)$

is in the region B, the character is $4r$,

„ A', „ $4i$,

„ C, „ $2r+2i$.

273. It is right to notice that the determination of the character is really made in what precedes; the determination of the analytical criteria of the different characters is a mere corollary; to obtain these it is only necessary to remark that

$z=+,\ x=+,\ y=+$ includes the whole of facultative region B,

that is, $(x,\ y,\ z)$ being each positive, the character is $4r$;

$$\left.\begin{aligned} z=+,\ x=+,\ y=-\\ x=-,\ y=-\\ x=-,\ y=-\end{aligned}\right\}\ \text{include each a part and together the whole of facultative region } A',$$

that is, z being $+$, but (x, y) not each positive, the character is $4i$;

$$\left.\begin{array}{l} z=-,\ x=+,\ y=+ \\ \ \ ,,\ \ \ \ \ \ x=+,\ y=- \\ \ \ ,,\ \ \ \ \ \ x=-,\ y=- \end{array}\right\} \begin{array}{l}\text{include each a part and together the whole} \\ \text{of facultative region } C,\end{array}$$

$$z=-,\ x=-,\ y=+ \quad \text{does not include any facultative space,}$$

that is, z being $-$, the character is $2r+2i$; and the combination of signs $z=-$, $x=-$, $y=+$ is one which does not exist.

The results thus agree with those furnished by Sturm's theorem; and in particular the impossibility of $z=-$, $x=-$, $y=+$ appears from Sturm's theorem, inasmuch as his combination would give a gain instead of a loss of changes of sign.

Article Nos. 274 to 285.—*Determination of the characters of the quintic equation.*

274. Passing now to the case of the quintic, I write

$$\begin{aligned} J &= G, \\ K &= Q, \\ D &= Q', \\ L &= -U, \\ I &= W; \end{aligned}$$

viz. J is the quartinvariant, K and D are octinvariants (D the discriminant), L is 12-thic invariant, and I is the 18-thic or skew invariant. Hence also J, D, $2^{11}L-J^3$ are invariants of the degrees 4, 8, 12 respectively; and forming the combinations

$$x=\frac{2^{11}L-J^3}{J^3},\ y=\frac{D}{J^2},\ z=J,$$

I assume that $(x, y; z)$ are auxiliars, reserving for the concluding articles of the present memoir the considerations which sustain this assumption.

275. The separation into regions is effected as follows:—We have identically (see *ante*, No. 255)

$$16I^2=JK^4+8LK^3-2J^2LK^2-72JL^2K-432L^3+J^3L^2;$$

or putting for K its value $=\frac{1}{128}(J^2-D)$, this is

$$\begin{aligned} 2^{32}I^2 = {} & J(J^2-D)^4+\&\text{c.}, \\ = {} & (J^3-2^{11}L)^2(J^3-3^3.2^{10}L) \\ & +DJ(-4J^6+61.2^{10}J^3L+144.2^{20}L^2) \\ & +D^2J^2(\ \ 6J^3-2^{10}.29L) \\ & +D^3\ \ (-4J^3-\ \ \ 2^{10}L) \\ & +D^4J; \end{aligned}$$

or writing as above

$$x = \frac{2^{11} L - J^3}{J^3}, \; y = \frac{D}{J^2},$$

whence also

$$1 + x = \frac{2^{11} L}{J^3},$$

this is

$$\begin{aligned} 2^{32} \frac{I^2}{J^9} = & - x^2 \left\{ \tfrac{3}{2}(1+x) - 1 \right\} \\ & + y \left\{ 36(1+x)^2 - \tfrac{61}{2}(1+x) - 4 \right\} \\ & - y^2 \left\{ \tfrac{29}{2}(1+x) - 6 \right\} \\ & + y^3 \left\{ -\tfrac{1}{2}(1+x) - 4 \right\} \\ & + y^4, \end{aligned}$$

or, what is the same thing,

$$\begin{aligned} 2 \,.\, 2^{32} \frac{I^2}{J^9} = & - 3x^3 - x^2 \\ & + y(72x^2 + 205x + 125) \\ & + y^2(-29x - 17) \\ & + y^3(-x - 9) \\ & + y^4 \,.\, 2 \\ = & \quad \phi(x, y) \text{ suppose.} \end{aligned}$$

276. Hence also writing $z = J$, we have

$$z\phi(x, y) = 2 \,.\, 2^{32} \frac{I^2}{J^8} = +,$$

or the equation of the bounding surface may be taken to be

$$z\phi(x, y) = 0,$$

that is, the bounding surface is composed of the plane $z = 0$, and the cylinder $\phi(x, y) = 0$. Taking the plane of the paper for the plane $z = 0$, the cylinder meets this plane in a curve $\phi(x, y) = 0$, which is Professor Sylvester's Bicorn: this curve divides the plane into certain regions, and if we attend to the solid figure and instead of the curve consider the cylinder, then to each region of the plane there correspond *in solido* two regions, one in front of, the other behind the plane region, and of these regions *in solido*, one is facultative, the other is non-facultative (viz. for given values of (x, y), whatever be the sign of $\phi(x, y)$, then for a certain sign of z, $z\phi(x, y)$ will be positive or the solid region will be facultative, and for the opposite sign of z, $z\phi(x, y)$ will be negative or the region will be non-facultative). It hence appears that we may attend only to the plane regions, and that (the proper sign being attributed to z, that is to J) each of these may be regarded as facultative. It is to be added that the discriminatrix is in the present case the plane $y = 0$, or, if we attend only to the plane figure, it is the line $y = 0$; so that in the plane figure the separation into regions is effected by means of the Bicorn and the line $y = 0$.

277. Reverting to the equation of the Bicorn, and considering first the form at infinity, the intersections of the curve by the line infinity are given by the equation $y^3(2y-x)=0$, viz. there is a threefold intersection $y^3=0$, and a simple intersection $2y-x=0$; the equation $y^3=0$ indicates that the intersection in question is a point of inflexion, the tangent at the inflexion (or stationary tangent) being of course the line infinity; the visible effect is, however, only that the direction of the branch is ultimately parallel to the axis of x. The equation $2y-x=0$ indicates an asymptote parallel to this line, and the equation of the asymptote is easily found to be $2y-x+5=0$.

278. The discussion of the equation would show that the curve has an ordinary cusp; and a cusp of the second kind, or node-cusp, equivalent to a cusp and node; the curve is therefore a unicursal curve, or the coordinates are expressible rationally in terms of a parameter ϕ; we in fact have

$$x=\frac{-(\phi+2)(\phi^3-\phi^2+2\phi-4)}{\phi^3(\phi+1)},\quad y=\frac{(\phi+2)^2(\phi-3)}{\phi^2(\phi+1)},$$

whence also

$$\frac{dy}{dx}=\tfrac{1}{2}\phi(\phi+2).$$

279. The curve may be traced from these equations (see Plate, fig. 2, where the bicorn is delineated along with a cubic curve afterwards referred to): as ϕ extends from an indefinitely small positive value ϵ through infinity to $-1-\epsilon$, we have the upper branch of the curve, viz.

$\phi=\epsilon$, gives $x=\infty$, $y=-\infty$, point at infinity, the tangent being horizontal,

$\phi=\infty$, gives $x=-1$, $y=+$, the node-cusp, tangent parallel to axis of y,

$\phi=-2$, gives $x=0$, $y=0$, the tangent at this point being the axis of x,

$\phi=-1-\epsilon$, gives $x=\infty$, $y=+$, point at infinity along the asymptote;

and as ϕ extends from $x=-1+\epsilon$ to $x=-\epsilon$, we have the lower branch, viz.

$\phi=-1+\epsilon$, gives $x=-\infty$, $y=-\infty$, point at infinity along the asymptote,

$\phi=-\frac{3}{4}$, $x=-76\frac{23}{27}$, $y=-41\frac{2}{3}$; the cusp, shown in the figure out of its proper position (observe that for $x=-76\frac{23}{27}$, we have for the asymptote $y=-40\frac{25}{27}$, so that the distance below the asymptote is $=\frac{20}{27}$; Professor Sylvester's value $y=-25$ for the ordinate of the cusp is an obvious error of calculation).

$\phi=-\epsilon$, $x=-\infty$, $y=-\infty$, point at infinity, the tangent being horizontal.

The class of the curve is $=4$.

280. The node-cusp counts as a node, a cusp, an inflexion, and a double tangent; the node-cusp absorbs therefore $(6+8+1=)$ 15 inflexions, and the other cusp 8 inflexions; there remains therefore $(24-15-8=)$ 1 inflexion, viz. this is the inflexion at infinity, having the line infinity for tangent; there is not, besides the tangent at the node-cusp, any other double tangent of the curve.

281. The form of the Bicorn, so far as it is material for the discussion, is also shown in the Plate, fig. 3, and it thereby appears that it divides the plane into three regions; viz. these are the regions PQR and S, for each of which $\phi(x, y)$ is $=-$, and the region TU, for which $\phi(x, y)$ is $=+$; that is, for PQR and S we must have $J=-$, and for TU we must have $J=+$. Hence in connexion with the bicorn, considering the line $y=0$, we have the six regions P, Q, R, S, T, U. It has just been seen that for P, Q, R, S we have $J=-$, and for T, U we have $J=+$; and the sign of J being given, the equations $x=\dfrac{2^{11}L-J^3}{J^3}$, $y=\dfrac{D}{J^2}$, then fix for the several regions the signs of $2^{11}L-J^3$ and D, as shown in the subjoined Table; by what precedes each of the six regions has a determinate character, which for R, S, and U (since here D is $=-$) is at once seen to be $3r+2i$, and which, as will presently appear, is ascertained to be $5r$ for P and $r+4i$ for Q and T.

282. We have thus the Table

$$\begin{array}{llll}
P, & D=+, & J=-, & 2^{11}L-J^3=+\,\}\,5r, \\
Q, & D=+, & J=-, & 2^{11}L-J^3=-\,\Big\}\,r+4i, \\
T, & D=+, & J=+, & 2^{11}L-J^3=\pm \\
R, & D=-, & J=-, & 2^{11}L-J^3=\pm \\
S, & D=-, & J=-, & 2^{11}L-J^3=+\,\Big\}\,3r+2i; \\
U, & D=-, & J=+, & 2^{11}L-J^3=\pm
\end{array}$$

so that we have the kingdom $5r$ consisting of the single region P, the kingdom $r+4i$ consisting of the regions Q and T, and the kingdom $3r+2i$ consisting of the regions R, S, and U.

283. For a given equation if D is $=-$, the character is $3r+2i$; if $D=+$, $J=+$, the character is $r+4i$; if $D=+$, $J=-$, then, according as $2^{11}L-J^3$ is $=+$ or is $=-$, the character is $5r$ or $r+4i$. But in the last case the distinction between the characters $5r$ and $r+4i$ may be presented in a more general form, involving a parameter μ, arbitrary between certain limits. In fact drawing upwards from the origin, as in Plate, fig. 3, the lines $x-2y=0$ and $x+y=0$, and between them any line whatever $x+\mu y=0$, the point (x, y), assumed to lie in the region P or Q, will lie in the one or the other region according as it lies on the one side or the other side of the line in question, viz. in the region P if $x+\mu y$ is $=-$, in the region Q if $x+\mu y$ is $=+$. But we have

$$x+\mu y=\frac{2^{11}L-J^3+\mu JD}{J^3},$$

and J being by supposition negative, the sign of $2^{11}L-J^3+\mu JD$ is opposite to that of $x+\mu y$. The region is thus P or Q according to the sign of $2^{11}L-J^3+\mu JD$; and completing the enunciation, we have, finally, the following criteria for the number of real roots of a given quintic equation, viz.

