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ON THE NON-EUCLIDIAN GEOMETRY.

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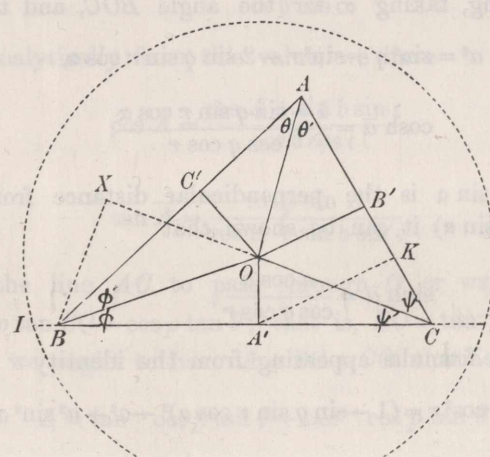
THE theory of the Non-Euclidian Geometry as developed in Dr Klein's paper "Ueber die Nicht-Euklidische Geometrie" may be illustrated by showing how in such a system we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the "hyperbolic" case of plane geometry; viz. the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to the points *within* the circle.

I use the simple letters $a, A, ..$ to denote (linear or angular) distances measured in the ordinary manner; and the same letters, with a superscript stroke, $\bar{a}, \bar{A}, ..$ to



denote the same distances measured according to the theory. The radius of the absolute is for convenience taken to be $=1$; the distance of any point from the centre can therefore be represented as the sine of an angle.

The distance $\overline{BC}$, or say $\overline{a}$, of any two points B, C is by definition as follows:

Radius of circle $= 1$:

In $\triangle\ ABC$, sides are a, b, c:

angles „ A, B, C:

OA, OB, OC are $= \sin p$, $\sin q$, $\sin r$:

OA', OB', OC' „ „ $\sin a$, $\sin b$, $\sin c$:

$\measuredangle BOC$, COA, AOB „ „ α, β, γ.

$$\overline{a} = \tfrac{1}{2} \log \frac{BI \,.\, CJ}{BJ \,.\, CI},$$

(where I, J are the intersections of the line BC with the circle); that is,

$$e^{\overline{a}} + e^{-\overline{a}}, \text{ or } 2 \cosh \overline{a} = \sqrt{\frac{BI \,.\, CJ}{BJ \,.\, CI}} + \sqrt{\frac{BJ \,.\, CI}{BI \,.\, CJ}}, = \frac{BI \,.\, CJ + BJ \,.\, CI}{\sqrt{BI \,.\, BJ}\sqrt{CI \,.\, CJ}},$$

where the numerator is

$$BI\,(BJ - BC) + CI\,(BC + CJ), \quad = BI \,.\, BJ + CI \,.\, CJ + BC\,(CI - BI),$$
$$= BI \,.\, BJ + CI \,.\, CJ + BC^2.$$

Hence taking a for the distance BC, and $\sin q$, $\sin r$, for the distances OB, OC respectively, we have $BI \,.\, BJ = \cos^2 q$, $CI \,.\, CJ = \cos^2 r$; and the formula is

$$\cosh \overline{a} = \frac{\cos^2 q + \cos^2 r + a^2}{2 \cos q \cos r},$$

or, what is the same thing, taking α for the angle BOC, and therefore

$$a^2 = \sin^2 q + \sin^2 r - 2 \sin q \sin r \cos \alpha,$$

we have

$$\cosh \overline{a} = \frac{1 - \sin q \sin r \cos \alpha}{\cos q \cos r}.$$

In a similar manner, if $\sin \mathfrak{a}$ is the perpendicular distance from O on the line BC (that is, $a \sin \mathfrak{a} = \sin q \sin r \sin \alpha$) it can be shown that

$$\sinh \overline{a} = \frac{a \cos \mathfrak{a}}{\cos q \cos r},$$

the equivalence of the two formulæ appearing from the identity

$$\cos^2 q \cos^2 r = (1 - \sin q \sin r \cos \alpha)^2 - a^2 + a^2 \sin^2 \mathfrak{a},$$

which is at once verified.

Next for an angle; we have by definition

$$\overline{A} = \frac{1}{2i} \log \frac{\sin BAI \,.\, \sin CAJ}{\sin CAI \,.\, \sin BAJ},$$

where AI, AJ are the (imaginary) tangents from A to the circle; or writing for shortness BI &c. instead of BAI, &c. (the angular point being always at A),

$$\bar{A} = \frac{1}{2i} \log \frac{\sin BI \,.\, \sin CJ}{\sin CI \,.\, \sin BJ},$$

consequently

$$e^{i\bar{A}} - e^{-i\bar{A}} = 2i \sin \bar{A}$$

$$= \sqrt{\frac{\sin BI \,.\, \sin CJ}{\sin CI \,.\, \sin BJ}} - \sqrt{\frac{\sin CI \,.\, \sin BJ}{\sin BI \,.\, \sin CJ}}, = \frac{\sin BI \sin CJ - \sin BJ \,.\, \sin CI}{\sqrt{\sin BI \,.\, \sin BJ}\sqrt{\sin CI \,.\, \sin CJ}},$$

