CAYLEY

MATHEMATICAL PAPERS

8

1895

# MATHEMATICAL PAPERS.

# THE COLLECTED

# MATHEMATICAL PAPERS

OF

ARTHUR CAYLEY, Sc.D., F.R.S.,

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Ribaucour C.R. t. 75 - 1872 pp. 533-536. Referring to my Note remarks that the condition can be ~~expressed~~ (by means of the imaginary coordinates of M. Ossian Bonnet) expressed in a simple form. Communicated by him to the Philomathic Society, May 1870. I reproduce this investigation, altho' it is not easy to present it ~~in a quite intelligible form: we take~~ ~~The investigation is as follows: he takes~~ $p = f(x, y, z)$ to represent a family of surfaces belonging to a triply orthotomic system. Considering two neighbouring surfaces (A) & (A') corresponding to the values $z$ & $z + dz$; A & A' the two points where they meet the trajectories of the surfaces; AT, A'T' the tangents to the curves of curvature of the same system at A, A' respectively. Then according to the remark of M. Lévy, it is to be expressed that these lines meet, and this is done by expressing that along the trajectory AA', the variation of the angle of AT with the osculating plane at A is equal to the angle of the osculating planes at A, A' respectively.

Let B' be the spherical image of A', the plane OBB' is parallel to the osculating plane at A of the trajectory, and the angle of the two osculating planes measures the geodesic curvature of BB': denote this by $d\gamma$

Let $\beta$ be the angle of BB' with BX, $\theta$ the angle of AT with BX, $\beta - \theta$ is the angle of AT with the osculating plane at A of the trajectory: $d\beta - d\theta = d\gamma$

Introducing the symmetric imaginary coordinates $x$ & $y$ we write

$$a = \frac{dp}{\lambda^2 dx},\quad b = \frac{dp}{\lambda^2 dy},\quad c = \frac{1}{\lambda^2}\frac{d^2 p}{dx\,dy}$$

$$ds^2 = 4\lambda^2 \frac{da}{dx}\frac{db}{dy}\,dx\,dy$$

But $dx$ & $dy$ being the increments of $x, y$ corresponding to $dz$ in the passage from A to A', then by a theorem of M. Liouville

$$d\gamma = d\beta - i\left(\frac{d\lambda}{\lambda dx}dx - \frac{d\lambda}{\lambda dy}dy\right);$$

the condition thus is

$$d\theta = i\left(\frac{d\lambda}{\lambda dx}dx - \frac{d\lambda}{\lambda dy}dy\right)$$

and the formula $e^{-2i\theta} = \pm\sqrt{\frac{da}{dx}} \div \sqrt{\frac{db}{dx}}$

enables this to be written in the definitive form

$$dx\frac{d}{dx}l\left(\frac{db}{dy} \div \lambda^4\frac{da}{dx}\right) - dy\frac{d}{dy}l\left(\frac{da}{dx} \div \lambda^4\frac{db}{dy}\right) + dz\,\frac{d}{dz}l\frac{db}{dy} - \frac{d}{dz}l\left(\frac{da}{dx}\right) = 0.$$

We have

$$dx\left(\tfrac{1}{2}p + c\right) + dy\frac{db}{dy} + dz\frac{db}{dz} = 0$$

$$dx\frac{da}{dx} + dy\left(\tfrac{1}{2}p + c\right) + dz\frac{da}{dz} = 0$$

and thence eliminating $dx, dy, dz$ we have

$$\begin{vmatrix} \frac{d}{dx}l\left(\frac{db}{dy} \div \lambda^4\frac{da}{dx}\right), & \frac{d}{dy}l\left(\frac{db}{dy} \div \lambda^4\frac{da}{dx}\right), & \frac{d}{dz}l\left(\frac{db}{dy} \div \frac{da}{dx}\right) \\ \tfrac{1}{2}p + c, & \frac{db}{dy}, & \frac{db}{dz} \\ \frac{da}{dx} & \tfrac{1}{2}p + c & \frac{da}{dz} \end{vmatrix} = 0$$

which defines the triply orthotomic system.

# PREFATORY NOTE.

THE death of Professor Cayley, which occurred on the 26th of January, 1895, has deprived the later part of this volume, as it will deprive the succeeding volumes, of the advantage of his supervision. The Syndics of the Press desired that the collection of the papers should be completed; and on the 15th of February, they asked me to undertake the duty of editing the remaining volumes. I willingly acceded to their request.

Professor Cayley had himself passed the first thirty-eight sheets of this volume for press; his illness prevented him from even revising any succeeding sheets. He had prepared one Note for the volume: it is printed at the end. The remaining volumes must appear without Notes and References: the reason being that he did not prepare these Notes in advance but only when the corresponding papers came before him in the proof-sheets.

The actual manuscript of the Note has been reproduced in facsimile upon the kind of paper which he regularly used during his mathematical investigations. As it refers to the memoir that ends only in the last sheet but one which he passed for press, it is one of the last pieces of his writing.

He left no instructions as to the Collection of his Mathematical Papers; the statement, prefixed to the first volume, is the only account of his method of arrangement. A comparison of the contents of the first seven volumes with the list of his papers in the Royal Society's *Catalogue of Scientific Papers* has enabled me to make out the detailed course of the method which will, of course, be followed in the remaining volumes.

The Syndics expressed their desire that I should insert some biographical notice of Professor Cayley in a volume of the series. Accordingly, one is inserted in the present volume; it is a reprint (with only slight verbal changes) of the notice which was written for the *Proceedings of the Royal Society.* And I have ventured to add a complete list of the lectures which he announced from year to year after his return to Cambridge in 1863 as Sadlerian Professor.

A. R. FORSYTH.

8 *June*, 1895.

# ARTHUR CAYLEY.

[From the Obituary Notices in the *Proceedings of the Royal Society*, vol. LVIII. 1895.]

ARTHUR CAYLEY was the second son of Henry Cayley and Maria Antonia Doughty; he was born at Richmond, in Surrey, on 16 August, 1821.

The family, to whose fame so much honour has been added by one of the greatest mathematicians of all time, is of old origin and illustrious descent. Its name, like not a few English names, is derived from a locality in Normandy; there was a Castellum Cailleii, near Rouen, held by baronial tenure. The head of the house appears to have come to England with William the Conqueror and to have settled in Norfolk, becoming Lord of Massingham, Cranwich, Brodercross, and Hiburgh in that county. The influence of the family increased and, by the time of Edward II., Sir Thomas de Cailli possessed estates also in Yorkshire. On his decease without issue, the Yorkshire property was transferred to a younger branch of the family and was inherited by a long succession of Cayleys who made their home at Thormanby. One of these was knighted, as Sir William Cayley, in 1641; in 1661 he was created a baronet in recognition of his services during the Civil Wars, the title surviving to the present day. The fourth son of Sir William, Cornelius, settled at York; and the eldest son of the latter, also Cornelius, born in 1692, was a barrister and in 1725 was appointed Recorder of Kingston-upon-Hull, an office which he held until a few years before his death in 1779. Probably the advantages offered by Hull, then, as now, the greatest port on the northern coast of England, suggested commerce as an occupation for some members of the Recorder's large family; two of his sons became Russia merchants, settling in St Petersburg. The younger of these, being the fifth son of the Recorder, was Henry Cayley, born in 1768; he married, in 1814, Maria Antonia Doughty, a daughter of William Doughty. The eldest son of this marriage died in infancy. The youngest son, Charles Bagot, was a scholar, possessed of linguistic genius; he was particularly interested in the Romance Languages and he made verse-translations of Homer's *Iliad*, Dante's *Divine Comedy*, and the *Sonnets* of Petrarch. The second son was Arthur, the subject of the present sketch; he was born during a visit of his parents to England. Before passing to the details of his life, it may be added that the second of his father's sisters married Edward Moberly—also a Russia merchant living in St Petersburg—and was the mother of the late Dr. George Moberly, Bishop of Salisbury.

Mr. Henry Cayley took his young family to Russia and remained there for a few years. On retiring from business in 1829, he returned to England and settled into residence at Blackheath. Arthur was sent soon afterwards to a private school there, kept by the Rev. G. B. F. Potticary; and when he was fourteen he was transferred to King's College School, London. At a very early age he had begun to show some of those preferences by which the existence of mathematical ability is wont to reveal itself; he had a great liking for numerical calculations and he developed a great aptitude for them.

In his new school the boy showed himself to be possessed of remarkable ability: his power of grasping a new subject very rapidly and of seizing its central principles was certainly unusual. An old friend tells of an examination in chemistry: the subject had not been studied by Cayley before, but he soon acquired sufficient knowledge to carry off the medal from the professedly chemical students, to their surprise and mortification *. But it was most of all by the indications of mathematical genius that he astonished his teachers. It had been Mr. Cayley's intention to educate his son with the view of placing him in his former business—an intention not abandoned without reluctance. The impression, however, produced upon his teachers could not lightly be set aside; and the advice of the Principal to send him to Cambridge, where his abilities promised to secure brilliant distinction, was adopted.

Accordingly, he went to Cambridge. He was entered at Trinity College on 2nd May, 1838, as a pensioner, and he began residence in the succeeding October at the unusually early age of seventeen. He passed through the ordinary stages in the career of a successful student of mathematics. Like the other able undergraduates of his period, he "coached" with William Hopkins of Peterhouse, who has been described as a great and stimulating teacher—a description justified by the high achievements of a long line of distinguished and grateful pupils.

Cayley's fame grew rapidly: and, as is the way of Cambridge undergraduates, he soon was pointed out as the future Senior Wrangler of the year. It is interesting to find a record of him written about this time and published not long afterwards by an acquaintance†, who says that:—

> "As an undergraduate he had generally the reputation of a mere mathematician, which did him great injustice, for he was really a man of much varied information, and that on some subjects the very opposite of scientific—for instance, he was well up in all the current novels, an uncommon thing at Cambridge where novel-reading is not one of the popular weaknesses."

* It may be added that he maintained his interest in chemistry throughout his life, and acquired a considerable knowledge of it. When he was at Baltimore, in 1882, lecturing at the Johns Hopkins University by special invitation, he attended Professor Remsen's lectures with a pleasure which found expression in his letters home to his children in England. And on one occasion, at Professor Remsen's request, he lectured to the chemistry class on the hydrocarbon "trees" (*Brit. Assoc. Report*, 1875, pp. 257—305).