If $D=-$, the character is $3r+2i$,

If $D=+$, $J=+$, then it is $r+4i$.

But if $D=+$, $J=-$, then μ being any number at pleasure between the limits $+1$ and -2, both inclusive, if

$$2^{11}L-J^3+\mu JD=+\text{, the character is } 5r,$$
$$2^{11}L-J^3+\mu JD=-\text{, ,, ,, ,, } r+4i.$$

284. The characters $5r$ of the region P and $r+4i$ of the regions Q and T may be ascertained by means of the equation $(a,\ 0,\ c,\ 0,\ e,\ 0\between\theta,\ 1)^5=0$, that is

$$\theta\,(a\theta^4+10c\theta^2+5e)=0\,;$$

there is always the real root $\theta=0$, and the equation will thus have the character $5r$ or $r+4i$ according as the reduced equation $a\theta^4+10c\theta^2+5e=0$ has the character $4r$ or $4i$. It is clear that $(a,\ e)$ must have the same sign, for otherwise θ^2 would have two real values, one positive, the other negative, and the character would be $2r+2i$. And $(a,\ e)$ having the same sign, then the character will be $4r$, if θ^2 has two real positive values, that is, if $ae-5c^2$ is $=-$, and the sign of c be opposite to that of a and e, or, what is the same thing, if ce be $=-$; but if these two conditions are not satisfied, then the values of θ^2 will be imaginary, or else real and negative, and in either case the character will be $4r$.

285. Now, for the equation in question, putting in the Tables $b=d=f=0$, we find

$$\begin{aligned} D &= 256\ ae^3\,(ae-5c^2)^2,\\ J &= \ 16\ ce\,(ae+3c^2),\\ 2^{11}L-J^3 &= \ 2^{12}\,ce^3\,\{2\,(ae-c^2)^4-c^2\,(ae+3c^3)^3\}\\ &= \ 2^{12}\,ce^3\,(ae-5c^2)\,(2a^3e^3+a^2c^2e^2+8ac^4e+5c^6). \end{aligned}$$

We have by supposition $D=+$, that is, $ae=+$; hence J has the same sign as ce; whence if $J=+$, then also $ce=+$, and the character is $4i$; that is the character of the region T is $r+4i$. But if $J=-$, then also $ce=-$. But ae being $=+$, the sign of $2^{11}L-J^3$ is the same as that of $ce\,(ae-5c^2)$, and therefore the opposite of that of $ae-5c^2$: hence $D=+$, $J=-$, the quartic equation has the character $4r$ or $4i$ according as $2^{11}L-J^3$ is $=+$ or $=-$. Hence the region P has the character $5r$ and the region Q the character $r+4i$; and the demonstration is thus completed.

Article Nos. 286 to 293.—HERMITE'S *new form of* TSCHIRNHAUSEN'S *transformation, and application thereof to the quintic.*

286. M. Hermite demonstrates the general theorem, that if $f(x,\ y)$ be a given quantic of the n-th order, and $\phi\,(x,\ y)$ any covariant thereof of the order $n-2$, then considering the equation $f(x,\ 1)=0$, and writing

$$z=\frac{\phi\,(x,\ 1)}{f_x{}'\,(x,\ 1)}$$

(where $f_x'(x, 1)$ is the derived function of $f(x, 1)$ in regard to x), then eliminating x, we have an equation in z, the coefficients whereof are all of them invariants of $f(x, y)$.

287. In particular for the quintic $f(x, y) = (a, b, c, d, e, f \between x, y)^5$, if

$$\phi_1(x, y),\quad \phi_2(x, y),\quad \phi_3(x, y),\quad \phi_4(x, y)$$

are any four covariant cubics, writing

$$z = \frac{t\,\phi_1(x, 1) + u\,\phi_2(x, 1) + v\,\phi_3(x, 1) + w\,\phi_4(x, 1)}{f_x'(x, 1)}$$

(viz. the numerator is a covariant cubic involving the indeterminate coefficients t, u, v, w) then, in the transformed equation in z, the coefficients are all of them invariants of the given quintic. Conducting the investigation by means of a certain canonical form, which will be referred to in the sequel, he fixes the signification of his four covariant cubics, these being respectively covariant cubics of the degrees 3, 5, 7, and 9, defined as follows; viz. starting with the form

$$-3\left|\begin{array}{cccc} y^3, & -y^2x, & yx^2, & -x^3 \\ a\,, & b\,, & c\,, & d \\ b\,, & c\,, & d\,, & e \\ c\,, & d\,, & e\,, & f \end{array}\right|,$$

$$= -3D, = -3(A, B, C, D \between x, y)^3, \text{ or } (-3A, -B, -C, -3D \between x, y)^3, \text{ suppose,}$$

and considering also the quadric covariant

$$(\alpha, \beta, \gamma \between x, y)^2 = B,$$

then $\phi_1, \phi_2, \phi_3, \phi_4$ are derived from the form

$$(A, B, C, D \between \zeta x - \eta(\beta x + 2\gamma y),\ \zeta y + \eta(2\alpha x + \beta y))^3,$$

viz. we have

$$\phi_1(x, y) = -3(A, B, C, D \between x, y)^3,$$

$$\phi_2(x, y) = +3(A, B, C, D \between x, y)^2(-\beta x - 2\gamma y,\ 2\alpha x + \beta y),$$

$$\{\phi_3(x, y)\} = -3(A, B, C, D \between x, y)(-\beta x - 2\gamma y,\ 2\alpha x + \beta y)^2,$$

$$\{\phi_4(x, y)\} = +3(A, B, C, D \between\ \ -\beta x - 2\gamma y,\ 2\alpha x + \beta y)^3,$$

where $\{\phi_3(x, y)\}$ and $\{\phi_4(x, y)\}$ are the functions originally called by him $\phi_3(x, y)$ and $\phi_4(x, y)$: those ultimately so called by him are

$$^{(1)}\ \phi_3(x, y) = 4\{\phi_3(x, y)\} + J\phi_1(x, y), \qquad (J = G),$$

$$\phi_4(x, y) = 4\{\phi_4(x, y)\} + 3J\phi_2(x, y) + 96\psi_1(x, y),$$

[1] M. Hermite, p. 17, has erroneously written $\phi_3(x, y) + 4A\phi_1(x, y)$, instead of $4\phi_3(x, y) + A\phi_1(x, y)$; the latter expression is that which he really makes use of, and the formula in the text is correct.

where $\psi_1(x,\ y)$ is the cubicovariant $(-27A^2D+9ABC-2B^3, ..\between x,\ y)^3$ of $\phi_1(x,\ y)$, $=(-3A, -B, -C, -3D\between x,\ y)^3$, *ut suprà.*

The covariant $\phi_2(x,\ y)$ has the property that if the given quintic $(a, ...\between x,\ y)^5$ contains a square factor $(lx+my)^2$, then $\phi_2(x,\ y)$ contains the factor $lx+my$: $\{\phi_3(x,\ y)\}$ and $\{\phi_4(x,\ y)\}$ are covariants not possessing the property in question, and they were for this reason replaced by $\phi_3(x,\ y)$ and $\phi_4(x,\ y)$ which possess it, viz. $\phi_3(x,\ y)$ contains the factor $lx+my$, and $\phi_4(x,\ y)$ contains $(lx+my)^3$, being thus a perfect cube when the given quintic contains a square factor.

288. The covariants $\phi_1(x,\ y)$ and $\phi_2(x,\ y)$ are included in my Tables, viz. we have

$$\phi_1(x,\ y)=-3D \text{ of } 142,\ 143,$$

$$\phi_2(x,\ y)=-K \quad ,, \quad ,,$$

(observe that in K the first coefficient vanishes if $a=0$, $b=0$, which is the property just referred to of $\phi_2(x,\ y)$); the other two covariants, as being of the degree 7 and 9, are not included in my Tables, but I have calculated the leading coefficients of these covariants respectively, viz.

Table No. 85 gives leading coefficient (or that of x^3) in $\phi_3(x,\ y)$, and

Table No. 86 gives leading coefficient (or that of x^3) in $\phi_4(x,\ y)$ [and by means thereof we have the values of the covariants in question].

The coefficients in question vanish for $a=0$, $b=0$, that is, $\phi_3(x,\ y)$ and $\phi_4(x,\ y)$ then each of them contain the factor y; if the remaining coefficients of $\phi_4(x,\ y)$ were calculated, it should then appear that for $a=0$, $b=0$, those of x^2y, xy^2 would also vanish, and thus that $\phi_4(x,\ y)$ would be a mere constant multiple of y^3.

Table No. 85 [= leading coefficient of $16BJ-15DG$].

a^3cef^2	+ 1	$a^2b^2ef^2$	− 1	ab^3df^2	+ 64	b^4cf^2	...
$a^3d^2f^2$	+ 15	a^2bcdf^2	− 94	ab^3e^2f	− 54	b^4def	− 144
a^3de^2f	− 32	a^2bce^2f	+ 86	$ab^2c^2f^2$	− 48	b^4e^3	+ 135
a^3e^4	+ 16	a^2bd^2ef	+ 106	ab^2cdef	+ 184	b^3c^2ef	+ 108
		a^2bde^3	− 96	ab^2ce^3	− 135	b^3cd^2f	+ 288
		$a^2c^3f^2$	+ 63	ab^2d^3f	− 272	b^3cde^2	− 450
		a^2c^2def	− 188	$ab^2d^2e^2$	+ 243	b^3d^3e	+ 80
		$a^2c^2e^3$	+ 32	abc^3ef	− 66	b^2c^3df	− 360
		a^2cd^3f	+ 60	abc^2d^2f	+ 212	$b^2c^3e^2$	+ 135
		$a^2cd^2e^2$	+ 68	abc^2de^2	+ 148	$b^2c^2d^2e$	+ 360
		a^2d^4e	− 36	$abcd^3e$	− 412	b^2cd^4	− 160
				abd^5	+ 144	bc^5f	+ 108
				ac^4df	− 36	bc^4de	− 180
				ac^4e^2	− 48	bc^3d^3	+ 80
				ac^3d^2e	+ 124		
				ac^2d^4	− 48		
	± 32		± 415		± 1119		± 1294

Table No. 86 [= leading coefficient of S'].