where the numerator is

$$\sin BI \sin (BJ - BC) - \sin BJ \sin (BI + BC) = \sin BC \sin IJ,$$

or say $= \sin A \sin IJ$. Moreover taking the distance OA to be $= \sin p$, and the perpendicular distances from O on the lines AB, AC to be $\sin \mathfrak{c}$ and $\sin \mathfrak{b}$ respectively, then if for a moment the angle IJ is put $= 2\omega$, we have $\sin p \sin \omega = 1$: moreover

$$\sin BI \sin BJ = \sin (\omega - BO) \sin (\omega + BO) = \sin^2 \omega - \sin^2 BO;$$

and $\sin p \sin BO = \sin \mathfrak{c}$; that is, $\sin BI \sin BJ = \dfrac{1 - \sin^2 \mathfrak{c}}{\sin^2 p}, = \dfrac{\cos^2 \mathfrak{c}}{\sin^2 p}$: and similarly $\sin CI \sin CJ = \dfrac{\cos^2 \mathfrak{b}}{\sin^2 p}$; also

$$\sin IJ = -\sin 2\omega = 2 \sin \omega \cos \omega = \frac{2}{\sin p} \frac{i \cos p}{\sin p};$$

whence the required formula

$$\sin \bar{A} = \frac{\cos p \sin A}{\cos \mathfrak{b} \cos \mathfrak{c}}.$$

In the same way, or analytically from this value, we have

$$\cos \bar{A} = \frac{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}{\cos \mathfrak{b} \cos \mathfrak{c}},$$

and thence also

$$\tan \bar{A} = \frac{\cos p \sin A}{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}.$$

In particular, taking the line AC to pass through O, or writing in the formula $\mathfrak{b} = 0$, we have $\tan \overline{BO} = \cos p \tan BO = \cos p \tan \theta$; that is, $\overline{BO} = \tan^{-1} . \cos p \tan \theta$; and similarly $\overline{CO} = \tan^{-1} \cos p \tan \theta'$; we ought to have $\bar{A} = \overline{BO} + \overline{CO}$, that is,

$$\bar{A} = \tan^{-1} \cos p \tan \theta + \tan^{-1} \cos p \tan \theta'$$

which, observing that $\sin p \sin \theta = \sin \mathfrak{c}$ and $\sin p \sin \theta' = \sin \mathfrak{b}$, also $A = \theta + \theta'$, is in fact equivalent to the above formula for $\tan \bar{A}$.

Observe in particular that when A is at the centre, p is $= 0$, and the formula becomes $\bar{A} = \theta + \theta'$, $= A$, or say for an angle at the centre, $\bar{O} = O$.

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I return to the expression for $\cosh \overline{a}$; in explanation of its meaning, let the distances $\overline{OB}$, $\overline{OC}$ be $\overline{q}$, $\overline{r}$ respectively and let the angle $\overline{BOC}$ be $\overline{\alpha}$; to find $\overline{q}$ we have only to take C at O, that is, in the formula for $\cosh \overline{a}$ to write $r=0$, we thus find $\cosh \overline{q} = \dfrac{1}{\cos q}$: and similarly $\cosh \overline{r} = \dfrac{1}{\cos r}$, whence also

$$\cos q = \text{sech}\, \overline{q}, \quad \sin q = i \tanh \overline{q},$$

$$\cos r = \text{sech}\, \overline{r}, \quad \sin r = i \tanh \overline{r},$$

also, as seen above, $\overline{\alpha} = \alpha$; the formula thus is

$$\cosh \overline{a} = \frac{1 + \tanh \overline{q} \tanh \overline{r} \cos \overline{\alpha}}{\text{sech}\, \overline{q}\, \text{sech}\, \overline{r}}$$

$$= \cosh \overline{q} \cosh \overline{r} + \sinh \overline{q} \sinh \overline{r} \cos \alpha,$$

or, what is the same thing, it is

$$\cos \overline{\alpha} = \frac{\cosh \overline{a} - \cosh \overline{q} \cosh \overline{r}}{\sinh \overline{q} \sinh \overline{r}},$$

viz. as will presently appear, this is the formula for $\cos \overline{BOC}$ in the triangle BOC.