† Bristed, *Five Years in an English University* (second edition, 1852), p. 95.

It may be added that Cayley declared the story about him in the tripos, recorded by Bristed, to be quite apocryphal.

So also was another story, belonging to a later part of his life, according to which he is reported to have said that "the object of law was to say a thing in the greatest number of words, and of mathematics to say it in the fewest": this view, and the possibility of his ever having held it, he repudiated entirely.

Novel-readers are more frequent in Cambridge now than they appear to have been in 1842, and Cayley in his later days avoided reading some of the modern novels; but it is worth noting, as will subsequently be seen more in detail, that he had this "popular weakness" all his life.

He was admitted a scholar of the College on 1st May, 1840, winning his scholarship at the earliest time when it was possible to do so: and he secured a first class in each of the annual examinations of the College. No record of marks for the first and the second years is given in the Trinity Head Examiner's Book; but in the third year the marks are given and, as he then scored more than twice the marks of the second candidate, the Head Examiner separated him from the rest of the first class by drawing a line under his name. This presage of his powers was confirmed in the following year, 1842, when he graduated as Senior Wrangler; the Examiners were so definitely satisfied that he was first as to dispense in his case with the viva voce tests which at that time were a customary part of the Tripos. And in due course the first Smith's Prize was awarded to him in the succeeding examination.

Cayley's own "year" at Trinity was a distinguished one; for, in addition to himself, it contained Mr. (now the Right Honourable) George Denman, for many years a Judge of the High Court of Justice, and Mr. Hugh Andrew Johnstone Munro, one of the foremost of Latin Scholars of any period. And the distinction of Cayley's contemporaries in neighbouring years is marked: it is impossible to avoid noticing the names of some of the graduates in the Mathematical Tripos about that time. Sylvester and Green (second and fourth wranglers respectively in 1837), Leslie Ellis (senior in 1840), Stokes (senior in 1841), Cayley (senior in 1842), Adams (senior in 1843), Thomson—now Lord Kelvin—(second in 1845), constitute an extraordinary succession of mathematicians of whom England is justly proud. Their achievements in mathematical science have done much to render their University one of the acknowledged chief mathematical schools of the world.

Cayley was elected a Fellow of Trinity and admitted to fellowship on 3rd October, 1842, at an age younger than any other Fellow of the College, at least in the present century; and he was promoted from the position of Minor Fellow to that of Major Fellow on 2nd July, 1845, the year in which he proceeded to his M.A. degree. He was an Assistant Tutor of the College for three years; but such a post was then of an almost nominal character, and there appears to be no indication that any of the mathematical teaching of the College fell to him. He did, indeed, accept some private pupils: his lifelong friend, Canon Venables, has given a pleasant account* of a reading-party which Cayley took to Aberfeldie in 1842.

His pupils, however, did not tie him strictly to Cambridge, for it appears that the latter half of the year 1843 was devoted to continental rambles. The summer was spent in Switzerland, where his zest for walking and for mountain-climbing, a pleasure that never failed while his health lasted, found an active outlet: he had become a member of the Alpine Club in its comparatively early days. The last four months of the year were spent in Italy, partly in the North and in Florence, partly in Rome and Naples. It may have been on this tour that he acquired his love for

* *Guardian*, 6th Feb. 1895, p. 201.

both painting and architecture. The works of painters such as Masaccio, Giovanni Bellini, Perugino, and Luini, then first became known to him; they proved a delight at the time and remained a happy remembrance with him.

These and other continental journeys from time to time, while he remained in residence as a Fellow of his College, were his relaxations. He had no formal lecturing and he did not attempt to obtain a large number of private pupils. The leisure that he thus secured was turned to the best, and to him the most pleasant, of uses, in carrying out mathematical researches. It was, indeed, as an undergraduate that Cayley began the marvellous series of publications which, extending over more than fifty years of his life, have been concerned with practically every branch of pure mathematics as well as with theoretical dynamics and physical astronomy.

The time seemed ripe for the outburst of some mathematical activity. By the efforts of Herschel, Peacock, and Whewell, Cambridge teaching had been set free from the bonds that restricted methods of procedure to those which had proved effective in Newton's days; and the struggle to secure the admittance of analytical methods had been successfully completed. One sign of the new freedom was the foundation of the *Cambridge Mathematical Journal*, in 1837, by D. F. Gregory and Leslie Ellis. Before that time, practically the only English means of publication open to mathematicians was in the *Philosophical Transactions of the Royal Society;* and young writers, whether modest or not about the value of their researches, might well have hesitated before seeking publication in a quarter that exacts so high a standard. The new journal then founded was open to young students and gave them an opportunity, previously difficult to obtain, of making their researches known; and it proved a great stimulus to the intellectual activity of those members of the University. Only four volumes of the journal appeared; but it was continued, first under the name of the *Cambridge and Dublin Mathematical Journal*, and, subsequently down to the present time, under that of the *Quarterly Journal of Pure and Applied Mathematics.* Though the opportunities of publication, which now are afforded to mathematicians both in England and abroad, are vastly more numerous than they were half a century ago, the undoubted service rendered to English mathematics by the initial venture of the two young Cambridge men should not be forgotten.

It was in the second volume of this journal that Cayley's earliest paper, written in 1841, was printed: and two other papers bearing the same date—it was the year before his degree—are included in the third volume. Though the results are not remarkable, the freshness and the independence of these early investigations are worthy of notice. Cayley had evidently read with enquiring and critical care the *Mécanique Analytique* of Lagrange, some of the work of Laplace, and several memoirs in the two continental journals of the time, those of Liouville and Crelle. These achievements of an undergraduate of nineteen or twenty, which are rarely accomplished now and were still rarer in his day, recall Abel's dictum*:—

"Si l'on veut faire des progrès dans les mathématiques, il faut étudier les maîtres et non pas les écoliers."

* *Niels-Henrik Abel* (par Bjerknes, Paris, 1885), p. 173.

It was as certainly one of the characteristics of Cayley to find a stimulus to new developments in the main ideas of other writers as it was one of his characteristics to be able to follow out his own ideas with the insistent unwearying patience of an investigator creating a new work complete. And it is interesting to see how this faculty of receiving inspiration reveals itself from the beginning of his career.

Once free from the necessity of preparing for his Tripos and his Fellowship examination, he was able to throw himself into the work of production. His activity may be estimated from the fact that he produced three papers in 1842, eight in 1843, four in 1844, and thirteen in 1845. Moreover, these papers deal with a great variety of subjects. Thus he makes his first investigations in the numerative calculus of plane curves: he initiates his discussions about geometry of $n$ dimensions: he founds the theory of invariants and covariants: and he elucidates the connexion between doubly-infinite products and elliptic functions. Some of these early papers are now classical; and the briefest inspection of them is sufficient to reveal the suggestiveness and the easy strength of the young mathematician who was not yet in his twenty-fifth year.

Even by this date the opportunities of publication in England had become inadequate to his needs. Curiously enough, he does not appear to have sent any paper to the Royal Society until the year 1852, when Sylvester communicated the "Analytical Researches connected with Steiner's Extension of Malfatti's Problem*" to that Society. Later in the same year, Cayley was elected a Fellow of the Society, and thereafter many of his papers appear in its *Philosophical Transactions*. Before 1852, there were few journals either at home or abroad which did not receive communications from him: and even in the quite early years of his researches, several of his papers, written in French, appeared in Liouville's journal and in Crelle's journal. As societies and journals grew in number, so the area over which his papers spread became ever wider.

At first, after winning his Trinity Fellowship, he remained at Cambridge, and his time must then have been largely at his own disposal. This freedom, in his circumstances, could last for only a limited time because, unless he either entered holy orders or devoted himself to teaching in some permanent post (if obtainable) in the College, the Fellowship could be held for not more than seven years after his M.A. degree—a period that would expire in 1852. He was unwilling to take holy orders—not that there was any religious obstacle in his way, for he was not harassed either by philosophical doubts or critical difficulties. His simple reason for remaining a layman was that, though devout in spirit and an active Churchman, he felt no vocation for the sacred office.

In consequence, it became necessary to choose some profession. Cayley selected the law, left Cambridge in 1846, entered at Lincoln's Inn, and became a pupil of the famous conveyancer, Mr. Christie. A story of their first interview, that Mr. Christie used to tell in after years, is an illustration of the modesty and the lack of self-

* Cayley's *Collected Mathematical Papers*, vol. II. No. 114. Subsequent references to this series will be made in the form *C. M. P.*

assertiveness which were leading features of Cayley's character: and this impression is confirmed by the recollections of a fellow-pupil, Mr. T. C. Wright, who says:—

> "... We fellow-pupils knew that Arthur Cayley had been the Senior Wrangler of his year, and that he possessed extraordinary abilities; but they were not indicated by his personal bearing, and the retiring modesty of his disposition prevented him from ever alluding to the honours he had won at Cambridge. He had one of the most unsophisticated minds I have ever known; jokes, and the badinage of the pupil-room, seemed to be delightful novelties to him, and his face beamed with amusement as he listened to them without taking much part in the conversation, being content to devote his time assiduously to work which I suspect was not altogether congenial to his taste...."

But if the modest, almost shy, man did not display his honours, he could not conceal his powers; and very soon his clearness of head, his almost intuitive grasp of the principles of any subject that came before him, his capacity for work and his power of concentration, made him a favourite pupil. He was called to the Bar on 3rd May, 1849, and thereafter he had no occasion to wait for business. Mr. Christie was always ready to supply him with at least as much conveyancing work as he was willing to undertake: but no advice, no encouragement, no opening however favourable, least of all any wish for fame or fortune, could tempt him to subside into a large practice. He restricted himself to "devilling" for Mr. Christie, and he limited the amount of work he would undertake in this way, always refusing work that came to him at first hand. There is no doubt that, had he remained at the Bar and devoted himself to its business, he could have made a great legal reputation and a substantial fortune: even as it was, some of his drafts* have been made to serve as models. But the spirit of research possessed him; it was not merely will but an irresistible impulse that made the pursuit of mathematics, not the practice of law, his chief desire. To achieve this desire, he reserved with jealous care a due portion of his time; and he regarded his legal occupations mainly as the means of providing a livelihood.

