495.

ON THE ENVELOPE OF A CERTAIN QUADRIC SURFACE.

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To find the envelope of the quadric surface

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

where the coefficients vary subject to the conditions

$$\begin{cases} a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0, \\ \dfrac{p^2}{a} + \dfrac{q^2}{b} + \dfrac{r^2}{c} + \dfrac{s^2}{d} = 0, \end{cases}$$

$(\alpha, \beta, \gamma, \delta)$ and (p, q, r, s) being respectively constant.

We have in the usual manner

$$x^2 + \lambda\alpha^2 + \mu\frac{p^2}{a^2} = 0,$$

$$y^2 + \lambda\beta^2 + \mu\frac{q^2}{b^2} = 0,$$

$$z^2 + \lambda\gamma^2 + \mu\frac{r^2}{c^2} = 0,$$

$$w^2 + \lambda\delta^2 + \mu\frac{s^2}{d^2} = 0$$

and thence $a^2 = \dfrac{-\mu p^2}{x^2 + \lambda\alpha^2}$, &c., and substituting these values μ disappears and we have

$$p\sqrt{(x^2+\lambda\alpha^2)} + q\sqrt{(y^2+\lambda\beta^2)} + r\sqrt{(z^2+\lambda\gamma^2)} + s\sqrt{(w^2+\lambda\delta^2)} = 0,$$

$$\frac{\alpha^2 p}{\sqrt{(x^2+\lambda\alpha^2)}} + \frac{\beta^2 q}{\sqrt{(y^2+\lambda\beta^2)}} + \frac{\gamma^2 r}{\sqrt{(z^2+\lambda\gamma^2)}} + \frac{\delta^2 s}{\sqrt{(w^2+\lambda\delta^2)}} = 0,$$

from which λ is to be eliminated; the second equation is here the derived function of the first in regard to λ, so that rationalising the first equation, the result is, as will be shown, of the form $(*\between\lambda, 1)^4 = 0$, and the result is obtained by equating to zero the discriminant of the quartic function.

Denoting for shortness the first equation by

$$A + B + C + D = 0,$$

the rationalised form is

$$(A^4 + B^4 + C^4 + D^4 - 2A^2B^2 - 2A^2C^2 - 2A^2D^2 - 2B^2C^2 - 2B^2D^2 - 2C^2D^2)^2 - 64A^2B^2C^2D^2 = 0,$$

which is of the form

$$-(\mathfrak{A} + 2\mathfrak{B}\lambda + \mathfrak{C}\lambda^2)^2 + (\mathrm{a, b, c, d, e}\between 1, \lambda)^4 = 0,$$

where

$$\begin{aligned}
\mathfrak{A} &= p^4x^4 \ldots - 2p^2q^2x^2y^2 \ldots,\\
\mathfrak{B} &= p^4\alpha^2x^2 \ldots - p^2q^2(\alpha^2y^2 + \beta^2x^2) \ldots,\\
\mathfrak{C} &= p^4\alpha^4 \ldots - 2p^2q^2\alpha^2\beta^2 \ldots,\\
\mathrm{a} &= 8\,.\,x^2y^2z^2w^2,\\
4\mathrm{b} &= 8\,.\,\alpha^2y^2z^2w^2 + \ldots,\\
6\mathrm{c} &= 8\,.\,\alpha^2\beta^2z^2w^2 + \ldots,\\
4\mathrm{d} &= 8\,.\,\alpha^2\beta^2\gamma^2w^2 + \ldots,\\
\mathrm{e} &= 8\,.\,\alpha^2\beta^2\gamma^2\delta^2.
\end{aligned}$$

Writing I', J' for the two invariants we find without difficulty

$$\begin{aligned}
I' &= I - \tfrac{4}{3}P + \Delta^2,\\
J' &= J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3,
\end{aligned}$$

where

$$\begin{aligned}
I &= \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2,\\
J &= \mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} - \mathrm{c}^3 + 2\mathrm{bcd},\\
\Delta &= \mathfrak{A}\mathfrak{C} - \mathfrak{B}^2,\\
P &= \mathrm{a}\mathfrak{C}^2 - 4\mathrm{b}\mathfrak{B}\mathfrak{C} + 2\mathrm{c}(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) - 4\mathrm{d}\mathfrak{A}\mathfrak{B} + \mathrm{e}\mathfrak{A}^2,\\
Q &= (\mathrm{ce} - \mathrm{d}^2)\mathfrak{A}^2 + (\mathrm{ae} + 2\mathrm{bd} - 3\mathrm{c}^2)\,.\,\tfrac{1}{3}(\mathfrak{A}\mathfrak{C} + 2\mathfrak{B}^2) + (\mathrm{ac} - \mathrm{b}^2)\mathfrak{C}^2\\
&\quad - 2(\mathrm{ad} - \mathrm{bc})\mathfrak{B}\mathfrak{C} \qquad\qquad - 2(\mathrm{be} - \mathrm{cd})\mathfrak{A}\mathfrak{B}.
\end{aligned}$$

The result thus is

$$(I - P + \tfrac{4}{3}\Delta^2)^3 - 27\,(J - Q + \tfrac{1}{3}\Delta P - \tfrac{8}{27}\Delta^3)^2 = 0,$$

or, what is the same thing, it is

$$\begin{aligned}(I-P)^3 - 27\,(J-Q)^2 - 9\Delta P\,(J-2Q)&\\ +\ \Delta^2\,(4I^2 - 8IP + P^2)&\\ +\,8\Delta^3\,(J-2Q)&\\ +\ \Delta^4\,.\,\tfrac{16}{3}I = 0,&\end{aligned}$$

where the left-hand side is of the order 24 in (x, y, z, w). I apprehend that the order should be $=12$ only; for writing (x, y, z, w) in place of (x^2, y^2, z^2, w^2), the equations which connect (a, b, c, d) express that these quantities are the coordinates of a point on a plane cubic; and the problem is in fact that of finding the reciprocal of the plane cubic: this is a sextic cone, or restoring (x^2, y^2, z^2, w^2) instead of (x, y, z, w), we should have a surface of the order 12. I cannot explain how the reduction is effected.