a^4cef^3	$+\ 9$	a^3bef^3	$-\ 9$	$a^2b^3df^3$	$+\ 120$	ab^4cf^3	$-\ 576$	b^6f^3	$+\ 192$
$a^4d^2f^3$	$+\ 21$	a^3bcdf^3	$-\ 162$	$a^2b^3e^2f^2$	$-\ 21$	ab^4def^2	$+\ 672$	b^5cef^2	$-\ 1440$
a^4def^2	$-\ 78$	$a^3bce^2f^2$	$+\ 99$	$a^2b^2c^2f^3$	$+\ 486$	ab^4e^3f	$-\ 359$	$b^5d^4f^2$	$-\ 192$
a^4e^4f	$+\ 48$	$a^3bd^2ef^2$	$+\ 309$	$a^2b^2cdef^2$	$-\ 2160$	$ab^3c^2ef^2$	$+\ 3456$	b^5de^2f	$-\ 1080$
		a^3bde^3f	$+\ 12$	$a^2b^2ce^3f$	$+\ 1023$	$ab^3cd^2f^2$	$-\ 864$	b^5e^4	$+\ 2025$
		a^3be^5	$-\ 240$	$a^2b^2d^3f^2$	$+\ 120$	ab^3cde^2f	$+\ 2094$	$b^4c^2df^2$	$+\ 2592$
		$a^3c^3f^3$	$-\ 81$	$a^2b^2d^2e^2f$	$-\ 1053$	ab^3ce^4	$-\ 3915$	$b^4c^2e^2f$	$+\ 3546$
		$a^3c^2def^2$	$+\ 1026$	$a^2b^2de^4$	$+\ 1314$	ab^3d^3ef	$+\ 528$	b^4cd^2ef	$+\ 5280$
		$a^3c^2e^3f$	$-\ 768$	$a^2bc^2ef^2$	$-\ 1863$	$ab^3d^2e^3$	$-\ 45$	b^4cde^3	$-\ 13500$
		$a^3cd^3f^2$	$-\ 738$	$a^2bc^2d^2f^2$	$+\ 2538$	$ab^2c^3df^2$	$-\ 2592$	b^4d^4f	$-\ 4800$
		$a^3cd^2e^2f$	$-\ 564$	$a^2bc^2de^2f$	$+\ 2340$	$ab^2c^3e^2f$	$-\ 9747$	$b^4d^3e^2$	$+\ 7800$
		a^3cde^4	$+\ 1056$	$a^2bc^2e^4$	$+\ 672$	$ab^2c^2d^2ef$	$-\ 8496$	$b^3c^4f^2$	$-\ 648$
		a^3d^4ef	$+\ 756$	a^2bcd^3ef	$+\ 2820$	$ab^2c^2de^3$	$+\ 26610$	b^3c^3def	$-\ 14040$
		$a^3d^3e^3$	$-\ 696$	$a^2bcd^2e^3$	$-\ 7812$	ab^2cd^4f	$+\ 8544$	$b^3c^3e^3$	$+\ 3075$
				a^2bd^5f	$-\ 3024$	ab^2cd^4e	$-\ 16650$	$b^3c^2d^3f$	$+\ 9120$
				$a^2bd^4e^2$	$+\ 4572$	ab^2d^5e	$+\ 720$	$b^3c^2d^2e^2$	$+\ 16350$
				$a^2c^4df^2$	$-\ 324$	abc^5f^2	$+\ 972$	b^3cd^4e	$-\ 19200$
				$a^2c^4e^2f$	$+\ 3888$	abc^4def	$+\ 24048$	b^3d^6	$+\ 4800$
				$a^2c^3d^2ef$	$-\ 8748$	abc^4e^3	$-\ 4464$	b^2c^5ef	$+\ 4860$
				$a^2c^3de^3$	$-\ 4800$	abc^3d^3f	$-\ 15984$	$b^2c^4d^2f$	$-\ 3240$
				$a^2c^2d^4f$	$+\ 4248$	$abc^3d^2e^2$	$-\ 30108$	$b^2c^4de^2$	$-\ 8100$
				$a^2c^2d^3e^2$	$+\ 14520$	abc^2d^4e	$+\ 35088$	$b^2c^3d^3e$	$+\ 9000$
				a^2cd^5e	$-\ 11448$	$abcd^6$	$-\ 8640$	bc^2d^5	$-\ 2400$
				a^2d^7	$+\ 2592$	ac^6ef	$-\ 7776$		
						ac^5d^2f	$+\ 5184$		
						ac^5de^2	$+\ 12960$		
						ac^4d^3e	$-\ 14400$		
						ac^3d^5	$+\ 3840$		
	± 78		± 3258		± 41253		± 124716		± 68640

[The values thus are $\phi_3(x, y) = 16BJ - 15DG$; $\phi_4(x, y) = S'$.]

289. The equation in z is of the form

$$z^5 + \frac{\mathfrak{A}}{D} z^3 + \frac{\mathfrak{B}}{D} z^2 + \frac{\mathfrak{C}}{D} z + \frac{\mathfrak{D}}{D} = 0,$$

where D is the discriminant of the quintic and $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ denote rational and integral functions of the coefficients (a, b, c, d, e, f). And the covariants $\phi_1(x, y)$, $\phi_2(x, y)$, $\phi_3(x, y)$, $\phi_4(x, y)$ having the values given to them above, the actual value of $\mathfrak{A}$ is obtained as a quadric function of the indeterminates (t, u, v, w), viz. this is

$$= [D_1t^2 - 6BDtv - D(D_1 - 10AB)v^2] + D[-Bu^2 + 2D_1uw + 9(BD - 10AD_1)w^2],$$

where $D_1 = 25AB + 16C$, these quantities, and the quantity $N (= D_1^2 - 10ABD_1 + 9B^2D)$ afterwards spoken of, being in the notation of the present Memoir as follows:

$$\begin{aligned}
A &= \quad J & &(= \quad G),\\
B &= -K & &(= -Q),\\
C &= \quad 9L + JK & &(= -9U + GQ),\\
D &= \quad D & &(= \quad Q'),\\
D_1 &= \quad 9(16L - JK), & &\\
N &= 1152(18L^2 - JKL - K^3) & &
\end{aligned}$$

290. If by establishing two linear relations between the coefficients (t, u, v, w) the equation $\mathfrak{A}=0$ can be satisfied (which in fact can be done by the solution of a quadric equation), then these quantities can be by means of the relations in question expressed as linear functions of any two of them, say of v and w; and then the next coefficient $\mathfrak{B}$ will be a cubic function $(v, w)^3$, and the equation $\mathfrak{B}=0$ will be satisfied by means of a cubic equation $(v, w)^3=0$, that is, the transformed equation in z can be by means of the solution of a quadric and a cubic equation reduced to the trinomial form

$$z^5+\frac{\mathfrak{C}}{D}z+\frac{\mathfrak{D}}{D}=0,$$

and M. Hermite shows that the equation $\mathfrak{A}=0$ can be satisfied as above very simply, and that in two different ways, viz.

291. 1°. $\mathfrak{A}=0$ if

$$\begin{aligned} D_1t^2-6BDtv-(\quad D_1-10AB\,)\,v^2&=0,\\ Bu^2-2D_1uw-(9BD\;-10AD_1)\,w^2&=0, \end{aligned}$$

that is, N denoting as above, if

$$t=\frac{3BD+\sqrt{ND}}{D_1}v,\qquad u=\frac{D_1+\sqrt{N}}{B}w.$$

292. 2°. Writing the expression for $\mathfrak{A}$ in the form

$$D_1(t^2-Dv^2+2Duw-10ADw^2)+BD(10Av^2-6tv-u^2+9Dw^2),$$

then $\mathfrak{A}=0$, if

$$\begin{aligned} t^2-Dv^2+2Duw-10ADw^2&=0,\\ 10Av^2-6tv-u^2+\;9\;\;Dw^2&=0. \end{aligned}$$

These equations, writing therein

$$t=\frac{1}{\sqrt{2}}\sqrt{D}T,\qquad u=U+5AW,\qquad v=\frac{1}{\sqrt{2}}V,\qquad w=W,$$

become

$$\begin{aligned} T^2-V^2+4UW&=0,\\ -5AV^2+3\sqrt{D}TV+U^2+10AUW+(25A^2-9D)\,W^2&=0, \end{aligned}$$

the first of which is satisfied by the values

$$T=\rho\,W-\frac{1}{\rho}U,\qquad V=\rho\,W+\frac{1}{\rho}U;$$

and then substituting for T and V, the second equation will be also satisfied if only

$$\rho^2=5A+3\sqrt{D}.$$

Article Nos. 293 to 295.—HERMITE'S *application of the foregoing results to the determination of the Character of the quintic equation.*

293. By considerations relating to the form

$$\frac{1}{D}\left\{[D_1t^2 - 6BDtv - D(D_1 - 10AB)v^2] + D[-Bu^2 + 2D_1uw + 9BD - 10AD_1w^2]\right\},$$

M. Hermite obtains criteria for the character of the quintic equation $f(x, 1) = 0$.

294. If $D = -$, the character is $3r + 2i$, but if $D = +$, then expressing the foregoing form as a sum of four squares affected with positive or negative coefficients, the character will be $5r$ or $2 + 4i$, according as the coefficients are all positive, or are two positive and two negative. Whence, if N denote as above, then for

$$D = +,\ N = -,\quad D_1 = +,\ B = -,\ \text{character is } 5r,$$

and

$$\left.\begin{array}{l} D = +,\ N = -,\ BD_1 = + \\ \\ D = +,\ N = + \end{array}\right\}\ \text{character is } r + 4i;$$

and further, the combination $D = +$, $N = -$, $D_1 = -$, $B = +$ cannot arise (Hermite's first set of criteria).

295. Again, from the equivalent form

$$\frac{1}{D}\left\{D_1(t^2 - Dv^2 + 2Duw - 10Aw^2) + BD(10Av^2 - 6tv - u^2 + 9Dw^2)\right\},$$

which, if ω, ω' are the roots of the equation $9\theta^2 - 10A\theta + D = 0$, is

$$= \frac{1}{D}\left\{\frac{D_1\omega - BD}{\omega - \omega'}\left[(t - 3\omega'v)^2 - \omega'\left(u - \frac{D}{\omega'}w\right)^2\right] + \frac{D_1\omega' - BD}{\omega' - \omega}\left[(t - 3\omega v)^2 - \omega\left(u - \frac{D}{\omega}w\right)\right]^2\right\};$$

then by similar reasoning it is concluded that

$$D = +,\ 25A^2 - 9D = +,\ A = -,\ N = -,\ \text{character is } 5r,$$

$$\left.\begin{array}{l} D = +,\ 25A^2 - 9D = +,\ A = -,\ N = +, \\ D = +,\ 25A^2 - 9D = +,\ A = +, \\ D = +,\ 25A^2 - 9D = -, \end{array}\right\}\quad ,, \qquad r + 4i.$$

(Hermite's second set of criteria.)