He remained at the Bar for fourteen years. Between two and three hundred papers are the mathematical outcome of that period; and they include some of the most brilliant of his discoveries. Among these papers are to be found the majority of his famous memoirs on quantics (particularly the sixth memoir, in which he develops his theory of geometry, and shows that all geometry can be made entirely descriptive), his work upon matrices, numerous contributions to the theory of symmetric functions of the roots of an equation, the elaborate calculations connected with the development of functions arising in the planetary and the lunar theories, and his valuable reports on theoretical dynamics. The enormous range over which his papers of these fourteen years extend is not more remarkable than the vigour of his contributions to knowledge; and a reference to them will show that he frequently recurs to some given problem, always adding something to the development.

* In Davidson's *Precedents and Forms in Conveyancing* (third edition, 1873), vol. III. Part II. p. 1067, the author adds a footnote, calling "attention to the remarkable skill exhibited in [a] settlement, the work of Mr. Arthur Cayley."

In judging of this persistent and unflagging activity, some account ought to be taken of his surroundings. It can hardly be that 2, Stone Court, from which many of his papers are dated, proved an inspiration to mathematical research. For part of the time, his friend Sylvester was in London—then as an actuary; and I have heard Cayley describe how Sylvester and he walked round the Courts of Lincoln's Inn discussing the theory of invariants and covariants which occupied (and occasionally absorbed) the attention of both of them during the fifties. And on matters which related to analytical geometry he was in frequent (but formal) correspondence with Salmon; indeed, the relation that existed between the two men developed ultimately into one of warm friendship and deep mutual regard: its sincerity can be gathered from the spirit animating Salmon's notice of Cayley, published in *Nature* in 1883, at the time when the latter was President of the British Association. But, with special exceptions of the types indicated, his work was so largely of the kind that is called path-breaking that he was bound to do it alone: he did it with a simple unconscious courage and with unfailing resolution.

It may easily be imagined that his links with life at Cambridge had now become slight. During the earliest of the years spent at the bar, he had returned on a few occasions. In 1848, the year before his call, he was the junior mathematical examiner in the regular annual examinations of Trinity; in 1849, and also in 1850, he was the senior mathematical examiner in the same examinations. In 1851 he was Senior Moderator for the Mathematical Tripos; one of the wranglers, Lightfoot, becoming subsequently his friend, and his colleague in the University, before going to his great work in the diocese of Durham as Bishop. In 1852 he was Senior Examiner for the Tripos, the senior wrangler of the year being Tait (also afterwards one of his intimate friends), now Professor of Natural Philosophy at Edinburgh. These seem to have been the only occasions when he was recalled to Cambridge; and they did not require any permanent connexion with the College or the University. He was settled in London, his allegiance divided between law and mathematics.

A change, however, in the statutes of the University offered an opportunity for his return to Cambridge; a professorship of pure mathematics was established upon an old foundation. Lady Mary Sadleir (who endowed the Croonian Lecture Fund of the Royal College of Physicians of London and also that of the Royal Society in memory of her first husband, Dr. William Croone, a physician and one of the earliest Fellows of the Royal Society) had, by her will, dated 25th September, 1701, and proved 6th November, 1706, given to the University an estate, which was to be used as an endowment of lectureships in algebra at nine of the colleges in Cambridge. These posts were duly established. The great developments of analysis, which took place at the end of the last century and during the first half of the present century, gradually proved that the restriction to algebra prevented the lectureships from being as adequate an encouragement to the advancement of mathematics as they were designed to be at the time of their establishment. Moreover, the lecturers had ceased to attract undergraduates to their lectures: so that the purpose of the foundation was not being fulfilled. Consequently, in 1857, a proposal was made by the Council of the Senate of the University that a new direction should be given to the endowment by the establishment of a professorship, to be called the Sadlerian Professorship of Pure Mathematics: the duty of the professor was "to explain and teach the principles of pure mathematics, and to apply himself to the advancement of that science." The proposal was approved by the Senate on 3rd December, 1857, and the new statute was sanctioned by an Order of the Queen in Council on 7th March, 1860. Some time had to elapse before certain provisional arrangements could be completed, and it was not until after three years that the University was in a position to act.

On 10th June, 1863, Cayley was elected Sadlerian professor: he held the chair for the rest of his life. The stipend attached to the professorship was modest, though it was improved in the course of subsequent legislation; these changes, however, could not have been foreseen at the time when Cayley was elected. Yet he had no hesitation about returning to Cambridge: for the post enabled him to devote his life to the pursuit he liked best. He never showed the slightest regret at having neglected the prospects of distinction at the bar, or at having chosen to return to his University; and he always expressed perfect satisfaction and content with his life in Cambridge, which was one of great happiness.

His appointment as Sadlerian professor marks a turning point in his life. Henceforward he lived, for the most part, in the quiet of the University; yet it was by no means in seclusion, for he took his share in administration, which claims a part (often too large a part) of the leisure of men fitted for this necessary duty. But he was not burdened by heavy claims arising out of his official position: and he was directed by the statutes governing him to do what was, as a matter of fact, his ideal in life. No man could have been better suited than Cayley was to fulfil the charge of the statutes: his knowledge and his power of research pointed him out as the obvious choice of the electors.

He settled in Cambridge at once. On 8th September, 1863, he married Susan, daughter of Robert Moline, of Greenwich. This is not the place to dwell upon his domestic life; but it is impossible to omit in silence all reference to its singular happiness, based upon the affection felt by its members for one another. Friends and visitors who have been in that home will not soon forget the kindness and the gracious courtesy of the welcome they received, or the atmosphere of peace into which they were raised. Sometimes in the old garden by the river-side, more often in the drawing-room, the talk went on; the professor himself listening, attentive and watchful, frequently taking only a slight share, but ever ready to join in. No cynicism or paradox in speech was ventured upon in his presence; no harshness of judgment was tolerated without a quiet protest; no sense of bustle or ambition was felt there; in all things the charm of an old-world home, centred round him. His widow and their two children, Mary and Henry, remain to mourn their loss.

His teaching duty was limited to the delivery of one course of lectures in the academic year, and he usually chose the Michaelmas term. This practice was maintained for twenty-three years until he was placed under the new statutes, which in 1882 had come into operation so far as concerned all future appointments. After that change, he delivered two courses of lectures, one in the Michaelmas term, the other in the Lent term. An inspection of the list of his lectures* shows that he chose his

* The list is given on pp. xlv, xlvi.

subjects by preference from analytical geometry, dynamics (in his view, theoretical dynamics is a portion of pure mathematics), differential equations, theory of equations, Abelian functions, elliptic functions, and modern algebra. The titles of the lectures, as announced, were sometimes vague, nor were they intended to limit his range; in all cases he went far beyond the boundary that so frequently limits Cambridge studies. Thus a course of lectures on differential equations, announced for the Michaelmas term in 1879, was chiefly concerned with conformal representation, polyhedral functions, and Schwarz's investigations on the hypergeometric series.

For many years he dispensed with the use of blackboard and chalk in his class-room; this was possible because his class usually was small. He brought his work written out upon the blue draft-paper,* which was regularly used by him in all his writing of mathematics; the exposition consisted partly of verbal explanations made as he showed the manuscript, partly of details written out at the moment. A change came in 1881, when his class amounted to fifteen or sixteen: he was then obliged to use the blackboard, and he subsequently maintained the new practice. Occasionally his older habit of explaining his manuscript recurred—he then placed it upon the board. This was especially the case when he brought carefully prepared diagrams, such as those used in the modular-function division of the plane: these diagrams were made much clearer by the use of water-colours to distinguish different sets of regions, and their preparation evidently gave him pleasure.

But, as may be surmised, his influence as a teacher was overshadowed by his influence as an investigator. Those whom he affected by his lectures belonged for the most part to the mathematical teachers in Cambridge: the number of undergraduates whom he influenced was small, though, when any one of them did come under his influence, the effect was well marked. His starting point in any subject was usually beyond the range of all other than quite advanced students; but to any able undergraduate who was willing to devote time, not merely to the comprehension of the matter in the lectures but also to collateral reading, the lectures were stimulating and inspiring. This effect was partly due to the easy strength with which he worked, partly to the spirit in which he approached old and new subjects alike; an independent suggestiveness and a singular freshness marked his views, and gave an added interest to his exposition even of a well-known theory. One reason of this freshness may be found in the fact that his lectures consisted of the current researches upon which he was engaged at the time; sometimes, even, a lecture would be devoted to results which he had obtained since the preceding lecture. Though the titles of the courses occasionally recur from one year to another, the same course was never given twice. The new matter in any course, once given, was usually incorporated in a paper or memoir; and when the same subject was nominally lectured upon again, it was a distinct part of the subject—old notes were never used a second time.

It was not alone by his lectures that he acted as professor. Students, seeking help or desiring to interest him in their work, found him always willing to give them the benefit of his advice, his criticism, and his knowledge. Nor was it merely mathematicians in Cambridge whom he helped in this way. He was continually consulted by

* It was the customary "scribbling paper" of his undergraduate days.

foreigners, who appreciated the promptness no less than the fulness of information in his replies.

It frequently happens that a man of genius, great enough to leave a distinct impress of his originality upon his science, finds it irksome to study what others have written. With the growth of all sciences during the last fifty years, especially—it may be said—with the growth of pure mathematics in that time, the tendency of workers is to become specialists in their own subject and, perhaps, in subjects immediately cognate with it, and to acquire only a slight acquaintance with what is being done outside the circle of their limited interests. Not so was Cayley: he was singularly learned in the work of other men, and catholic in his range of knowledge. Yet he did not read a memoir completely through: his custom was to read only so much as would enable him to grasp the meaning of the symbols and understand its scope. The main result would then become to him a subject of investigation: he would establish it (or test it) by algebraical analysis and, not infrequently, develop it so as to obtain other results. This faculty of grasping and testing rapidly the work of others, together with his great knowledge, made him an invaluable referee; his services in this capacity were used through a long series of years by a number of societies to which he almost was in the position of standing mathematical adviser.