From the above formulæ

$$\cosh \overline{a} = \frac{1 - \sin q \sin r \cos \alpha}{\cos q \cos r},$$

and

$$\sin \overline{A} = \frac{\cos p \sin A}{\cos \mathfrak{b} \cos \mathfrak{c}}, \quad \cos \overline{A} = \frac{\cos A + \sin \mathfrak{b} \sin \mathfrak{c}}{\cos \mathfrak{b} \cos \mathfrak{c}},$$

and the like formulæ for $\overline{b}$, $\overline{c}$, $\overline{B}$, $\overline{C}$, it may be shown that in the triangle ABC we have

$$\cosh \overline{a} = \frac{\cos \overline{A} + \cos \overline{B} \cos \overline{C}}{\sin \overline{B} \sin \overline{C}}.$$

In fact, substituting the foregoing values, this equation becomes

$$\frac{(1 - \sin^2 \mathfrak{a})(\cos A + \sin \mathfrak{b} \sin \mathfrak{c}) + (\cos B + \sin \mathfrak{c} \sin \mathfrak{a})(\cos C + \sin \mathfrak{a} \sin \mathfrak{b})}{\sin B \sin C \cos \mathfrak{b} \cos \mathfrak{c}} = \frac{1 - \sin q \sin r \cos \alpha}{\cos q \cos r},$$

that is,

$$\cos A + \cos B \cos C - \sin^2 \mathfrak{a} \cos A + \sin \mathfrak{a} \sin \mathfrak{b} \cos B + \sin \mathfrak{a} \sin \mathfrak{c} \cos C + \sin \mathfrak{b} \sin \mathfrak{c}$$
$$= \sin B \sin C (1 - \sin q \sin r \cos \alpha),$$

or, what is the same thing,

$$\sin^2 \mathfrak{a} (\cos B \cos C - \sin B \sin C) + \sin \mathfrak{a} \sin \mathfrak{b} \cos B + \sin \mathfrak{a} \sin \mathfrak{c} \cos C + \sin \mathfrak{b} \sin \mathfrak{c}$$
$$= -\sin B \sin C \sin q \sin r \cos \alpha,$$

that is,

$$(\sin \mathfrak{a} \cos B + \sin \mathfrak{c})(\sin \mathfrak{a} \cos C + \sin \mathfrak{b}) = \sin B \sin C (\sin^2 \mathfrak{a} - \sin q \sin r \cos \alpha),$$

a relation which I proceed to verify.

We may, from the formulæ

$$a^2 = \sin^2 q + \sin^2 r - 2\sin q \sin r \cos\alpha, \quad a\sin\mathfrak{a} = \sin q \sin r \sin\alpha, \ \&c.,$$

but, more simply, geometrically as presently shown, deduce

$$\sin\mathfrak{a}\cos B + \sin\mathfrak{c} = \frac{1}{a}\sin B \sin q\,(\sin q - \sin r\cos\alpha),$$

$$\sin\mathfrak{a}\cos C + \sin\mathfrak{b} = \frac{1}{a}\sin C \sin r\,(\sin r - \sin q\cos\alpha),$$

and thence

$$(\sin\mathfrak{a}\cos B + \sin\mathfrak{c})(\sin\mathfrak{a}\cos C + \sin\mathfrak{b}) = \frac{1}{a^2}\sin B\sin C\sin q\sin r\begin{Bmatrix}\sin q\sin r\,(1+\cos^2\alpha)\\ -\cos\alpha\,(\sin^2 q + \sin^2 r)\end{Bmatrix}$$

$$= \frac{1}{a^2}\sin B\sin C\sin q\sin r\,(\sin q\sin r\sin^2\alpha - a^2\cos\alpha)$$

$$= \sin B\sin C\,(\sin^2\mathfrak{a} - \sin q\sin r\cos\alpha),$$

which is the equation in question. For the subsidiary equations used in the demonstration, observe that the four points O, X, A', B lie in a circle, and consequently that $CO\,.\,CX = CA'\,.\,CB$; or multiplying each side by $\sin C$, then $CO\,.\,CX\,.\sin C = A'K\,.\,CB$, that is,

$$\sin r\,(\sin r - \sin q\cos\alpha)\sin C = a\,(\sin\mathfrak{a}\cos C + \sin\mathfrak{b}),$$

and the other of the equations in question is proved in the same manner.

From the formula for $\cosh\bar{a}$ we find

$$\sinh\bar{a} = \frac{1}{\sin\bar{B}\sin\bar{C}}\,\Delta,$$

where

$$\Delta^2 = -(1 - \cos^2\bar{A} - \cos^2\bar{B} - \cos^2\bar{C} - 2\cos\bar{A}\cos\bar{B}\cos\bar{C}),$$

whence also

$$\sinh\bar{a} : \sinh\bar{b} : \sinh\bar{c} = \sin\bar{A} : \sin\bar{B} : \sin\bar{C};$$

and we can also obtain

$$\cos\bar{A} = \frac{\cosh\bar{a} - \cosh\bar{b}\cosh\bar{c}}{\sinh\bar{b}\sinh\bar{c}}\ \&c.$$

So that the formulæ are in fact similar to those of spherical trigonometry with only $\cosh\bar{a}$, $\sinh\bar{a}$ &c. instead of $\cos a$, $\sin a$ &c. The before-mentioned formula for $\cos\bar{\alpha}$ in terms of $\bar{a}$, $\bar{q}$, $\bar{r}$ is obviously a particular case of the last-mentioned formula for $\cos\bar{A}$.

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