Article Nos. 296 to 303.—*Comparison with the Criteria* No. 283: *the Nodal Cubic.*

296. For the discussion of Hermite's results, it is to be observed that in the notation of the present Memoir we have

$$\begin{aligned} A &= J, \\ B &= -K = -\tfrac{1}{128}(J^2 - D), \\ D &= D, \\ D_1 &= 16L - JK = \tfrac{1}{128}(2^{11}L - J^3 + JD), \\ N &= 18L^2 - JKL - K^3 \\ &= \frac{1}{2^{21}}\{3^2 \cdot 2^{22}L^2 - 14JL(J^2 - D) - (J^2 - D)^3\}, \end{aligned}$$

or, putting as above,

$$x = \frac{2^{11}L - J^3}{J^3},\ y = \frac{D}{J^2},\ \text{and therefore}\ 1 + x = \frac{2^{11}L}{J^3},\ 1 - y = \frac{J^2 - D}{J^2},$$

we have

$$\begin{aligned} A &= J, \\ B &= \tfrac{1}{128}J^2(y-1), \\ D &= J^2 y, \\ D_1 &= \tfrac{1}{128}J^3(x+y), \\ N &= \frac{1}{2^{21}}J^6\{9(1+x)^2 - 8(1+x)(1-y) - (1-y)^3\}, \\ &= \frac{1}{2^{21}}J^6 \cdot \{y^3 - 3y^2 + 8xy + 9x^2 + 11y + 10x\}. \end{aligned}$$

It thus becomes necessary to consider the curve

$$\psi(x, y) = y^3 - 3y^2 + 8xy + 9x^2 + 11y + 10x = 0,$$

the equation whereof may also be written

$$9x + 4y + 5 = (y-1)\sqrt{25 - 9y}.$$

297. This is a cubic curve, viz. it is a divergent parabola having for axis the line $9x + 4y + 5 = 0$, and its ordinates parallel to the axis of x; and having moreover a node at the point $x = -1$, $y = +1$, that is, at the node-cusp of the bicorn; the curve is thus a nodal cubic; we may trace it directly from the equation, but it is to be noticed that *quà* nodal cubic it is a unicursal curve; the coordinates x, y are therefore rationally expressible in terms of a parameter ψ; and it is easy to see that we in fact have

$$\begin{aligned} 81(x+1) &= \psi^2(\psi - 8), \\ 9(y-1) &= -\psi(\psi - 8), \end{aligned}$$

whence also

$$\frac{dy}{dx} = \frac{-18(\psi - 4)}{\psi(3\psi - 16)}.$$

298. We see that

$\psi=\infty$, gives $x=\infty$, $y=-\infty$, point at infinity, the direction of the curve parallel to axis of x.

$\psi=9$, „ $x=0$, $y=0$, the origin.

$\psi=8$, „ $x=-1$, $y=+1$, the node, tangent parallel to axis of y.

$\psi=\frac{16}{3}$, „ $x=\frac{4325}{2187}$, $y=\frac{209}{81}$, tangent parallel to the axis of y.

$\psi=4$, „ $x=-\frac{145}{81}$, $y=\frac{25}{9}$, tangent parallel to axis of x.

$\psi=0$, „ $x=-1$, $y=+1$, the node.

$\psi=-1$, „ $x=-\frac{10}{9}$, $y=0$.

$\psi=-16$, „ $x=-76\frac{23}{27}$, $y=-41\frac{2}{3}$, the cusp of the bicorn.

$\psi=-\infty$, „ $x=-\infty$, $y=-\infty$, point at infinity, direction of curve parallel to axis of x.

299. The Nodal Cubic is shown along with the Bicorn, Plate, fig. 2; it consists of one continuous line, passing from a point at infinity, through the cusp of the bicorn, on to the node-cusp, then forming a loop so as to return to the node-cusp, again meeting the bicorn at the origin, and finally passing off to a point at infinity, the initial and ultimate directions of the curve being parallel to the axis of x.

300. It may be remarked that, inasmuch as one of the branches of the cubic touches the bicorn at the node-cusp, the node-cusp counts as $(4+2=)$ 6 intersections; the intersections of the cubic with the bicorn are therefore the cusp, the node-cusp, and the origin, counting together as $(2+6+1=)$ 9 intersections, and besides these the point at infinity on the axis of x, counting as 3 intersections. This may be verified by substituting in the equation of the cubic the bicorn ϕ-values of x and y. But to include all the proper factors, we must first write the equation of the cubic in the homogeneous form

$$(9x+8y+5z)^2 z-(y-z)^2(25z-9y)=0,$$

and herein substitute the values

$$x : y : z=-(\phi+2)(\phi^3-\phi^2+2\phi-4) : (\phi+2)^2(\phi-3)\phi : (\phi+1)\phi^3;$$

the result is found to be

$$\phi^3\{(\phi+1)(4\phi^2+6\phi-9)^2-(2\phi+3)^2(4\phi^3+4\phi^2+18\phi+27)\}=0,$$

that is

$$-9\phi^3(\phi+2)(4\phi+3)^2=0;$$

and considering this as an equation of the order 12, the roots are $\phi=0$, 3 times, $\phi=-2$, 1 time; $\phi=-\frac{3}{4}$, 2 times, and $\phi=\infty$, 6 times.

301. The cubic curve divides the plane into 3 regions, which may be called respectively the loop, the antiloop, and the extra cubic; for a point within the loop or antiloop, $\psi(x, y)$ is $=-$, for a point in the extra cubic $\psi(x, y)$ is $=+$. If in conjunction with the cubic we consider the discriminatrix, or line $y=0$, then we have in all six regions, viz. y being $=+$, three which may be called the loop, the triangle,

and the upper region; and y being $=-$, three which may be called the right, left, and under regions respectively; the triangle and the other region form together the antiloop.

302. It is now easy to discuss Hermite's two sets of criteria; the first set becomes

$$y=+,\quad y-1=-,\quad J(x+y)=+,\quad \psi(x,y)=-,\quad \text{character } 5r,$$

$$\left.\begin{array}{ll} y=+, & J(y-1)(x+y)=+,\quad \psi(x,y)=- \\ y=+, & \psi(x,y)=+ \end{array}\right\}\ \text{character } r+4i,$$

$$y=+,\quad y-1=+,\quad J(x+y)=-,\quad \psi(x,y)=-,\quad \text{cannot exist.}$$

Referring to the Plate, fig. 4, which shows a portion of the cubic and the bicorn, then 1° the conditions $y=+, \psi(x, y)=-$ imply that the point (x, y) is within the loop or within the triangle of the cubic; the condition $y-1=-$ brings it to be within the triangle, and for any point within the triangle we have $x+y=-$, whence also the condition $J(x+y)=+$ becomes $J=-$; hence the conditions amount to $J=-$, (x, y) within the triangle; but by the general theory (x, y), being within the triangle, that is, in the region P or T, if $J=-$, will of necessity be within the region P; so that the conditions give $J=-$, (x, y) within the region P; the corresponding character being $5r$, which is right.

2°. $y=+, \psi(x, y)=-$, the point (x, y) must be within the loop, or within the triangle; if (x, y) is within the loop, then $y-1=+$, $x+y=1$, and the condition $J(y-1)(x+y)=+$ becomes $J=-$, that is, we have $J=-$ and (x, y) within the loop, that is, in the region T. And again, if (x, y) be within the triangle, then $y-1=-$, $x+y=+$, and the condition $J(y-1)(x+y)=+$ still gives $J=-$; but $J=-$, and (x, y) within the triangle, that is, in the region T or P, will of necessity be in the region T; so that in either case we have $J=-$, (x, y) in the region T, which agrees with the character $r+4i$.

3°. $y=+, \psi(x, y)=+$, (x, y) is in the upper region, that is, in the region Q or T; if (x, y) is in the region Q, then of necessity $J=-$, and if in the region T, then of necessity $J=+$, that is, we have

$$J=-,\ (x, y) \text{ in the region } Q, \text{ or}$$

$$J=+,\ (x, y) \text{ in the region } P,$$

which agrees with the character $r+4i$.

And it is to be observed that the portions of T under 2° and 3° respectively make up the whole of the region T, and that 3° relates to the whole of the region Q, so that the conditions allow the point (x, y) to be anywhere in Q or T, which is right.

4°. $y=+, \psi(x, y)=-$, (x, y) is in the loop or the triangle, and then $y-1=+$ implies that it is in the loop, whence $x+y=+$, and the condition $J(x+y)=-$ becomes $J=-$; we should therefore if the combination existed have $J=-$, (x, y) within the loop, that is, in the region T; but this is impossible.

303. Hermite's second set of criteria are

$$y=+,\ \tfrac{25}{9}-y=+,\ J=-,\ \psi(x,y)=-,\ \text{character } 5r,$$

$$\left.\begin{array}{l} y=+,\ \tfrac{25}{9}-y=+,\ J=-,\ \psi(x,y)=+ \\ y=+,\ \tfrac{25}{9}-y=+,\ J=+, \\ y=+,\ \tfrac{25}{9}-y=-, \end{array}\right\}\text{character } r+4i.$$

1°. If $y=+$, $\psi(x, y)=-$, then the point (x, y) must be situate within the loop or within the triangle; and recollecting that at the highest point of the loop we have $y=\frac{25}{9}$, the condition $\frac{25}{9}-y=+$ is satisfied for every such point, and may therefore be omitted. The conditions therefore are $J=-$, (x, y) within the loop, that is, in the region T, or within the triangle, that is, in the region P or the region T; but for any point of T the general theory gives $J=+$, and the conditions are therefore $J=-$, (x, y) within the region P; which agrees with the character $5r$.

2°. $y=+$, $\psi(x, y)=+$, that is, (x, y) is within the upper region, that is, in the region Q or T; and $\frac{25}{9}-y=+$, (x, y) will be within the portions of Q and T which lie beneath the line $y=\frac{25}{9}$; but $J=-$, and therefore (x, y) cannot lie in the region T; hence the conditions amount to $J=-$, (x, y) within that portion which lies beneath the line $y=\frac{25}{9}$ of the region Q.

3°. $y=+$, $\frac{25}{9}-y=+$, (x, y) lies beneath the line $y=\frac{25}{9}$, viz. in one of the regions P, Q or T; but $J=+$, (x, y) cannot lie in the region P or Q; hence the conditions give $J=+$, (x, y) within the portion which lies beneath the line $y=\frac{25}{9}$ of the region T.

4°. $y=+$, $\frac{25}{9}-y=-$, that is, (x, y) lies above the line $y=\frac{25}{9}$, and therefore in one of the regions T or Q; and by the general theory, according as (x, y) lies in T or in Q, we shall have $J=+$ or $J=-$, hence the conditions give

$J=-$, (x, y) within the portion which lies above the line $y=\frac{25}{9}$, of the region Q.

$J=+$, (x, y) within the portion which lies above the line $y=\frac{25}{9}$, of the region T.

2°, 3°, and 4°, each of them agree with the character $r+4i$, and together they imply $J=-$, (x, y) anywhere in the region Q, or else $J=+$, (x, y) anywhere in the region T; which is right.

Article Nos. 304 to 307.—HERMITE'S *third set of Criteria; comparison with* No. 283, *and remarks.*

304. In the concluding portion of his memoir, M. Hermite obtains a third set of criteria for the character of a quintic equation; this is found by means of the equation for the function

$$a^4(\theta_0-\theta_1)(\theta_1-\theta_2)(\theta_2-\theta_3)(\theta_3-\theta_4)(\theta_4-\theta_0)$$

of the roots $(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$ of the given quintic equation $(a, b, c, d, e, f\between\theta, 1)^5=0$. The function in question has 12 pairs of equal and opposite values, or it is determined by an equation of the form $(u^2, 1)^{12}=0$, which equation is decomposable, not rationally but by the adjunction thereto of the square root of the discriminant, into two equations of the form $(u^2, 1)^6=0$; viz. one of these is

$$\begin{aligned}&u^{12}\\ &+u^{10}(\mathrm{a}+3\sqrt{\Delta})\\ &+u^8\left[\tfrac{1}{4}(\mathrm{a}-\sqrt{\Delta})^2+\Delta\right]\\ &-u^6\ \mathrm{d}\\ &+u^4\left[\tfrac{1}{4}(\mathrm{a}+\sqrt{\Delta})^2+\Delta\right]\Delta\\ &+u^2(\mathrm{a}-3\sqrt{\Delta})\Delta^2\\ &+\quad\Delta^3=0,\end{aligned}$$

and the other is of course derived from it by reversing the sign of $\sqrt{\Delta}$. I have in the equation written (a, d) instead of Hermite's writing capitals A, D; the sign $-$ of the term in u^6 instead of $+$, as printed in his memoir, is a correction communicated to me by himself. The signification of the symbols is in the author's notation

$$\begin{aligned}\mathrm{a}&=5^4A,\\ \mathrm{d}&=4\,.\,5^9\left(AD-\tfrac{80}{9}D_1\right),\\ \Delta&=5^5D,\end{aligned}$$

whence, in the notation of the present memoir, the expressions of these symbols are

$$\begin{aligned}\mathrm{a}&=\ \ 5^4J,\\ \mathrm{d}&=-\tfrac{1}{2}5^{10}\left(2^{11}L-J^3-\tfrac{3}{5}JD\right),\\ \Delta&=\ \ 5^5D.\end{aligned}$$

305. From the equation in u, taking therein the radical $\sqrt{\Delta}$ as positive, M. Hermite obtains ($\mathrm{d}<0$ a mistake for $\mathrm{d}>0$) the following as the necessary and sufficient conditions for the reality of all the roots,

$$\Delta=+,\quad \mathrm{a}+3\sqrt{\Delta}=-,\quad \mathrm{d}=+,\quad \text{character } 5r$$

(Hermite's third set of criteria).