Concurrently with his teaching, he continued his investigations. He wrote only one book—a *Treatise on Elliptic Functions,* published in 1876, which was intended to bridge over the gap from Legendre's *Traité des Fonctions Elliptiques* to Jacobi's *Fundamenta Nova*; it contains a considerable amount of new matter. But paper after paper was published in a long unfailing succession almost until his death; their tale amounts to more than 800. Happily for the convenience of mathematicians, the republication of his papers in collected form was undertaken by the Cambridge University Press,—perhaps the most enduring, certainly not the least fitting, monument of his fame. The request was made to him in 1889 by the Syndics of the Press; he willingly acceded to it and deeply appreciated, both then and afterwards, what he regarded as a great compliment to himself. Seven large quarto volumes, under his own editorship, have already appeared. The preparation of them was always a great happiness to him; and, especially in the later years of his life, it gave him an occupation in his science which was still within the range of his failing strength. At the time when the collection was begun it was estimated that ten volumes would suffice for the purpose, but it is now evident that ten will be certainly insufficient. The Syndics of the Press intend to complete the series of volumes; it is a matter of regret that the illustrious author of the papers has not lived to complete it himself.

Even his teaching and investigations did not fully occupy his time. For the first few years after his return he was left comparatively free from a large share in administration, but gradually it was assigned to him. As he became better known for his effective business capacity, his share in administration grew until he came to be regarded as an indispensable member of the Council of the Senate. He was elected a member of that body on 7th November, 1876, and with the exception of some six months when he was absent in America, he continued a member of it until 1892, when failing health compelled him to resign. During this period of service he was re-elected three times. Party feeling ran rather strongly at times during the discussions that led to the new statutes; but both parties included his name among their lists of nominations—an adequate proof that he possessed the confidence of the Senate. He was free from party bias, and he became established in his position of strength by his fairmindedness, his sound judgment, and his calm temperament. He would listen to a discussion, speaking only when he had something of importance to add; when speaking he was listened to with full attention. More frequently he would take no part in the discussion until his opinion was asked, as was usually the case in difficult questions; his opinion was always valued and sometimes final. Similarly, on syndicates, his co-operation was much sought, and in particular the services which he rendered to the Library Syndicate and the Press Syndicate were of substantial importance. He also took great interest in the movement for the higher education of women. In the early days of Girton College he gave direct help in teaching, and for some years he was Chairman of the Council of Newnham College, in the progress of which he took the keenest interest even to the last.

But, with all his general aptitude for business, he was perhaps most specially helpful by his legal knowledge. The training he had undergone and the knowledge he had acquired at the bar ultimately proved invaluable. His opinion on legal matters was sought by the University, by his own college, and by the scientific societies with which he was connected; when given, it frequently had the effect of a judicial decision. His powers of drafting were constantly being called into requisition; he responded to the calls upon him and, with unstinted generosity, placed his time and skill at the disposal of these bodies, so that the new statutes of Trinity College, and not a few of the statutes and ordinances of the University, owe much to him.

One other illustration, at once of his general business capacity and of the confidence reposed in him, may be given. The elections for representatives of the Universities in the House of Commons are still conducted openly and by means of voting papers, delivered either by the elector himself or by another elector whom he has nominated; objections may be raised against any voting paper, but they must be decided at once. In Cambridge the Vice-Chancellor, being the returning officer, nominates a number of assessors to act with him in the case of a contested election. At a bye-election in 1882, when the candidates were Mr. H. C. Raikes and Professor James Stuart, Cayley was nominated as presiding officer at one of the polling places. His imperturbable firmness, his calm courtesy, and the justice of his decisions secured for his effectiveness in this capacity the admiration of the University.

This brief account of his participation in business affairs is necessary; without some such indication a proper estimate of his position in Cambridge cannot be framed. And it also may help to show that his supremacy in the subjects of his investigations neither made him a recluse, nor limited his other interests, nor restricted his practical usefulness.

The merits of such a man were recognised by the only means at the disposal of a grateful and appreciative University. He was elected an honorary Fellow of Trinity College on 22nd May, 1872, at the same time as Dr. Lightfoot, Mr. James

Spedding, and Professor Clerk Maxwell; and on 11th October, 1875, he was made an ordinary Fellow, a position which he retained for the rest of his life. His friends subscribed for a presentation portrait,* painted by Lowes Dickenson in 1874; it now hangs in the College Hall. The simplest of inscriptions is on its frame, but the humorous lines which Clerk Maxwell† wrote at the time should not readily be forgotten. The graver element, seldom absent from his verses, is not entirely repressed even by his wit, and the lines were based upon a deep admiration of the man

> "Whose soul, too large for vulgar space,
> In *n* dimensions flourished unrestricted."

His bust, by Mr. Henry Wiles, was given to Trinity College by a donor who wished to remain anonymous. It was placed in the beautiful library of the College on 3rd December, 1888, an honour that has been conferred during life in only two other cases—Tennyson and Sedgwick.

After the new statutes came into operation, the Senate on 27th May, 1886, decided that the Sadlerian Professorship should at once be made subject to the improved provisions, a decision which, though it increased the amount of lecturing required, gave him the benefit of the full stipend. At the same time the Lucasian Professorship, held by Professor Stokes, was also made subject to the new statutes; and it was currently believed that the Lowndean Professorship would have been included in the proposal had Professor Adams been willing to have the change made. There was a wish on the part of members of the University to give some recognition to the glory conferred upon the mathematical school by Stokes, Adams, and Cayley; one possibility remained. The opportunity came in 1888 when Prince Edward (as he was known in Cambridge), afterwards Duke of Clarence, received the degree of LL.D. Such an occasion is customarily marked by the conferment of a number of honorary degrees upon distinguished men; among them, on this particular occasion, were the three professors who had been colleagues for a quarter of a century. On the 9th of June in that year a great assembly gathered to see these degrees conferred upon the recipients. It need hardly be said that the men singled out for honour received ovations on being presented; among the most enthusiastic ovations were those accorded to the three professors.

Nor were external bodies and learned societies, both at home and abroad, backward in recognising the merits of his work; the honours he received were numerous and came from all quarters. Honorary degrees were conferred upon him by several universities as well as his own, among them being Oxford, Dublin, Edinburgh, Göttingen, Heidelberg, Leyden, and Bologna. President Carnot nominated him an Officer of the Legion of Honour. He was either a Fellow or a foreign corresponding member of most of the scientific societies of the Continent, among them being the French Institute, the Academies of Berlin, Göttingen, St. Petersburg, Milan, Rome, Leyden, Upsala, and Hungary. He was also a Fellow of the Royal Society of Edinburgh, of the Royal Irish

* A photographic reproduction of the portrait is prefixed to vol. VI. of the *C. M. P.*

† See Campbell and Garnett's *Life of James Clerk Maxwell*, p. 636.

Academy, and of the Royal Astronomical Society. He had been President of the Cambridge Philosophical Society, and he sat on its Council for many years; also President of the London Mathematical Society and of the Royal Astronomical Society. He was elected a Fellow of the Royal Society on 3rd June, 1852, and he served as a member of its Council for six periods of office. In 1859 he received from the Royal Society a Royal medal, and in 1882 the Copley Medal, the highest scientific distinction it is in its power to bestow. When the De Morgan Medal was instituted in connexion with the London Mathematical Society, the first award was fitly made to Cayley. And from Leyden he received the Huyghens Medal.

Mention should be made of one other honour which he received: it is of a kind seldom conferred. The high opinion of his work which was held in America was indicated by an invitation in 1881 to deliver a course of lectures in the Johns Hopkins University, Baltimore, where his friend and fellow investigator, Sylvester, was then professor. He accepted the invitation, and left England in December of that year. During the next five months he lectured on Abelian and Theta Functions; the substance of these lectures was incorporated in a memoir subsequently published in the *American Journal of Mathematics**. He returned to England in June, 1882, bringing back pleasant remembrances of kindnesses and friendships.

His life, spent in mathematical research and in the quiet round of activity in the University, offered little of either interest or incident to make his name known by the outside world to the same extent or in the same way as the names of many scientific men, engaged in other lines of enquiry, are known. Once, however, in his life circumstances brought him prominently into notice. In 1883 he was President of the British Association for the Advancement of Science, the meeting being held at Southport; and, in that capacity at the opening of the meeting, he had to deliver a formal address, an abstract of which appeared as usual in the leading newspapers of the country.

In the early days of the Association, the President's address frequently reviewed the whole field of science; but as knowledge has developed, a tendency has set in, according to which each later President has confined himself more particularly to those matters within whose range he is an authority. And, subject to this restriction, it is hoped that the address may be legitimately popular. There have been critics of presidential addresses prepared to assert that science was sacrificed to popularity; there have been immense audiences convinced that popularity was sacrificed to science. Taken together, the presidential addresses, some severe and others popular, form an interesting series of reviews of the successive stages in scientific achievements.

Cayley's address belonged to the severely scientific class. From the nature of his subject—the progress of mathematics, more particularly of pure mathematics—it was bound to have this character. Few of the members of a regular Association audience have more than a slight acquaintance with pure mathematics; and, consequently, it is impossible to deliver to such a gathering an address which, in a reasonable time, can give them any real idea of the condition or the progress of the science. Cayley felt

* Vol. v. (1883), pp. 137—179; vol. vii. (1885), pp. 101—167.

this and confessed to the feeling in a passage which is perhaps the best known in the address:—

> "It is difficult to give an idea of the vast extent of modern mathematics. The word 'extent' is not the right one: I mean extent crowded with beautiful detail—not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower. But, as for everything else, so for a mathematical theory—beauty can be perceived but not explained."

But he also felt that the respect due to the Association requires its President to deal with that branch of science about which, as he knows it best, he is best fitted to tell them, so that different subjects may thus in turn be brought before successive meetings.

> "So much the worse," he added, "it may be, for a particular meeting; but the meeting is the individual which on evolution principles must be sacrificed for the development of the race."

Granting then the inevitably stern character (as popularly estimated) that must mark any proper exposition of his subject, the address is one of singular interest. It undoubtedly made a great impression. Parts of it were incomprehensible to all but mathematicians: still, there was much which others could understand and, understanding, found excellent. Even leader-writers at the time recognised its lucidity, its finish, its native elegance, and its instructive and stimulating essence. To mathematicians it counts for much. Not merely is it a valuable historical review of various mathematical theories; but the exposition possesses all the freshness, the independence of view, the suggestiveness and the amazing knowledge that were so characteristic of Cayley. And, consequently, it can often be recurred to with unfailing profit.

