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NOTE ON THE THEORY OF INVARIANTS.

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If two binary quantics $(a,..)(x, y)^n$, $(a',..)(x', y')^n$ are linearly transformable the one into the other, and if for the first of them P, Q are any two invariants whatever of the same degree, and P', Q' are the like invariants for the second of them, then we have

$$P : Q = P' : Q',$$

(or, what is the same thing, the absolute invariants have the same values for the two functions respectively); and the entire system of these equations constitutes only a $(n-3)$ fold relation between the two sets of coefficients. But the converse theorem, viz. that if the entire system of equations is satisfied, the two functions are linearly transformable the one into the other, is only true *sub modo.*

For instance, considering the two binary sextics

$$(0, 0, 0, d, e, f, g)(x, y)^6 \text{ and } (0, 0, 0, d', e', f', g')(x', y')^6,$$

or, what is the same thing,

$$(20d, 5e, 2f, g)(x, y)^3 y^3 \text{ and } (20d', 5e', 2f', g')(x', y')^3 y'^3,$$

the invariants of the two functions respectively are each and all of them $=0$, and yet the two functions are not in general linearly transformable the one into the other.

For they can be transformable only by the substitution

$$x = \lambda x' + \mu y', \quad y = \rho y';$$

or, what is the same thing, only if the cubic functions are transformable by the substitution $x = x' + ay'$, $y = y'$; and forming for these the seminvariants $ac - b^2$ and $a^2d - 3abc + 2b^3$ for the cubic $(a, b, c, d)(x, y)^3$, we have as the necessary condition for the transformability

$$(8df - 5e^2)^3 : (8d^2g - 12def + 5e^3)^2 = (8d'f' - 5e'^2)^3 : (8d'^2g' - 12d'e'f' + 5e'^3)^2.$$

To deduce this result from the theory of the sextic function, I observe that denoting by A, B, C, Δ, the values of the quadrinvariant, the sextinvariant, and the discriminant, as given in Salmon's *Higher Algebra*, Ed. 2, pp. 202—211, then in the particular case $a=0$, $b=0$, we have

$$\begin{array}{llll} A = -10\,d^2 & B = \quad d^4 & C = -\ 8\,d^6 & \Delta = 0, \\ \quad +15\,ce, & \quad -3\,cd^2e & \quad +36\,cd^4e & \\ & \quad +\ c^2e^2, & \quad -39\,c^2d^2e^2 & \\ & & \quad -\ 8\,c^3e^3, & \end{array}$$

and hence forming the new invariants

$$\mathrm{B} = 100\,B - A^2,$$
$$\Gamma = 1000\,C - 1200\,AB + 4A^3,$$

the values of these in the same particular case $a=b=0$ are

$$\begin{array}{ll} \mathrm{B} = \ 25\,c^2(8\,df - 5e^2) & \Gamma = -2500\,c^3(\ 8\,d^2g - 12\,def + 5e^3) \\ \quad -100\,c^3g, & \quad +3000\,c^4(10\,eg\ - 9f^2). \end{array}$$

Taking now A, B, C, Δ as the invariants of the sextic, one of the conditions for the transformation is $B^3 : C^2 = B'^3 : C'^2$.

In the particular case $a=b=c=0$ and $a'=b'=c'=0$, the invariants vanish and the equation is satisfied identically. But if we assume in the first instance only $a=b=0$, $a'=b'=0$, then the terms contain the common factors c^6 and c'^6 respectively; and throwing these out, and then writing $c=0$, $c'=0$, we obtain the condition previously found in a different manner.

It will be observed that the condition is of the original form $P : Q = P' : Q'$, but with the difference that P, Q and the corresponding functions P', Q', are not invariants. As possessing the foregoing property these functions may however be called "imperfect invariants," it being understood that an imperfect invariant is not an invariant, and is not in any case included in the term "invariant" used without qualification.

And we may now establish the general theory as follows: Consider the similarly constituted special forms $(a, ..)(x, y, z, ..)^n$ and $(a', ..)(x', y', z', ..)^n$: to fix the ideas the coefficients $(a, ..)$ may be regarded as homogeneous functions of the elements $(\alpha, \beta, ..)$ which are either independent, or homogeneously connected together in any manner; and then the coefficients $(a', ..)$ will be the like functions of the elements $(\alpha', \beta', ..)$ which are either independent or (as the case may be) homogeneously connected in the like manner.

The entire series of functions $P, Q, ...$ of $(\alpha, \beta, ..)$, which are such that P, Q being of the same degree, and P', Q' being the like functions of (α', β'), we have for the linearly transformable functions $(a, ..)(x, y, z, ..)^n$ and $(a', ...)(x', y', z', ..)^n$ the relation

$$P : Q = P' : Q',$$

may be called the "perfect and imperfect invariants" of $(a, ..)(x, y, z, ..)^n$; and the relation in question be briefly referred to by the expression that the perfect and imperfect invariants are proportional.

We have then the theorem that if the two functions $(a, ..)(x, y, z, ..)^n$ and $(a', ..)(x', y', z', ..)^n$ are linearly transformable the one into the other, the two functions have their perfect and imperfect invariants proportional; and *conversely* the theorem, that two functions which have their perfect and imperfect invariants proportional, are linearly transformable the one into the other.

There is thus a wide field of inquiry in regard to the imperfect invariants, even of a binary function, but still more so as to those of a ternary or quaternary function representing a curve or surface possessed of singularities.

We have in what precedes the explanation of an error into which I fell in my paper "On the transformation of plane curves," *Proc. Lond. Math. Soc.*, vol. I. No. 3, Oct. 1865, [384], see Arts. Nos. 27—30. Considering a given curve of deficiency D and, by means of a system of $D-3$ points chosen at pleasure on the curve, transforming this into a curve of the order $D+1$ with deficiency D; then for any two of the transformed curves (that is, two curves obtained by means of different systems of the $D-3$ points) I showed that these had the same absolute invariants—or in the language of the present paper, that they had their invariants proportional, and I thence inferred that the two transformed curves were linearly transformable the one into the other—whereas, to sustain this conclusion, it is necessary that the two curves should have their perfect and imperfect invariants proportional; and this was in no wise proved. That the two transformed curves are not in fact linearly transformable the one into the other has since been shown *a posteriori* by Dr Brill in the particular case $D=4$. Riemann's conclusions, with which my own were at variance, are thus correct.

I remark that if a binary function of an odd or even degree $n=2p+1$ or $=2p$, has $p+1$ equal factors, then the invariants all of them vanish; but the equality of the $p+1$ factors implies only a p-fold relation between the coefficients; that is, the vanishing of all the invariants gives only a p-fold relation between the coefficients, viz. the relation is $\frac{1}{2}(n-1)$ fold or $\frac{1}{2}n$-fold according as n is odd or even. Thus for a sextic function the equations $A=0$, $B=0$, $C=0$, $\Delta=0$ constitute only a 3-fold relation between the coefficients.

Similarly if the function has p equal factors, then every invariant is a mere numerical multiple of a power of one and the same function Θ; so that the vanishing invariants can be at once formed. And we have thus only a $(p-1)$ fold relation between the coefficients, viz. the relation is $\frac{1}{2}(n-3)$ fold or $\frac{1}{2}(n-2)$ fold according as n is odd or even.

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