306. It is clear that $\mathrm{a}+3\sqrt{\Delta}=-$ is equivalent to ($\mathrm{a}=-$ and $\mathrm{a}^2-9\Delta=+$), and we have $\mathrm{a}^2-9\Delta=5^5(125J^2-9D)$, so that these conditions for the character $5r$ are

$$D=+,\quad J=-,\quad 125J^2-9D=+,\quad 2^{11}L-J^3-\tfrac{3}{5}JD=+.$$

Now, writing as above,

$$x=\frac{2^{11}L-J^3}{J^3},\quad y=\frac{D}{J^2},$$

these are $y=+$, $J=-$, $\frac{125}{9}-y=+$, $x-\frac{3}{5}y=-$; the conditions $y=+$, $J=-$ imply that (x, y) is in the region P or the region Q; and the condition $x-\frac{3}{5}y=-$ (observe the line $x-\frac{3}{5}y=0$ lies between the lines $x+y=0$, $x-2y=0$, and so does not cut either the region P or the region Q) restricts (x, y) to the region P; and for every point of P y is at most $=1$, and the condition $\frac{125}{9}-y=+$ is of course satisfied. The condition, $125J^2-9D=+$, is thus wholly unnecessary, and omitting it, the conditions are

$$D=+,\quad J=-,\quad 2^{11}L-J^3-\tfrac{3}{5}JD=0,\quad \text{character } 5r,$$

which, $-\frac{3}{5}$ being an admissible value of μ, agrees with the result *ante*, No. 283.

307. It may be remarked in passing that if 12345 is a function of the roots $(x_1, x_2, x_3, x_4, x_5)$ of a quintic equation, which function is such that it remains unaltered by the cyclical permutation 12345 into 23451, and also by the reversal (12345 into 15432) of the order of the roots, so that the function has in fact the 12 values

$$\alpha_1=12345,\quad \beta_1=24135,$$

$$\alpha_2=13425,\quad \beta_2=32145,$$

$$\alpha_3=14235,\quad \beta_3=43125,$$

$$\alpha_4=21435,\quad \beta_4=13245,$$

$$\alpha_5=31245,\quad \beta_5=14325,$$

$$\alpha_6=41325,\quad \beta_6=12435,$$

then $\phi(\alpha, \beta)$ being any unsymmetrical function of (α, β), the equation having for its roots the six values of $\phi(\alpha, \beta)$ (viz. $\phi(\alpha_1, \beta_1), \phi(\alpha_2, \beta_2)\ldots\phi(\alpha_6, \beta_6)$) can be expressed rationally in terms of the coefficients of the given quintic equation and of the square root of the discriminant of this equation. In fact, v being arbitrary, write

$$L=\Pi_6\{v-\phi(\alpha, \beta)\},\quad M=\Pi_6\{v-\phi(\beta, \alpha)\},$$

then the interchange of any two roots of the quintic produces merely an interchange of the quantities L, M; that is,

$$L+M \text{ and } (L-M)\div\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4, x_5)$$

are each of them unaltered by the interchange of any two roots, and are consequently expressible as rational functions of the coefficients; or observing that $\zeta^{\frac{1}{2}}(x_1, x_2, x_3, x_4, x_5)$ is a multiple of $\sqrt{D}$, we have L a function of the form $P+Q\sqrt{D}$; the equation $L=0$, the roots whereof are $v=\phi(\alpha_1, \beta_1)\ldots v=\phi(\alpha_6, \beta_6)$, is consequently an equation of the form $P+Q\sqrt{D}=0$, viz. it is a sextic equation $(*\between v, 1)^6=0$, the coefficients of which are functions of the form in question. Hence in particular

$$u^2=12345=(x_1-x_2)^2(x_2-x_3)^2(x_3-x_4)^2(x_4-x_5)^2(x_5-x_1)^2$$

is determined as above by an equation $(*\between u^2, 1)^6=0$. Another instance of such an equation is given by my memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order," *Phil. Trans.* vol. CLI. (1861), pp. 263—276, [268].

Article Nos. 308 to 317.—HERMITE'S *Canonical form of the quintic.*

308. It was remarked that M. Hermite's investigations are conducted by means of a canonical form, viz. if $A\ (=J,\ =G)$ be the quartinvariant of the given quintic $(a, b, c, d, e, f \between x, y)^5$, then he in fact finds (X, Y) linear functions of (x, y) such that we have

$$(a, b, c, d, e, f \between x, y)^5 = (\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda' \between X, Y)^5$$

(viz. in the transformed form the two mean coefficients are equal; this is a convenient assumption made in order to render the transformation completely definite, rather than an absolutely necessary one); and where moreover the quadricovariant B of the transformed form is

$$= \sqrt{A}XY,$$

or, what is the same thing, the coefficients $(\lambda, \mu, \sqrt{k}, \sqrt{k}, \mu', \lambda')$ of the transformed form are connected by the relations

$$\left.\begin{aligned} \lambda\mu' - 4\mu\sqrt{k} + 3k &= 0, \\ \lambda'\mu - 4\mu'\sqrt{k} + 3k &= 0, \\ \lambda\lambda' - 3\mu\mu' \quad + 2k &= \sqrt{A}, \end{aligned}\right\}$$

the advantage is a great simplicity in the forms of the several covariants, which simplicity arises in a great measure from the existence of the very simple covariant operator $\dfrac{d}{dX} \cdot \dfrac{d}{dY}$ (viz. operating therewith on any covariant we obtain again a covariant).

309. Reversing the order of the several steps, the theory of M. Hermite's transformation may be established as follows:

Starting from the quintic

$$(a, b, c, d, e, f \between x, y)^5,$$

and considering the quadricovariant thereof

$$(\alpha, \beta, \gamma \between x, y)^2 \qquad B$$

$((\alpha, \beta, \gamma)$ are of the degree 2), and also the linear covariant

$$Px + Qy \qquad J$$

$((P, Q)$ are of the degree 5), we have

$$\beta^2 - 4\alpha\gamma = A, \qquad G$$

and moreover

$$(\alpha, \beta, \gamma \between Q, -P)^2 = -C,$$

viz. the expression on the left hand, which is of the degree 12, and which is obviously an invariant, is $=-C$, where C is (*ut suprà*)

$$C = 9L + JK = -9U + GM.$$

The Jacobian of the two forms, viz.

$$\begin{vmatrix} 2\alpha x+\beta y, & \beta x+2\gamma y \\ P & , \quad Q \end{vmatrix},$$

$$= x\,(2\alpha Q-\beta P)+y\,(\beta Q-2P\gamma),$$

is a linear covariant of the degree 7, say it is

$$= P'x+Q'y,$$

and it is to be observed that the determinant $PQ'-P'Q$ of the two linear forms is $=-2\,(\alpha,\beta,\gamma\between Q,-P)^2$, that is, it is $=2C$.

310. Hence writing

$$T=\frac{1}{2\sqrt{C}}\,(Px+Qy)=\frac{1}{2\sqrt[4]{A}}(\quad X+Y),$$

$$U=\frac{1}{2\sqrt{C}}(P'x+Q'y)=\frac{\sqrt[4]{A}}{2}\quad(-X+Y),$$

whence also

$$X=T\sqrt[4]{A}-\frac{U}{\sqrt[4]{A}},$$

$$Y=T\sqrt[4]{A}+\frac{U}{\sqrt[4]{A}},$$

the determinant of substitution from (X, Y) to (T, U) is $=2$, that from (T, U) to (x, y) is $\frac{1}{4C}\,2C, =\frac{1}{2}$, and consequently that from (X, Y) to (x, y) is $=1$.

We have

$$AT^2-U^2=\frac{1}{4C}\{(\beta^2-4\alpha\gamma)\,(Px+Qy)^2-(P'x+Q'y)^2\};$$

or putting for P', Q' their values, this is $=\frac{1}{4C}$ into $4\,(\alpha,\beta,\gamma\between Q,-P)^2\,(\alpha x^2+2\beta xy+\gamma y^2)$, that is, we have

$$AT^2-U^2=\alpha x^2+\beta xy+\gamma y^2;$$

and we have also

$$AT^2-U^2=\tfrac{1}{4}\sqrt{A}\,[(X+Y)^2-(X-Y)^2]=\sqrt{A}XY,$$

consequently

$$\alpha x^2+\beta xy+\gamma y^2=AT^2-U^2=\sqrt{A}XY.$$

311. We have

$$x=\frac{1}{\sqrt{C}}(\quad Q'T-QU),$$

$$y=\frac{1}{\sqrt{C}}(-P'T+PU),$$

so that, pausing a moment to consider the transformation from (x, y) to (T, U), we have

$$(a, b, c, d, e, f \between x, y)^5 = \frac{1}{\sqrt{C^5}}(a, b, c, d, e, f \between Q'T - QU, \quad -P'T + PU)^5$$

$$= \frac{1}{\sqrt{C^5}}(\mathrm{a, b, c, d, e, f} \between T, U)^5 \text{ suppose,}$$

where (a, b, c, d, e, f) are *invariants*, of the degrees 36, 34, 32, 30, 28, 26 respectively; it follows that b, d, f each of them contain as a factor the 18-thic invariant I, the remaining factors being of the orders 16, 12, 8 respectively.

312. That (a, b, c, d, e, f) are invariants is almost self-evident; it may however be demonstrated as follows. Writing

$$\{y\partial_x\} = \ a\partial_b + 2b\partial_c + 3c\partial_d + 4d\partial_e + 5e\partial_f, \ = \delta \text{ suppose,}$$
$$\{x\partial_y\} = 5b\partial_a + 4c\partial_b + 3d\partial_c + 2e\partial_d + f\partial_e, \ = \delta_1 \quad ,,$$

then $Px + Qy$, $P'x + Q'y$ being covariants, we have $\delta P = 0$, $\delta Q = P$, $\delta P' = 0$, $\delta Q' = P'$, whence, treating T, U as constants, $\delta(Q'T - QU) = P'T - PU$, $\delta(-P'T + PU) = 0$. Hence

$$\begin{aligned}
&\delta(a, b, c, d, e, f \between Q'T - QU, \ -P'T + PU)^5 \\
&\quad = 5(a, b, c, d, e \between Q'T - QU, \ -P'T + PU)^4 . (-P'T + PU) \\
&\qquad + 5(a, b, c, d, e \between \quad ,, \quad ,, \quad)^4 . (\ P'T - PU) \\
&\qquad + 5(b, c, d, e, f \between \quad ,, \quad ,, \quad)^4 \quad . \ 0,
\end{aligned}$$

the three lines arising from the operation with δ on the coefficients (a, b, c, d, e, f) and on the facients $Q'T - QU$ and $-P'T + PU$ respectively; the third line vanishes of itself, and the other two destroy each other, that is,

$$\delta(a, b, c, d, e, f \between Q'T - QU, -P'T + PU)^5 = 0, \text{ and similarly}$$
$$\delta_1(a, b, c, d, e, f \between Q'T - QU, -P'T + PU)^5 = 0,$$

or the function $(a, b, c, d, e, f \between Q'T - QU, -PT + PU)^5$, treating therein T and U as constants, is an invariant, that is, the coefficients of the several terms thereof are all invariants.