After this event, his life pursued the unbroken tenor of its scientific course. Ever thinking, working, writing, he maintained the flow of his papers with the same unslackening vigour, and he showed the same sympathetic encouragement of others, as had marked him before the scientific world had tried to acknowledge his genius by showering its honours upon him.

It is now some years since the painful internal malady, which ultimately was the cause of death, began to show itself. At first, its action was slow; and there was reasonable hope that his naturally strong constitution would enable him to throw it off. Unfortunately these hopes were not realised; its growth was steady, its undermining influence persistent. Change of scene was tried once or twice, but without good effect; and it soon appeared that Cambridge itself troubled him least. Three years ago his friends saw that his health began to fail: he had occasional attacks of severe illness which confined him to his bed for weeks together, each of them leaving him gravely frailer than before. Gradually he became confined to his house and his garden; he could see only very few friends, and usually even them only for a short time. When they did see him, they found only too clearly how rare and brief were his intervals of relief from pain, though occasionally his gentleness and his patience would almost delude them into hope.

The last of the severe attacks began on the 8th of January; he seemed to be getting better when, on the 21st, his strength suddenly began to collapse. He died about six o'clock on the evening of Saturday, 26th January, 1895. The funeral took place on the succeeding Friday when, in Trinity Chapel, a great assemblage, composed of members of the University, of representatives of the embassies of Russia and America, as well as of various learned societies and of personal friends, gathered to pay him their last homage of respect and reverence.

Sufficient has been said to show that Cayley was a man of general activities; but his scientific work and his public duties by no means exhausted or limited his general interests.

It has already been stated that, as an undergraduate, he was fond of reading novels; this practice remained with him all his days. He preferred a novel of the old orthodox type with a "happy ending"; and though his greatest delight was in the older novels, a modern book, such as *Beside the Bonnie Briar Bush* (which he read quite late in 1894), met with words of warm praise. He had a good memory, and used to discuss plots and characters with considerable animation. The two novelists, by whose works many English people are divided into one or other of two classes, did not affect him much; Thackeray he read but did not like, and he would not read Dickens. His favourite authors were Scott and Jane Austen; all their works had been read by him many times, and they were read aloud to him during the long period of his illness. *Guy Mannering* and *The Heart of Midlothian*, among Scott's, and *Persuasion*, among Jane Austen's, were the books he liked the best. He also was fond of George Eliot's novels, particularly of *Romola*. Indeed, though he had aversions, his taste was somewhat general. Commendation of a book was enough to make him willing to try it; and there was only one limitation to his range of novel-reading—he had an instinctive abhorrence of anything that suggested either coarseness or vulgarity.

His English reading was not confined to novels. He had a keen liking for many of Shakespeare's plays, notably *Much Ado About Nothing*, and some of the historical dramas. He delighted in Milton's shorter poems, though he would not tolerate *Paradise Lost*. Scott's poems were frequently read; and he had a great appreciation of Byron's *Tales* and of Coleridge's *Ancient Mariner*. Grote's *History of Greece* and Macaulay's *History of England* he read repeatedly and with zest; and he never seemed tired of Lockhart's *Life of Scott*.

He was also a good linguist. He knew French well; it was a second writing-language to him, as will be seen from the large number of papers, written in French, which occur in his collected mathematical papers. He read (but he did not talk) German and Italian with ease, and his Greek remained fresh throughout his life. This last power may have been due to the admiration he felt for Plato; he referred to the *Republic* and the *Theætetus* in his Presidential Address; and, on the afternoon of the day of the "Greek division"* in the Senate House, I remember finding him at home reading the *Gorgias*.

He had the keenest interest, amounting almost to a passionate delight, in travelling; cities of historic or artistic fame delighted him equally with beautiful scenery. Long after he had become an invalid, he found a fascination in guide-books and maps; and all his younger friends will recall the sympathetic zeal with which he entered into their projected journeys, and the happy pleasure he took in hearing them speak of recent journeyings and in recalling, with a wonderful vivid memory, his own experiences and ideas about places they had visited.

Reference has been made to his early pleasure in the old Italian masters. Yet, if any inferences can be drawn from the likings of his later years, architecture attracted him even as much as pictures. He had a true feeling and a clear judgment as to genuine excellence: he sketched well, and had a quick eye for proportions, perspective, light and shade. One of his relaxations was to make coloured sketches of buildings that he liked, notably sepia drawings of some of the great Gothic cathedrals and churches of northern France. He kept up his practice of water-colour painting all his life, and in his closing years it proved a great solace to him at times when his strength was so far reduced that he could not work. He had great happiness in looking at architectural pictures and at books on architecture, one of his favourites among the latter being Street's *Brick and Marble in the Middle Ages*.

Financial matters and accounts also interested him; and only a few months before his death he published a brief pamphlet on book-keeping by double entry, which he has been known to declare one of the two perfect sciences. He could not resist some reference to the subject in his Presidential Address, making the remark that the notion of a negative magnitude "is used in a very refined manner in book-keeping by double-entry."

His bearing was gentle, and it was marked by a courtesy that was unfailing. On questions of administration and in discussions, his opinions were stated clearly and quietly. Not that he did not hold decided views or that he would abate one jot of his firm, even chivalrous, defence of what he held to be right; but there was a judicial temper in his mind which prevented the subjective element in a discussion from disturbing his equanimity. The even balance of his mind enabled him to recognise and appreciate the position of one who differed from him, and his quiet "I do not think so" was all the more effective because its very calmness excluded the slightest suggestion of hostile spirit.

His figure was spare: until his illness, he could easily endure the fatigue of long walks, in which he delighted, especially in hill country. In later years it became rather bent, and he had the appearance of being frail. His head was very impressive,

* In 1891 a proposal was made by the Council of the Senate for the appointment of a Syndicate to enquire, among other things, into the expediency of allowing alternatives for one of the two classical languages in the Previous Examination. Many members of the Senate were convinced that the adoption of an alternative would lead to the extinction of the study of Greek except in the greater public schools; they consequently opposed the proposal, which, on 29th October, 1891, was rejected by a great majority (525 to 185).

It may be added that Cayley was in the minority. He allowed his signature to be added to a letter which was sent to the London newspapers as an appeal for assistance in defeating the attempt to resist enquiry.

as may be seen from his portrait and from photographs. In repose, and when his attention was not concentrated upon what was passing, his face had a grave air and the blue-grey eyes suggested that he was far away in thought; but when attentive or amused, and when expressing pleasure, the eyes became singularly keen and a peculiar charm lightened up the whole face.

He was absolutely modest. The honours conferred on him in full profusion never injured in the least degree the grand simplicity of his character, never gave rise to the slightest trace of vanity, which was alien to his nature. He rarely spoke of them, and, when he did, it never was as of honours: they pleased him, but, perhaps, rather as recognition of his work than as tributes to the worker. If any one expressed appreciation of any of his papers, owing to the help it had given, he would reply very quietly: but he did not stint the expression of his pleasure at advances beyond his own results when they were made by others. Public appearances were rather distressing to him at first, for his disposition was retiring and he could be reserved; but as time wore on, duty often compelled him to take part in them. In such cases he accepted the claim and discharged it with a straightforward simplicity that was entirely devoid of self-consciousness; but he gladly avoided demonstrations whenever it was possible.

In the spirit of his work one great quality was his generosity to others, particularly to young men, whose work he was always willing to recognise. He ignored the fact that he was a great mathematician—probably it never occurred to him to think of his doings: but it may be doubted whether this unconsciousness of his greatness ever proved at once more fascinating or more bewildering than when he was discussing scientific results with young men. He so evidently had his wishes centred on a single-hearted desire for the right result that it was difficult to conceive him approaching a question merely as a learner: yet he was ever a learner. There are few men, if any, with not even a tithe of his scientific achievements, who have had less of controversy or have had such immunity from questions as to priority of discovery. This arose not merely from the indisputable priority of his results: it was partly owing to his nature. Salmon says of him:—

> "His motto has always been 'esse quam videri,' and I do not know any one to whom it would be more repulsive to engage in a personal contest by claiming for himself a particle of honour or of money more than was spontaneously conceded. He would be apt to take for his model the patriarch Isaac, who, when the Philistines claimed a well which he had dug, went on and dug another, and when they claimed that, too, went on and dug a third":

an exceedingly happy description of the man the tide of whose genius was

> "Too full for sound or foam."

Some account of his work, some estimate of its character, some indication of the original contributions made by him to his science, may not improperly be given here. It is, of course, impossible to predict what his permanent influence will be upon mathematics, or what opinion coming generations of workers will hold of him: certainly, by his own contemporaries, he was deemed one of the greatest mathematicians the world has seen. Bertrand, Darboux, and Glaisher have compared him to Euler, alike for his range, his analytical power, and, not least, for his prolific production of new views and fertile theories. There is hardly a subject in the whole of pure mathematics at which he has not worked. Some new subjects owe their existence to him; to others he has made very definite contributions, so that their boundaries have been enlarged often to an enormous extent; there are few upon which he has not left the mark of his genius.

In several of the notices that appeared at his death he was described as a great explorer. Such he undoubtedly was, but he was more. He not merely discovered new countries but he also opened them up, so that others were able to enter into some possession of those regions without undergoing the difficulties that he had overcome. And if the metaphor may be carried further, he had the restlessness of the explorer: he could not long remain satisfied with an achievement concluded, but must try his fortune again and elsewhere.

Varying opinions have been expressed as to Cayley's style; the variations are largely due to preconceived views of what a mathematical paper should be. It certainly is not easy to skim one of his papers; any attempt to do so leads to an inadequate estimate of what it usually establishes. It is not difficult to read one of his papers, even to grasp the contents well, provided proper care be devoted to it, because difficulties that occur are completely solved, and nothing lies in the background to cause doubt or suggest incompleteness. He has been well described by Glaisher as an unequalled master of analytical processes; it is especially in algebraical manipulation that his strength and his facility stand out in clear view. His success in this direction was achieved by a skill that cannot be explained by describing it as due to acquired knowledge, or to practice, or to long consideration and patient selection. It was rather an instinct for the management of the most complicated processes, and the way in which he controls the most elaborate calculations is sometimes little short of extraordinary.

As regards his methods, he does not seem to have cast about so as to choose one rather than another. As soon as he had thought of any method the possible effectiveness of which he could settle almost intuitively ("one's best things are done in five minutes," he once said to me, in confirmation of the satisfaction I was expressing at the fruitfulness of an idea that had occurred to me unexpectedly), the rest was the exercise of his powers. Among the methods he preferred, especially during the last twenty-five years of his life, was that of verification; in his hands it proved a weapon of great force. Indeed, only less remarkable than his algebraical skill, was the insight which enabled him to preserve the exact equivalence of all the equations in any particular process, so that he could have reversed each process merely by reversing the steps as they were made, and could have proceeded to the required theorem from the initial expression of an algebraical fact. Numerous instances of this quality in his work could be adduced; it will be sufficient to refer to some parts of his paper* "On the centro-surface of an ellipsoid."

But though Cayley was specially happy in the treatment of algebraical developments, an inadequate estimate of his genius would be obtained by supposing that he

* *C. M. P.* vol. VIII. No. 520: *Camb. Phil. Trans.* vol. XII. (1873), pp. 319—365.

was almost entirely an analyst. Much of his thinking, not a little of his writings, is completely geometrical; and his contributions to line geometry, his introduction of the Absolute into geometry, his continued recurrence to the methods in pure geometry invented by Poncelet and Chasles, should be sufficient to range him among geometricians.

Moreover, even in strictly analytical work, the synthetic element is often not far away though it does not always appear on the surface. In this connexion an acute suggestion, made by Salmon and perhaps based upon his remembrance of their mathematical correspondence that lasted through many years, is confirmed by one of the Notes Cayley himself added at the end of the second volume of his *Collected Mathematical Papers*. An enquiry sometimes begins by a comparatively easy problem which, when solved, leads to wider inferences; so that, ultimately in the development, considerable generalisations are effected. Now the usual writer, in publishing the results of such an enquiry, draws them up in a sequence that partly marks the order of their connected discovery: and, in doing so, he makes his work easier for his readers. But Cayley was not the usual writer. When he had reached his most advanced generalisations he proceeded to establish them directly by some method or other, though he seldom gave the clue by which they had first been obtained: a proceeding which does not tend to make his papers easy reading. An instance of the fact occurs* in his "Memoir on the Theory of Matrices," where he proves that a matrix satisfies an algebraical equation of his own order; he proves it by verification in simple cases, but he gives no clue as to his line of discovery. An instance of the method occurs in a note† added to one of his papers, where he says that the general equations

$$\{yd_x\} - yd_x = 0, \quad \{xd_y\} - xd_y = 0,$$

characteristic of covariants and invariants of binary quantics, were initially suggested by considering the relation of the quadratic $ax^2 + 2bxy + cy^2$ and its discriminant $ac - b^2$ to these equations. In the paper he drops linear transformation as connected with the covariantive property and defines a covariant as a function satisfying these two equations.

His literary style is direct, simple and clear. His legal training had an influence, not merely upon his mode of arrangement but also upon his expression; the result is that his papers are severe and present a curious contrast to the luxuriant enthusiasm which pervades so many of Sylvester's papers. He used to prepare his work for publication as soon as he had carried his investigations in any subject far enough for his immediate purpose. He found it an easy matter to do this part of his work, and thus differed widely in experience from those to whom the preparation of a paper is laborious even when the results to be incorporated have been obtained. As a matter of fact, he took the straightforward course of saying what he had to say in a clear and simple manner, fixing his mind upon the substance and never going out of his way in order to secure beautiful form for the presentation of results. Yet not infrequently his papers are so admirably written that they satisfy the exacting critics; thus it is perhaps not too much to affirm that his "Sixth Memoir on Quantics‡" could not be presented in more attractive form—a character due, however, to the tendency of

* *C. M. P.* vol. II. No. 152, pp. 482, 483; *Phil. Trans.* (1858), pp. 24, 25.

† *C. M. P.* vol. II. p. 600.

‡ *C. M. P.* vol. II. No. 158; *Phil. Trans.* (1859), pp. 61—90.

his method and to his results, but not acquired by any effort specially devoted to elaboration of clear expression. Again, a paper once written out was promptly sent for publication; this practice he maintained throughout his life. He undoubtedly formed projects for the immediate future; thus to the second edition * of his *Treatise on Elliptic Functions* he intended to add a couple of chapters, which, however, remained unwritten solely for the reason that all such projects were carried into effect only about the time when the need arose. The consequence is that he has left few arrears of unfinished or unpublished papers; his work has been given by himself to the world.

Only one other remark as to the form of his papers need be made. Readers must be struck with the number of exact references he makes to other writers. It was a practice about which he had very decided opinions: he wished not merely to make honourable acknowledgment of indebtedness but also to give indications of the history of the subject. In the latter particular he was always careful to insert in the reference the year in which the book or the paper had appeared; and he steadily urged others to insert dates in their references.

Cayley made additions to every important subject that lies within the range of pure mathematics. Their importance and their amount have varied in different subjects; thus on geometry his writings have a dominating influence: while on the general theory of functions, though he knew the subject well, he has left little mark, for he concerned himself chiefly with details such as the solution of more or less special problems in conformal representation. His papers in general have such value that he is the author most frequently quoted by the great body of current mathematicians. A full record of what he has done in pure mathematics could be made only by writing its history during the last half century; all that is attempted here consists of some brief indications of a selection among his more obviously important contributions to mathematical knowledge.

One of the subjects with which Cayley's name will probably be most closely associated is the theory of invariance. It is easy to cite simple cases of what is implied by an invariantive function: two will suffice.

It is known that, in solving an ordinary algebraical equation with literal coefficients, a certain functional combination of these coefficients (called the discriminant) must vanish in order that two roots of the equation may be equal; for example, the equation $ax^2 + 2bx + c = 0$, has equal roots if (and only if) the quantity $ac - b^2$ vanishes. When the variable is transformed from $x$ to $y$ by a relation $(l'x + m')y = lx + m$, where $l$, $m$, $l'$, $m'$ are constants, then evidently two values of $y$, corresponding to the two equal values of $x$, are equal. When $x$ is eliminated from the equation by means of the assumed relation, a new quadratic arises having $y$ for its variable; let it be $a'y^2 + 2b'y + c' = 0$, where $a'$, $b'$, $c'$ depend upon $a$, $b$, $c$ and $l$, $m$, $l'$, $m'$. The two values of $y$ determined by this equation are equal if (and only if) the quantity $a'c' - b'^2$ vanishes. But the equality of the two values of $y$ depends upon and is determined by the equality of the two values of $x$, the latter equality being secured if the quantity $ac - b^2$ vanishes. It follows that the vanishing of either of the quantities $a'c' - b'^2$ and $ac - b^2$ requires

* It was published four months after his death; only the earlier sheets had the benefit of his revision.

the vanishing of the other; and it is therefore inferred that, when neither of them vanishes, one of them contains the other as a factor. When the actual calculation is made, it is found that $a'c'-b'^2$ is the product of $ac-b^2$ and $(lm'-l'm)^2$, the latter being a quantity that depends solely upon the transforming relation. Consequently it appears that a combination of the coefficients in the original equation exists, such that when the equation is transformed by any relation of the type indicated and exactly the same combination of the new coefficients is constructed, the two combinations are equal to one another save as to a factor depending solely upon the transforming relation. Such a combination of the coefficients is called an invariant.

Again, it is known that every curve (of degree higher than two) possesses a number of points where a tangent to the curve not merely touches it but, having contact of one degree closer, crosses it; and it is found that all these points, called points of inflexion, also lie upon another curve uniquely derived from the first. When the curves are represented by means of equations, the statement is that the points of inflexion of a curve $U=0$ are given as the intersections of this curve with a curve $H=0$, the latter equation being uniquely derived from $U=0$. Now suppose that the axes, to which the curves have been referred, are changed to another system, so that new co-ordinates $x'$, $y'$ are connected with the former co-ordinates by relations

$$\frac{x'}{a_1x+b_1y+c_1}=\frac{y'}{a_2x+b_2y+c_2}=\frac{1}{ax+by+c}.$$

A new equation $U'=0$, obtained by eliminating $x$ and $y$ between these relations and $U=0$, will now represent the curve. The change thus made does not affect the geometrical properties of the curve; its points of inflexion are still given as its intersections with the curve $H=0$. But the points of inflexion of the curve represented by $U'=0$ are the intersections of this curve with another curve represented by $H'=0$, an equation derived from $U'=0$ in exactly the same way as $H=0$ is derived from $U=0$. It therefore appears that the associated curve $H'=0$ cuts the given curve in precisely the same points as the associated curve $H=0$, a result which suggests that the associated curves are the same. Now $H'=0$ has been derived from $U'=0$; but actual calculation shows that, if the relations between $x'$, $y'$ and $x$, $y$ be used to eliminate $x$, $y$ from $H=0$, the resulting equation is $H'=0$; in other words, the relations between $x'$, $y'$ and $x$, $y$ transform the equation $H=0$, derived from $U=0$, into the equation $H'=0$, derived in the same way from $U'=0$. Moreover, as in the case of the invariant, it is found that $H'$, a specially constructed function of $x'$, $y'$ and the coefficients in $U'$, is divisible by $H$, the same function of $x$, $y$ and the coefficients in $U$; the quotient being a quantity depending again only upon the constants in the transforming relations. Consequently it appears that a combination of the coefficients and the variables in the original equation exists such that, when the equation is transformed by means of relations of the type indicated, and exactly the same combination of the new coefficients and the new variables is constructed, the two combinations are equal to one another save as to a factor dependent solely upon the transforming relations. Such a combination of the coefficients and the variables is called a covariant.

The first notice of such a property appears to have been made by Lagrange. And Gauss discussed the invariance of the discriminants of certain expressions when the latter are subjected to linear transformations. Again, Boole in 1841 had shown that this invariantive property belongs to all discriminants, and he gave a method of deducing some other functions of this kind. Boole's paper suggested to Cayley a much more general subject—the permanence of invariantive form—so that he set himself the question of finding "all the derivatives of any number of functions which have the property of preserving their form unaltered after any linear transformation of the variables." The first set of results obtained by his investigations related to invariants; they appeared in his famous paper,* "On the Theory of Linear Transformations," published half a century ago. The second set of results related to covariants; they appeared in the paper,† "On Linear Transformations," published in the succeeding year. In these two papers Cayley demonstrated the general existence of a number of functions, both invariants and covariants (at first he called them hyperdeterminants), which preserve their form under linear transformation.