313. The expressions for the coefficients (a, b, c, d, e, f) are in the first instance obtained in the forms

$$\begin{aligned}
\mathrm{a} &= \ \ 2(L + 5MC + 10C^2), \\
\mathrm{b} &= -2(L' + 3M'C)A, \\
\mathrm{c} &= \ \ 2(L + MC - 2C^2)A^{-1}, \\
\mathrm{d} &= -2(L' - M'C), \\
\mathrm{e} &= \ \ 2(L - 3MC + 2C^2)A^{-2}, \\
\mathrm{f} &= -2(L' - 5M'C)A^{-1},
\end{aligned}$$

where, developing M. Hermite's expressions,

$72L=$	$24M=$	$24L'=$	$24M'=$
$A^7B \quad + \quad 1$ $A^6C^2 \quad + \quad 1$ $A^5B^2 \quad + \quad 6$ $A^4BC \quad - \quad 24$ $A^3B^2 \quad + \quad 9$ $A^3C^2 \quad - \quad 39$ $A^2B^2C \quad + \quad 9$ $ABC^2 \quad + \quad 108$ $C^3 \quad + \quad 72$	$A^4B \quad - \quad 1$ $A^3C \quad - \quad 1$ $A^2B^2 \quad - \quad 3$ $ABC \quad + \quad 12$ $C^2 \quad + \quad 24$	$ABI \quad + \quad 1$ $CI \quad + \quad 5$	$I+1$

and substituting these values, we find

$36a=$	$36b=$	$36c=$	$36d=$	$36e=$	$36f=$
$A^7B \quad + \quad 1$ $A^6C^2 \quad + \quad 1$ $A^5B^2 \quad + \quad 6$ $A^4BC \quad - \quad 39$ $A^3B^3 \quad + \quad 9$ $A^3C^2 \quad - \quad 54$ $A^2BC \quad - \quad 36$ $ABC^2 \quad + \quad 288$ $C^3 \quad + \quad 1152$	$A^2BI \quad - \quad 3$ $ACI \quad - \quad 24$	$A^6B \quad + \quad 1$ $A^5C \quad + \quad 1$ $A^4B^2 \quad + \quad 6$ $A^3BC \quad - \quad 27$ $A^2B^3 \quad + \quad 9$ $A^2C^2 \quad - \quad 42$ $BC^2 \quad + \quad 144$	$ABI \quad - \quad 3$ $CI \quad - \quad 12$	$A^5B \quad + \quad 1$ $A^4C \quad + \quad 1$ $A^3B^2 \quad + \quad 6$ $A^2BC \quad - \quad 15$ $AB^3 \quad + \quad 9$ $AC^2 \quad - \quad 30$ $B^2C \quad + \quad 36$	$BI \quad - \quad 3$

I have not thought it worth while to make in these formulæ the substitutions $A=J$, $B=-K$, $C=9L+JK$, which would give the expressions for (a, b, c, d, e, f) in terms of J, K, L.

314. Substituting for (x, y) their values in terms of (X, Y), we have

$$(a,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5$$

$$=(a,\ b,\ c,\ d,\ e,\ f \between \frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}}+Q\sqrt[4]{A}\right)X+\frac{1}{2\sqrt{C}}\left(\frac{Q'}{\sqrt[4]{A}}-Q\sqrt[4]{A}\right)Y,$$

$$\frac{1}{2\sqrt{C}}\left(\frac{-P'}{\sqrt[4]{A}}-P\sqrt[4]{A}\right)X+\frac{1}{2\sqrt{C}}\left(-\frac{P'}{\sqrt[4]{A}}+P\sqrt[4]{A}\right)Y$$

$$=(\lambda,\ \mu,\ \nu,\ \nu',\ \mu',\ \lambda' \between X,\ Y)^5 \text{ suppose,}$$

and by what precedes

$$\alpha x^2+\beta xy+\gamma y^2=\sqrt{A}XY;$$

this gives

$$\alpha\partial_y{}^2-\beta\partial_y\partial_x+\gamma\partial_x{}^2=-\sqrt{A}\partial_X\partial_Y,$$

and thence

$$(\alpha\partial_y{}^2-\beta\partial_y\partial_x+\gamma\partial_x{}^2)^2(a,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5$$

$$=A\partial_X{}^2\partial_Y{}^2(\lambda,\ \mu,\ \nu',\ \mu',\ \nu,\ \lambda' \between X,\ Y)^5$$

$$=120A(\nu X+\nu' Y);$$

the left-hand side is a linear covariant of the degree 5, it is consequently a mere numerical multiple of $Px+Qy$; and it is easy to verify that it is $=120(Px+Qy)$. (In fact writing $b=d=e=0$, the expression is $(3c^2\partial_y{}^2 - af\partial_y\partial_x)^2(ax^5+10cx^3y^2+fy^5)$, and the only term which contains x is $a^2f^2.\partial_y{}^2\partial_x{}^2.10cx^3y^2 = 120a^2cf^2.x$; but for $b=d=e=0$, Table J gives $Px=a^2cf^2x$, and the coefficient 120 is thus verified.) But $Px+Qy$ is $=\dfrac{\sqrt{C}}{\sqrt[4]{A}}(X+Y)$, and we have thus $A\nu = A\nu' = \dfrac{\sqrt{C}}{\sqrt[4]{A}}$, whence not only $\nu=\nu'$, $=\sqrt{k}$ suppose, but we have further $k=\dfrac{C}{\sqrt[4]{A^5}}$, a result given by M. Hermite.

315. Substituting for $\nu=\nu'$ the value $\sqrt{k}$, we have

$$(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5$$

$$\begin{aligned}=(a,\ b,\ c,\ d,\ e,\ f\between\ & \frac{1}{2\sqrt{C}}\Big(\frac{Q'}{\sqrt[4]{A}}+Q\sqrt[4]{A}\Big)X+\frac{1}{2\sqrt{C}}\Big(\frac{Q'}{\sqrt[4]{A}}-Q\sqrt[4]{A}\Big)Y,\\ & \frac{1}{2\sqrt{C}}\Big(\frac{-P'}{\sqrt[4]{A}}-P\sqrt[4]{A}\Big)X+\frac{1}{2\sqrt{C}}\Big(-\frac{P'}{\sqrt[4]{A}}+P\sqrt[4]{A}\Big)Y)^5,\end{aligned}$$

$$=(\lambda,\ \mu,\ \sqrt{k},\ \sqrt{k},\ \mu',\ \lambda'\between X,\ Y)^5,$$

and we have then $\alpha x^2+\beta xy+\gamma y^2=\sqrt{A}XY$, viz. the left-hand side being the quadricovariant of $(a,\ b,\ c,\ d,\ e,\ f\between x,\ y)^5$, the equation shows that the quadricovariant of the form $(\lambda,\ \mu,\ \sqrt{k},\ \sqrt{k},\ \mu',\ \lambda'\between X,\ Y)^5$ is $=\sqrt{A}XY$, and we thus arrive at the starting-point of Hermite's theory.

316. The coefficients $(\lambda,\ \mu,\ \sqrt{k},\ \sqrt{k},\ \mu',\ \lambda')$ of Hermite's form are by what precedes *invariants*; they are consequently expressible in terms of the invariants A, B, C (and I). M. Hermite writes

$$\lambda\lambda'=g,\quad \mu\mu'=h,$$

and he finds

$$\sqrt{A}=g-3h+2k,\quad \frac{B}{\sqrt{A^3}}=h-k,\quad \frac{C}{\sqrt{A^5}}=k,$$

or, what is the same thing,

$$g=\frac{A^3+3AB+C}{\sqrt{A^5}},\quad h=\frac{AB+C}{\sqrt{A^5}},\quad k=\frac{C}{\sqrt{A^5}},$$

which give g, h, k in terms of A, B, C, and then putting

$$\Delta=(9k^2+16hk-gh)^2-24hk^3,\ =\frac{I^2}{A^7}$$

(the equation $I^2=A^7\Delta$ is in fact equivalent to the before-mentioned expression of I^2

in terms of the other invariants), the coefficients $(\lambda, \mu, \mu', \lambda')$ are expressed in terms of g, h, k, that is of A, B, C, viz. we have

$$\left\{\begin{aligned}
72\sqrt{k^5}\lambda &= h(g-16k)^2 - 9k(g+16k) + (g-16k)\sqrt{\Delta},\\
24\sqrt{k^3}\mu &= 9k^2 + 16hk - gh - \sqrt{\Delta},\\
24\sqrt{k^3}\mu' &= 9k^2 + 16hk - gh + \sqrt{\Delta},\\
72\sqrt{k^5}\lambda' &= h(g-16k)^2 - 9k(g+16k) - (g-16k)\sqrt{\Delta};
\end{aligned}\right.$$

these values of $(\lambda, \mu, \mu', \lambda')$ could of course be at once expressed in terms of (J, K, L), but I have not thought it necessary to make the transformation.

317. It has been already noticed that the linear covariant $(C, = Px + Qy)$, was

$$= \sqrt{A}\,(\sqrt{k},\ \sqrt{k}\between X,\ Y),$$

it is to be added that the septic covariant $(P'x + Q'y)$ is

$$= \sqrt{A^3}\,(\sqrt{k},\ -\sqrt{k}\between X,\ Y),$$

and that the canonical forms of the cubicovariants $\phi_1(x, y)$, &c. are as follows:

$$\begin{aligned}
\phi_1(X, Y) &= \sqrt{A}\,(\mu,\ \ 3\sqrt{k},\ \ 3\sqrt{k},\ \ \mu'\between X,\ Y)^3,\\
\phi_2(X, Y) &= A\,(\mu,\ \ \sqrt{k},\ -\sqrt{k},\ -\mu'\between X,\ Y)^3,\\
\{\phi_3(X, Y)\} &= \sqrt{A^3}\,(\mu,\ -\sqrt{k},\ -\sqrt{k},\ \ \mu'\between X,\ Y)^3,\\
\{\phi_4(X, Y)\} &= A^2\,(\mu,\ -3\sqrt{k},\ \ 3\sqrt{k},\ -\mu'\between X,\ Y)^3,
\end{aligned}$$

$$\psi_1(X, Y) = \sqrt{A^3}\left(\begin{array}{l}
(2\sqrt{k^3} - 3\mu k + \mu'\mu^2),\\
3\,(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu k),\\
-3\,(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu' k),\\
-\ (2\sqrt{k^3} - 3\mu' k + \mu\mu'^2),
\end{array}\right)\!\!\between X,\ Y)^3,$$

$$\phi_3(X, Y) = \sqrt{A^3}\,(5\mu,\ -\sqrt{k},\ \sqrt{k},\ 5\mu'\between X,\ Y)^3,$$

$$\phi_4(X, Y) = \sqrt{A^3}\left(\begin{array}{l}
(7\sqrt{A}\mu + 96\,(2\sqrt{k^3} - 3\mu k + \mu'\mu^2)),\\
-3\,(3\sqrt{A}\ \sqrt{k} - 96\,(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu k)),\\
+3\,(3\sqrt{A}\ \sqrt{k} - 96\,(\sqrt{k^3} + \mu\mu'\sqrt{k} - 2\mu' k)),\\
-\ (7\sqrt{A}\mu' + 96\,(2\sqrt{k^3} - 3\mu' k + \mu\mu'^2))
\end{array}\right)\!\!\between X,\ Y)^3$$

or, as the last formula may also be written,

$$\phi_4(X, Y) = \sqrt{A^3}\left(\begin{array}{l}
((7g - 53h + 110k)\,\mu - 64\lambda\mu'\sqrt{k}),\\
-3\,((3g + 151h - 90k)\sqrt{k} - 64\lambda'\mu^2),\\
+3\,((3g + 151h - 90k)\sqrt{k} - 64\lambda\mu'^2),\\
-\ ((7g - 53h + 110k)\,\mu' - 64\lambda'\mu\sqrt{k})
\end{array}\right)\!\!\between X,\ Y)^3.$$

It is in fact by means of these comparatively simple canonical expressions that M. Hermite was enabled to effect the calculation of the coefficient $\mathfrak{A}$.