These discoveries of Cayley establish him as the founder of what is called sometimes modern algebra, sometimes invariants and covariants, sometimes theory of forms; the origination of the theory is incontestably his, and it is universally ascribed to him.

A discovery of this general importance and complete novelty soon attracted the attention of other workers. It is not too much to say that the subsequent investigations long absorbed the active interest of many mathematicians, and, as a result, the theory has influenced all that domain of mathematical science which is in any way connected with algebraical form. Among the first to enter the field was Sylvester, then living in London; he and Cayley were in constant communication, alike oral and written, and carried on their work in the most friendly relations with one another. Boole also resumed his investigations, and both he and Salmon made substantial additions to the theory. The continental mathematicians also had begun their important contributions, chief among them being Aronhold, Hesse, and, at a later date, Hermite. Aronhold, indeed, devised the so-called symbolical method, now the favourite method with German workers; in its origin it is nearly the same as the symbolical method introduced by Cayley, but the subsequent developments—due largely also to Clebsch and to Gordan—run on lines entirely different from Cayley's.

After a time, Cayley began his series of ten memoirs on quantics; they must rank among the most wonderful combinations of original researches and papers upon a single theory ever produced. They contain a splendid exposition of the theory as already established; they are full of original contributions to the subject, and as they take account of the work done by other authors, they have the further interest of showing how the subject grew between the appearance of the "Introductory Memoir" in 1854 and the appearance of the "Tenth Memoir" in 1878. This is hardly the

* *C. M. P.* vol. I. No. 13; *Camb. Math. Jour.* vol. IV. (1845), pp. 193—209.

† *C. M. P.* vol. I. No. 14; *Camb. and Dubl. Math. Jour.* vol. I. (1846), pp. 104—122. The two papers were rewritten, and appeared in *Crelle*, vol. XXX. (1846), pp. 1—37, under the title "Mémoire sur les Hyperdéterminants."

opportunity to write a history of the subject by apportioning among the various investigators the sections which they respectively originated;* yet reference should be made to two matters.

First, one of the problems that greatly interested Cayley was the determination of the complete asyzygetic system of irreducible invariants and covariants appertaining to a binary form, that is, the system such that every invariant and every covariant of the form can be expressed as a rational integral algebraical function of the members of the system, the coefficients in the function being numerical only. In his "Second Memoir on Quantics"† he had accurately determined the number (and their degrees) of the asyzygetic invariants for binary forms of orders 2, 3, 4, 5, 6; he had also accurately inferred the number (together with their degrees and their orders) of the asyzygetic covariants for binary forms of orders 2, 3, 4, all these concomitants being subsequently tabulated. But, in regard to the invariants of forms of order higher than 6 and the covariants of forms of order higher than 4, he came to the erroneous conclusion that the respective numbers are infinite. The error was not corrected until Gordan in his memoir‡, dated 8th June, 1868, and entitled "Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist," showed that the complete system for a binary quantic of any order contains only a limited number of members. Cayley at once returned to the question, and having found a source of error (it was the neglected interdependence of certain syzygies, reducing the numbers of invariants and covariants; the interdependence had not previously been suspected), he dedicated his "Ninth Memoir on Quantics§" (dated 7th April, 1870), to the correction of the error and a further development of the theory in the light of Gordan's results. His promptness in recognising and giving immediate prominence to the work of the younger author possibly prevented some controversy among unwise partisans; it was characteristic of the man.

And, secondly, though his series of memoirs was brought to an end with the tenth, his interest in the subject did not cease, and he frequently wrote upon parts of it under other titles. In particular, Captain P. A. MacMahon's discovery of a relation of a new character between seminvariants and symmetric functions (viz., that the leading coefficients of the covariants of a binary quantic are the same as the non-unitary partition symmetric functions of the roots of an equation connected with a modified quantic) proved of the keenest satisfaction to him. From time to time he wrote in the *American Journal of Mathematics* upon this subject and upon symmetric functions generally in this connexion, always sympathetic and appreciative of the advances made by others, able to grasp and assimilate

* Some information will be found in an appendix to Salmon's *Lessons on Higher Algebra*; also in the notes and references at the end of the second volume (pp. 598—601) of the *Collected Mathematical Papers*. A valuable and exhaustive report, containing a full history of the subject, was drawn up by Prof. Dr. Franz Meyer, and published under the title "Bericht über den gegenwärtigen Stand der Invariantentheorie" (*Jahresber. d. Deutschen Mathem.-Vereinigung*, I. 1892).

† *C. M. P.* vol. II. No. 250; *Phil. Trans.* (1856), pp. 101—126.

‡ *Crelle*, vol. LXIX. (1869), pp. 323—354.

§ *C. M. P.* vol. VII. No. 462; *Phil. Trans.* (1871), pp. 17—50.

their ideas, but using them as a master and not as a follower. It was not alone, however, to symmetric functions, upon which he had written long and important memoirs as early as 1857, but to many other cognate subjects that he extended his researches upon invariants and covariants. The theory of equations of the fifth and higher degrees, Sturm's functions, Tschirnhausen's transformation, partition of numbers, Arbogast's method of derivation, skew determinants*—to quote no others —are titles and subjects of papers, all of which contain investigations of great value. The reason that they are less known (if such be the case) than his other work in the same line of ideas is perhaps due to the fact that the direct theory of invariants and covariants was rapidly brought within the range of students through Salmon's *Lessons on Higher Algebra*, dedicated by the author to Cayley and Sylvester.

Another subject, of which he must be regarded as the creator, is the theory of matrices. His first memoir† upon this theory, "wherein," to quote Sylvester,‡ "he may be said to have laid the foundation-stone of multiple quantity," was published in 1858. A couple of isolated results had been obtained by Hamilton in 1852 through the methods of quaternions; but they were unknown to Cayley at the time of his memoir, and, owing to the connexion in which they occur, they have an entirely detached aspect.

A matrix may initially be defined as a symbol of linear operation; thus, when the equations

$$X = ax + by + cz, \quad Y = a'x + b'y + c'z, \quad Z = a''z + b''y + c''z$$

are expressed in the form

$$(X, Y, Z) = \left( \begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array} \right| \left\rangle (x, y, z) \right. = M(x, y, z),$$

the symbol $M$ is a matrix. Cayley was the first to discuss the theory of such symbols as subjects of functional operation and to dispense with the hitherto regular return at each stage to the equations of substitution in which the symbol first arises; in fact, he replaces the notion of substitutional operation by the notion of a new class of quantity.

Matrices (being of the same order or dimension) can be added like ordinary algebraical quantities; as regards multiplication, they are subject to the associative law, but not to the commutative law. Hence powers of a matrix (positive and negative, integral and fractional) can be obtained, and likewise algebraical functions of a matrix. It also follows that two general matrices are not convertible, that is, $LM$ is not the same as $ML$ save under special conditions; and it is a part of the theory to find the most general matrix convertible with a given matrix. The expression of this convertible

* His discoveries in this subject alone have done much to simplify the analytical investigations connected with Pfaff's problem and the allied theory.

† *C. M. P.* vol. II. No. 152; *Phil. Trans.* (1858), pp. 17—37.

‡ *Amer. Jour. Math.* vol. VI. (1884), p. 271.

matrix can be deduced by means of the fundamental equation which every matrix satisfies, viz., an algebraical equation of its own order, the coefficient of the highest term being unity, and the last term being the determinant of the constants in the matrix. All these results were given by Cayley in his initial memoir; and, at the same time, they were applied by him to obtain the most general automorphic linear transformation of a bipartite quadric function, an extension of the problem which requires the most general (orthogonal) substitution transforming the function $x^2+y^2+z^2+\ldots\ldots$ into the function $x'^2+y'^2+z'^2+\ldots\ldots$.

How fruitful the subject has proved may be inferred by noting the subsequent investigations of Sylvester, who has developed it on Cayley's lines, and has added to it many new ideas; of Tait, who developed the theory of quaternions on parallel lines: of the Peirces, father and son, whose researches on linear associate algebra* gave rise to the notion of matrices from a different source; of Clifford and Buchheim, who connected the theory with Grassmann's methods; of Laguerre, in whose memoir† the treatment of a "linear system" (the same as a Cayley matrix) is similar to Cayley's; and of many other writers, among whom Taber should be mentioned.

Connected with non-commutative algebraical quantities, Cayley's researches on the theory of groups require a passing notice. He devoted several papers to questions in this theory. Some of them relate to those groups of substitutions, the introduction of which by Galois made an epoch in the theory of equations, others of them relate to groups of homographic transformations, particularly those related to the polyhedral functions. But, so far as can be seen, he limited his published investigations to those groups which are finite and discontinuous.

Abstract geometry—the ideal geometry of $n$ dimensions—is a subject that he may almost be said to have created; no other name than his has been associated with its origin. More than anything else, it marks the line of difference between the kinds of homage accorded to him. Experts regard it as an illustration of his imaginative power: the unlearned regard it as an incomprehensible mystery.

It finds a place among his earliest investigations,‡ it was steadily present to his mind, illuminating many of his researches; and occasionally it found explicit treatment, e.g., in his "Memoir on Abstract Geometry,"§ and in his Presidential Address at Southport. The theory presents itself in two connexions: one, as a need in analysis, the other as a generalisation of the ordinary geometries of two dimensions and of three dimensions.

The former origin can be indicated in a brief statement. When an occasion arises for dealing with a number of variables, connected in any manner and regarded as either variable or determinate (wholly or partially), the nature of the relations among them is frequently indicated, and often is made more easily intelligible, by associating some geometrical interpretation with the given system of relations. Thus

* *Amer. Jour. Math.* vol. IV. (1881), pp. 97—229.

† "Sur le Calcul des Systèmes Linéaires (*Journal de l'Éc. Poly.* vol. XXV. 1867, pp. 215—264).

‡ *C. M. P.* vol. I. No, 11; *Camb. Math. Jour.* vol. IV. (1845), pp. 119—127.