Article Nos. 318 to 326.—*Theory of the imaginary linear transformations which lead to a real equation.*

318. An equation $(a, b, c, \ldots \between x, y)^n = 0$ is real if the ratios $a : b : c$, &c. of the coefficients are all real. In speaking of a given real equation there is no loss of generality in assuming that the coefficients $(a, b, c, \ldots)$ are all real; but if an equation presents itself in the form $(a, b, c, \ldots \between x, y)^n = 0$ with imaginary coefficients, it is to be borne in mind that the equation may still be real; viz. the coefficients may contain an imaginary common factor in such wise that throwing this out we obtain an equation with real coefficients.

In what follows I use the term *transformation* to signify a linear transformation, and speak of equations connected by a linear transformation as *derivable* from each other. An imaginary transformation will in general convert a real into an imaginary equation; and if the proposition were true universally,—viz. if it were true that the transformed equation was always imaginary—it would follow that a real equation derivable from a given real equation could then be derivable from it only by a real transformation, and that the two equations would have the same character. But any two equations having the same absolute invariants are derivable from each other, the two real equations would therefore be derivable from each other by a real transformation, and would thus have the same character; that is, all the equations (if any) belonging to a given system of values of the absolute invariants would have a determinate character, and the absolute invariants would form a system of auxiliars.

But it is not true that the imaginary transformation leads always to an imaginary equation; to take the simplest case of exception, if the given real equation contains only even powers or only odd powers of x, then the imaginary transformation $x : y$ into $ix : y$ gives a real equation. And we are thus led to inquire in what cases an imaginary transformation gives a real equation.

319. I consider the imaginary transformation $x : y$ into

$$(a + bi)\, x + (c + di)\, y \;:\; (e + fi)\, x + (g + hi)\, y,$$

or, what is the same thing, I write

$$x = (a + bi)\, X + (c + di)\, Y,$$

$$y = (e + fi)\, X + (g + hi)\, Y,$$

and I seek to find P, Q real quantities such that $Px + Qy$ may be transformed into a linear function $RX + SY$, wherein the ratio $R : S$ is real, or, what is the same thing, such that $RX + SY$ may be the product of an imaginary constant into a real linear function of (X, Y). This will be the case if

$$Px + Qy = (1 + \theta i)\, \{P\,(aX + cY) + Q\,(eX + gY)\},$$

that is if

$$P(bX+dY)+Q(fX+hY)=\theta\{P(aX+cY)+Q(eX+fY)\},$$

which implies the relations

$$bP+fQ=\theta(aP+eQ),$$
$$dP+hQ=\theta(cP+gQ),$$

or, what is the same thing,

$$(b\ a\theta)P+(f-e\theta)Q=0,$$
$$(d-c\theta)P+(h-g\theta)Q=0,$$

and if the resulting value of $P:Q$ be real, the last-mentioned equations give

$$(ag-ce)\theta^2-(ah+bg-cf-de)\theta+bh-df=0,$$

and θ being known, the ratio $P:Q$ is determined rationally in terms of θ.

320. The equation in θ will have its roots real, equal, or imaginary, according as

$$(ah+bg-cf-de)^2-4(ag-ce)(bh-df),$$

that is

$$\begin{aligned} &a^2h^2+b^2g^2+c^2f^2+d^2e^2 \\ &-2ahbg-2ahcf-2ahde-2bgcf-2bgde-2cfde \\ &+4adfgh+4bceh \end{aligned}$$

is $=+$, $=0$, or $=-$; and I say that the transformation is subimaginary, neutral, and superimaginary in these three cases respectively. In the subimaginary case there are two functions $Px+Qy$ which satisfy the prescribed conditions; in the neutral case a single function; in the superimaginary case no such function. But in the last-mentioned case there are two conjugate imaginary functions, $Px+Qy$, which contain as factors thereof respectively two conjugate imaginary functions $UX+VY$.

321. Hence replacing the original x, y, X, Y by real linear functions thereof, the subimaginary transformation is reduced to the transformation $x:y$ into $kX:Y$, where k is imaginary; and the superimaginary transformation is reduced to $x+iy:x-iy$ into $k(X+iY):(X-iY)$, where k is imaginary. As regards the neutral transformation, it appears that this is equivalent to

$$x=(a+bi)X+(c+di)Y,$$
$$y=\qquad\qquad(g+hi)Y,$$

with the condition $0=(ah+bg)^2-4agbh$, $=(ah-bg)^2$, that is, we have $ah-bg=0$, or without any real loss of generality $g=a$, $h=b$, or the transformation is

$$x=(a+bi)X+(c+di)Y,$$
$$y=\qquad\qquad(a+bi)Y,$$

that is, $x:y=X+kY:Y$, where k is imaginary.

322. The original equation after any real transformation thereof, is still an equation of the form

$$(a, \ldots \between x, y)^n = 0\,;$$

and if we consider first the neutral transformation, the transformed equation is

$$(a, \ldots \between X + kY, Y)^n = 0\,;$$

this is not a real equation except in the case where k is real.

323. For the superimaginary transformation, starting in like manner from $(a, \ldots \between x, y)^n = 0$, this may be expressed in the form

$$(\alpha + \beta i, \gamma + \delta i, \ldots, \gamma - \delta i, \alpha - \beta i \between x + iy, x - iy)^n = 0,$$

viz. when in a real equation $(x, y)^n = 0$ we make the transformation $x : y$ into $x + iy : x - iy$, the coefficients of the transformed equation will form as above pairs of conjugate imaginaries. Proceeding in the last-mentioned equation to make the transformation $x + iy : x - iy$ into $k(X + iY) : X - iY$, I throw k into the form

$$\cos 2\phi + i \sin 2\phi, \; = (\cos\phi + i\sin\phi) \div (\cos\phi - i\sin\phi)$$

(of course it is not here assumed that ϕ is real), or represent the transformation as that of $x + iy : x - iy$ into $(\cos\phi + i\sin\phi)(X + iY) : (\cos\phi - i\sin\phi)(X - iY)$; the transformed equation thus is

$$(\alpha + \beta i, \ldots \alpha - \beta i \between (\cos\phi + i\sin\phi)(X + iY), \; (\cos\phi - i\sin\phi)(X - iY))^n = 0.$$

The left-hand side consists of terms such as $(X^2 + Y^2)^{n-2s}$ into

$$(\gamma + \delta i)(\cos s\phi + i\sin s\phi)(X + iY)^s + (\gamma - \delta i)(\cos s\phi - i\sin s\phi)(X - iY)^s,$$

viz. the expression last written down is

$$= (\gamma\cos s\phi - \delta\sin s\phi)\{(X + iY)^s + (X - iY)^s\}$$

$$- (\gamma\sin s\phi + \delta\cos s\phi)\left\{\frac{(X + iY)^s - (X - iY)^s}{s}\right\},$$

and observing that the expressions in { } are real, the transformed equation is only real if $(\gamma\cos s\phi - \delta\sin s\phi) \div (\gamma\sin s\phi + \delta\cos s\phi)$ be real, that is, in order that the transformed equation may be real, we must have $\tan s\phi = \text{real}$; and observing that if $\tan s\phi$ be equal to any given real quantity whatever, then the values of $\tan\phi$ are all of them real, and that $\tan\phi$ real gives $\cos\phi$ and $\sin\phi$ each of them real, and therefore also ϕ real, it appears that the transformed equation is only real for the transformation

$$x + iy : x - iy = (\cos\phi + i\sin\phi)(X + iY) : (\cos\phi - i\sin\phi)(X - iY),$$

wherein ϕ is real; and this is nothing else than the *real* transformation $x : y$ into $X\cos\phi - Y\sin\phi : X\sin\phi + Y\cos\phi$. Hence neither in the case of the neutral transformation or in that of the superimaginary transformation can we have an imaginary transformation leading to a real equation.

324. There remains only the subimaginary transformation, viz. this has been reduced to $x : y$ into $kX : Y$, the transformed equation is

$$(a, \ldots \between kX,\ Y)^n = 0,$$

and this will be a real equation if some power k^p of k (p not greater than n) be real, and if the equation $(a, .. \between x, y)^n = 0$ contain only terms wherein the index of x (or that of y) is a multiple of p. Assuming that it is the index of y which is a multiple, the form of the equation is in fact $x^\alpha (x^p,\ y^p)^m = 0$, $(n = mp + \alpha)$, and the transformed equation is $X^\alpha (k^p X^p,\ Y^p)^m = 0$, which is a real equation.

325. It is to be observed that if p be odd, then writing $k^p = K$ (K real) and taking k' the real p-th root of K, then the very same transformed equation would be obtained by the real transformation $x : y$ into $k'X : Y$; so that the equation obtained by the imaginary transformation, being also obtainable by a real transformation, has the same character as the original equation.

326. Similarly if p be even, if K be real and positive, the equation $k^p = K$ has a real root k' which may be substituted for the imaginary k, and the transformed equation will have the same character as the original equation; but if K be negative, say $K = -1$ (as may be assumed without loss of generality), then there is no real transformation equivalent to the imaginary transformation, and the equation given by the imaginary transformation has not of necessity the same character as the original equation; and there are in fact cases in which the character is altered. Thus if $p = 2$, and the original equation be $x (x^2,\ y^2)^m = 0$, or $(x^2,\ y^2)^m = 0$, then making the transformation $x : y$ into $iX : Y$, the transformed equation will be $X (X^2,\ -Y^2)^m = 0$ or $(X^2, -Y^2)^m = 0$, giving imaginary roots $X^2 + aY^2 = 0$ corresponding to real roots $x^2 - ay^2 = 0$.

Article No. 327.—*Application to the auxiliars of a quintic.*

327. Applying what precedes to a quintic equation $(a, \ldots . \between x,\ y)^5 = 0$, this after any real transformation whatever will assume the form $(a', \ldots \between x',\ y')^5 = 0$; and the only cases in which we can have an imaginary transformation producing a real equation of an altered character is when this equation is $(a',\ 0,\ c',\ 0,\ e',\ 0 \between x',\ y')^5 = 0$ (c' not $= 0$), or when it is $(a',\ 0,\ 0,\ 0,\ e',\ 0 \between x',\ y')^5 = 0$, viz. when it is $x' (a'x'^4 + 10c'x'^2y'^2 + 5e'y'^4) = 0$, or $x'(a'x'^4 + 5e'y'^4) = 0$. In the latter case the transformation x', y' into $X \sqrt[4]{-1} : Y$ gives the real equation $X (a'X^4 - 5e'Y^4) = 0$. I observe however that for the form $(a', 0, 0, 0, e', 0 \between x, y)^4$, and consequently for the form $(a, \ldots \between x,\ y)^5$ from which it is derived we have $J = 0$; this case is therefore excluded from consideration. The remaining case is $(a', 0, c', 0, e', 0 \between x', y')^5 = 0$, which is by the imaginary transformation $x' : y'$ into $iX : Y$ converted into $(a',\ 0,\ -c',\ 0,\ e',\ 0 \between X,\ Y)^5 = 0$; for the first of the two forms we have $J = 16\, a'c'e'^2$, and for the second of the two forms $J = -16\, a'c'e'^2$, that is, the two values of J have opposite signs. Hence considering an equation $(a,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5 = 0$ for which J is not $= 0$, whenever this is by an imaginary transformation converted into a real equation, the sign of J is reversed; and it follows that, given the values of the absolute invariants and the value of J (or what is sufficient, the sign of J), the

different real equations which correspond to these data must be derivable one from another by real transformations, and must consequently have a determinate character; that is, the Absolute Invariants, and J, constitute a system of auxiliars.

ANNEX.—*Analytical Theorem in relation to a Binary Quantic of any Order.*

The foregoing theory of the superimaginary transformation led me to a somewhat remarkable theorem. Take for example the function

$$(a,\ b,\ c \between x+k,\ 1-kx)^2,$$

or, as this may be written,

$$\begin{array}{c|ccc} & k^2 & k & 1 \\ \hline x^2 & c, & 2b, & a \\ x & 2b, & 2a-2c, & -2b \\ 1 & a, & -2b, & c, \end{array} \quad \text{or} \quad \left(\begin{array}{ccc} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{array}\right.\!\!\between k,\ 1)^2\,(x,\ 1)^2,$$

then the determinant

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix}$$

is a product of linear functions of the coefficients (a, b, c); its value in fact is

$$= -2\,(a+c)\,(a+2bi+ci^2)\,(a-2bi+ci^2),\ = -2\,(a+c)\,[(a-c)^2+4b^2].$$

To prove this directly, I write

$$a' = a - 2bi + ci^2,$$
$$b' = a \qquad - ci^2,$$
$$c' = a + 2bi + ci^2,$$

and we then have

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix}\ \begin{vmatrix} 1, & 2, & 1 \\ i, & 0, & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}$$

$$= \begin{array}{l|l} & (1,\ i,\ i^2),\ (2,\ 0,\ -2i^2),\ (1,\ -i,\ i^2) \\ \hline (\ c, \quad 2b, \quad a) & \\ (2b, \quad 2a-2c, \quad -2b) & \\ (\ a, \quad -2b, \quad c) & \end{array},$$

$$= \begin{vmatrix} i^2a', & -2i^2b', & i^2c' \\ 2ia', & 0b', & -2ic' \\ a', & 2b', & c' \end{vmatrix} = a'b'c' \begin{vmatrix} i^2, & -2i^2, & i^2 \\ 2i, & 0, & -2i \\ 1, & 2, & 1 \end{vmatrix},$$

whence observing that the determinants

$$\begin{vmatrix} 1, & 2\,, & 1 \\ i\,, & 0\,, & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}, \quad \begin{vmatrix} i^2, & -2i^2, & i^2 \\ 2i, & 0\,, & -2i \\ 1\,, & 2\,, & 1 \end{vmatrix}$$

are as $1 : -2$, we have the required relation,

$$\begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix} = -2a'b'c', = -2(a+c)\{(a-c)^2+4b^2\}.$$

It is to be remarked that the determinant

$$\begin{vmatrix} 1, & 2\,, & 1 \\ i\,, & 0\,, & -i \\ i^2, & -2i^2, & i^2 \end{vmatrix}, \text{ taken as the multiplier of } \begin{vmatrix} c, & 2b, & a \\ 2b, & 2a-2c, & -2b \\ a, & -2b, & c \end{vmatrix}$$

is obtained by writing therein $a=b=c, =1$; and multiplying the successive lines thereof by $1, \frac{1}{2}i, i^2$ ($1, \frac{1}{2}, 1$ are the reciprocals of the binomial coefficients 1, 2, 1), the proof is the same, and the multiplier is obtained in the like manner for a function of any order; thus for the cubic $(a, b, c, d\between k+x, 1-kx)^3$,

$$\begin{array}{r|cccc} & k^3 & k^2 & k & 1 \\ \hline = x^3 & -\ d, & 3c, & -3b, & a \\ x^2 & 3c, & -6b+3d, & 3a-6c, & 3b \\ x & -3b, & 3a-6c, & 6b-3d, & 3c \\ 1 & a, & 3b, & 3c, & d \end{array}$$

the multiplier is obtained from the determinant by writing therein $a=b=c=d=1$, and multiplying the successive lines by $1, \frac{1}{3}i, \frac{1}{3}i^2, i^3$, viz. the multiplier is

$$\begin{vmatrix} -1, & 3, & -3, & 1 \\ i, & i, & -i, & i \\ -i^2, & -i^2, & i^2, & i^2 \\ i^3, & 3i^3, & 3i^3, & i^3 \end{vmatrix}$$

and the value of the determinant is found to be

$$9(a-3bi+3ci^2-di^3)(a-bi-ci^2+di^3)(a+bi-ci^2-di^3)(a+3bi+3ci^2+di^3),$$
$$= 9\left((a-3c)^2+(3b-d)^2\right)\left((a+c)^2+(b+d)^2\right).$$

But the theory may be presented under a better form; take for instance the cubic, viz. writing $\frac{x}{y}$ and $\frac{k}{l}$ for x and k respectively, we have $(a, b, c, d \between ky+lx, ly-kx)^3$

$$= \begin{array}{c|cccc} & k^3 & k^2l & kl^2 & l^3 \\ \hline x^3 & -d, & 3c, & -3b, & a \\ x^2y & 3c, & -6b+3d, & 3a-6c, & 3b \\ xy^2 & -3b, & 3a-6c, & 6b-3d, & 3c \\ y^3 & a, & 3b, & 3c, & d \end{array}$$

a bipartite cubic function $(* \between k, l)^3 (x, y)^3$; and the determinant formed out of the matrix is at once seen to be an invariant of this bipartite cubic function.

Assume now that we have identically

$$(a, b, c, d \between x, y)^3 = (a', b', c', d' \between \tfrac{1}{2}(x+iy), \tfrac{1}{2}(x-iy))^3,$$

viz. this equation written under the equivalent form

$$(a', b', c', d' \between X, Y)^3 = (a, b, c, d \between X+Y, i(X-Y))^3,$$

determines (a', b', c', d') as linear functions of (a, b, c, d), it in fact gives

$$\begin{aligned} a' &= (a, b, c, d \between 1, -i)^3 &&= a - 3bi + 3ci^2 - di^3, \\ b' &= (a, b, c, d \between 1, -i)^2 (1, i) &&= a - bi - ci^2 + di^3, \\ c' &= (a, b, c, d \between 1, -i)(1, i)^2 &&= a + bi - ci^2 - di^3, \\ d' &= (a, b, c, d \between 1, i)^3 &&= a + 3bi + 3ci^2 + di^3, \end{aligned}$$

then observing that $ky+lx \pm i(ly-kx) = (x \pm iy)(\mp ik + l)$, we have

$$(a, b, c, d \between ky+lx, ly-kx)^3 = (a', b', c', d' \between \tfrac{1}{2}(x+iy)(-ik+l), \tfrac{1}{2}(x-iy)(ik+l))^3,$$

and if in the expression on the right-hand side we make the linear transformations

$$\begin{aligned} x+iy &= x'\sqrt{2}, & -ik+l &= k'\sqrt{2}, \\ x-iy &= -iy'\sqrt{2}, & ik+l &= -il'\sqrt{2}, \end{aligned}$$

which are respectively of the determinant $+1$, the transformed function is

$$= (a', b', c', d' \between k'x', -l'y')^3,$$

that is, we have

$$(a, b, c, d \between ky+lx, ly-kx)^3 = (a', b', c', d' \between k'x', -l'y')^3.$$

The last-mentioned function is

$$\begin{array}{c|cccc} & k'^3 & k'^2l' & k'l'^2 & l'^3 \\ \hline x'^3 & a' & . & . & . \\ x'^2y' & . & -3b' & . & . \\ x'y'^2 & . & . & +3c' & . \\ y'^3 & . & . & . & -d' \end{array}$$

and (from the invariantive property of the determinant) the original determinant is equal to the determinant of this new form, viz. we have

$$\begin{vmatrix} -\ d, & 3c, & -3b, & a \\ 3c, & -6b+3d, & 3a-6c, & 3b \\ -3b, & 3a-6c, & 6b-3d, & 3c \\ a, & 3b, & 3c, & d \end{vmatrix} = 9a'b'c'd',$$

$$= 9\,[(a-3c)^2 + (3b-d)^2]\,[(a+c)^2 + (b+d)^2],$$

which is the required theorem. And the theorem is thus exhibited in its true connexion, as depending on the transformation

$$(a, \ldots \between x, y)^n = (a', \ldots \between \tfrac{1}{2}(x+iy), \tfrac{1}{2}(x-iy))^n.$$

ADDITION, *7th October*, 1867.

Since the present Memoir was written, there has appeared the valuable paper by MM. Clebsch and Gordan "Sulla rappresentazione tipica delle forme binarie," *Annali di Matematica*, t. I. (1867) pp. 23—79, relating to the binary quintic and sextic. On reducing to the notation of the present memoir the formula 95 for the representation of the quintic in terms of the covariants α, β, which should give for (a, b, c, d, e, f) the values obtained *ante*, No. 312, I find a somewhat different system of values; viz. these are

36a =	36b =	36c =	36d =	36e =	35f =
A^7B + 1	$^*A^4I$ − 1	A^6B + 1	$^*A^3I$ − 1	A^5B + 1	$^*A^2I$ − 1
A^6C + 1	A^2BI − 3	A^5C + 1	ABI − 3	A^4C + 1	ABI − 3
A^5B^2 + 6	$^*ACI$ + 24	A^4B^2 + 6	$^*CI$ + 12	A^3B^2 + 6	
A^4BC − 39		A^3BC − 27		A^2BC − 15	
A^3B^3 + 9		A^2B^3 + 9		AB^3 + 9	
A^3C^2 − 54		A^2C^2 − 42		AC^2 − 30	
A^2B^2C − 126		$^*AB^2C$ − 90		$^*B^2C$ − 54	
ABC^2 + 288		BC^2 + 144			
C^3 + 1152					

where I have distinguished with an asterisk the terms which have different coefficients in the two formulæ. I cannot at present explain this discrepancy.