§ *C. M. P.* vol. VI. No. 413; *Phil. Trans.* (1870), pp. 51—63.

the momental ellipsoid is of great use in the discussion of moments of inertia, in representing the motion of a body round a fixed point when there are no impressed forces, and in other questions in dynamics. Again, two non-homogeneous (or three homogeneous) variables can be regarded as the co-ordinates of a point in a two-dimensional geometry, such as that of a plane or the surface of a sphere or any analytical surface; and any equation among the co-ordinates is then interpreted as representing a curve (or curves, or portion of a curve or curves) upon the surface. Similarly, when there are three non-homogeneous (or four homogeneous) variables, they can be regarded as the co-ordinates of a point in a three-dimensional geometry, such as that of ordinary space; corresponding to an equation among the variables, there is a surface (or surfaces) in space; corresponding to two independent equations among the variables, there is a curve (or curves) in space; and corresponding to three independent equations, there is a point (or points) in space. In such cases the analytical relations can often, with great advantage, be exhibited as geometrical properties. When the number of non-homogeneous co-ordinates is greater than three (or the number of homogeneous co-ordinates is greater than four), the circumstances have greater need of such a representation, while there is a greater difficulty in constructing some geometrical illustration; and then it can be obtained in a corresponding form only by the idea of a space of the proper number of dimensions. To secure the possibility of such a representation, it is necessary to evolve the geometry of multiple space.

For example, there are four single theta-functions, and their squares are connected by linear homogeneous relations. In order to obtain other properties of the functions themselves, it is convenient to regard them as homogeneous co-ordinates of a point in (ordinary) space; the amplitude in space that then is to be selected is the quadri-quadric tortuous curve represented by those linear relations, viz., the curve which is common to two quadric cylinders with intersecting axes. Similarly there are sixteen double theta-functions, with corresponding linear relations among their squares. The associated geometry is fifteen-dimensional; the manifoldness in this space to be selected for the discussion of the properties is the quadri-quadric two-dimensional amplitude common to thirteen quadric hyper-cylinders.

An initial difficulty in the construction of an analytical geometry of $n$ dimensions is the expression of an amplitude of less than $n-1$ dimensions by means of equations that shall represent the complete amplitude, and nothing besides the amplitude. It occurs in ordinary solid geometry, the difficulty there being to obtain the expression of a tortuous curve in space by means of equations that represent it alone. For instance, a twisted cubic is frequently taken as the intersection of two quadrics having one common generator; but the equations of the quadrics taken together represent not the cubic curve alone but also the common generator. And the like for other cases.

Cayley's purpose in his "Memoir on Abstract Geometry," already referred to, was the exposition of some of the elementary principles of the subject. The paper is a remarkable instance of his power of presentation of abstract ideas, and of his clear precision of statement. Moreover, he makes it an explanatory paper; and, in view of the prevailing estimate of him as an analyst, it is worthy of notice that the paper does not contain a single equation, and contains only a few symbols. It is unnecessary to summarise its contents; the furthest stage reached is the establishment of the notion that underlies the principle of duality in geometry.

But though the necessity for hyperdimensional geometry can thus be met so far as it arises in connexion with analysis, it is a different matter when the geometry is to be regarded as the generalisation of the geometries of two-dimensional space and of three-dimensional space. Cayley's reply to his own question as to the meaning to be attached to hyperdimensional space is* that

> "It may be at once admitted that we cannot conceive of a fourth dimension of space; that space as we conceive of it, and the physical space of our experience, are alike three-dimensional; but we can, I think, conceive of space as being two- or even one-dimensional; we can imagine rational beings living in a one-dimensional space (a line) or in a two-dimensional space (a surface), and conceiving of space accordingly, and to whom, therefore, a two-dimensional space, or (as the case may be) a three-dimensional space would be as inconceivable as a four-dimensional space is to us."

By not a few people the first clause in this passage has been neglected and the later clauses have not always been read rightly; and his further remark, "I need hardly say that the first step is the difficulty, and that granting a fourth dimension we may assume as many more dimensions as we please," has left some readers rather puzzled as to whether Cayley had not, after all, some mysterious incommunicable conception of a fourth dimension. His position is stated in the first clause of the former passage; his conclusion is that hypergeometry is, and is only, a branch of mathematics.

Before passing from the consideration of his larger contributions to hypergeometry, it is proper to mention his introduction of the six co-ordinates of a line. These are six quantities connected by a homogeneous equation $af + bg + ch = 0$; and as only their ratios are used, they are thus equivalent to only four independent magnitudes, sufficient for the unique specification of a right line. They were first established, and primarily used by him, in connexion with his new analytical representation of curves in space;† and he often recurred to the subject, devoting in particular one paper‡ to the calculus of the six co-ordinates and to a discussion of Sylvester's involution of six lines. It should, however, be stated that these co-ordinates presented themselves independently to Plücker; the development of Plücker's theory as set forth in his memoir§ *On a New Geometry of Space*, and in his book‖ *Neue Geometrie des Raumes*, is entirely different from that obtained by Cayley, and it ought to be regarded as a separate creation. And it need hardly be remarked that while the introduction of a line, as an entity represented by a set of co-ordinates, leads to a new geometry of space, it is also clear that line-geometry can be regarded as a geometry of four dimensions.

* *Brit. Assoc. Report*, 1883, President's Address, p. 9.

† *C. M. P.* vol. IV. Nos. 284, 294.

‡ *C. M. P.* vol. VII. No. 435.

§ *Phil. Trans.* 1865, pp. 725—791.

‖ Leipzig, *Teubner*, 1868.

Another notion, entirely due to Cayley in its first form, is that of the Absolute; it was first introduced in his *Sixth Memoir on Quantics*,* which was devoted chiefly to his investigations on the generalised theory of metrical geometry.

It is a known property that the angle between two lines $AB$, $AC$, when multiplied by $2\sqrt{-1}$, is equal to the logarithm of the cross-ratio of the pencil made up of the lines $AB$, $AC$ and (conjugate imaginary) lines joining $A$ to the circular points at infinity; and the measure of the angle between two lines can thus be replaced by the consideration of a projective property of an extended system of lines. Other examples of similar changes could easily be quoted. The purpose of Cayley's theory was to replace metrical properties of a figure or figures by projective properties of an extended system composed of a given figure or figures and of an added figure.

But it is not solely owing to the generalisation of distance that the memoir is famous. It has revolutionised the theory of the so-called non-Euclidian geometry; and it has important bearings on the logical and philosophical analysis of the axioms of space-intuition. The independence and the importance of the ideas, originated by Cayley in this memoir, have never been questioned; but, as is often (and naturally) the case with the discoverer of a fertile subject, Cayley himself did not explain or foresee the full range of application of his new ideas. He did not recognise, at the time when his memoir was first published, the beautiful identification of his generalised theory of metrical geometry with the non-Euclidian geometry of Lobatchewsky and Bolyai. This fundamental step was taken by Klein in his admirable memoir†, *Ueber die sogenannte Nicht-Euklidische Geometrie*, which contains a considerable simplification in statement of Cayley's original point of view, and contributes one of the most important results of the whole theory. The work of the two mathematicians now being an organic whole, there is no advantage—at least here—in attempting to subdivide the subject for the purpose of specifying the exact share of each in its construction.

The scope of the Cayley-Klein ideas may briefly be gathered from the following sketch. Let $A_1$ and $A_2$ be two points, often called a point-pair; they are to be either both real or, if not both real, then conjugate imaginaries so far as their co-ordinates are concerned. Let $P$, $Q$, $R$ be three other points on the line $A_1 A_2$; and let the symbol $(PQ)$ denote

$$2\gamma \log \frac{A_1P \cdot A_2Q}{A_1Q \cdot A_2P} \text{ or } 2i\gamma \log \frac{A_1P \cdot A_2Q}{A_1Q \cdot A_2P},$$

according as $A_1$ and $A_2$ are a real point-pair, or an imaginary point-pair. Then it is manifest that

$$(PQ) + (QR) = (PR),$$

so that the functions $(PQ)$, $(QR)$, $(PR)$ satisfy the fundamental property of the distances between $P$ and $Q$, $Q$ and $R$, and $P$ and $R$. Consequently $(PQ)$ may be taken as a generalised conception of the distance between the points $P$ and $Q$.

* *C. M. P.* vol. II. No. 158; *Phil. Trans.* (1859), pp. 61—90.

† *Math. Ann.* vol. IV. (1871), pp. 573—625.

Now let a conic be described in a plane, either imaginary, say, of the form $x^2+y^2+z^2=0$ or real, say, of the form $x^2+y^2-z^2=0$. Choosing the latter case, let attention be confined to points lying within the conic, so that every straight line through a point cuts the conic in a real point-pair. Take two points, $P$ and $Q$; and let the line joining them cut the conic in two points, $A_1$ and $A_2$. Then $(PQ)$, as defined above (the constant $\gamma$ being the same for all such lines), is the generalised distance between $P$ and $Q$. This conic, which has been arbitrarily assumed, and upon which the generalised conception of distance depends, is termed by Cayley the Absolute.

Cayley, however, avoided the unsatisfactory procedure of using one conception of distance to define a more general conception. As he himself explains more fully,* he regarded the co-ordinates of points as some quantities which define the relative properties of points, considered without any reference to the idea of distance but conceived as ordered elements of a manifold. Thus if $\alpha_1$, $\beta_1$, $\gamma_1$ and $\alpha_2$, $\beta_2$, $\gamma_2$ be the co-ordinates of the point-pair $A_1$ and $A_2$, the co-ordinates of the points $P$ and $Q$ on the line $A_1A_2$ can be taken as $\lambda_1\alpha_1+\lambda_2\alpha_2$, $\lambda_1\beta_1+\lambda_2\beta_2$, $\lambda_1\gamma_1+\lambda_2\gamma_2$ and $\mu_1\alpha_1+\mu_2\alpha_2$, $\mu_1\beta_1+\mu_2\beta_2$, $\mu_1\gamma_1+\mu_2\gamma_2$ respectively. The function $(PQ)$ can then be defined as

$$2\gamma\log\frac{\lambda_2\mu_1}{\lambda_1\mu_2} \text{ or } 2i\gamma\log\frac{\lambda_2\mu_1}{\lambda_1\mu_2};$$

the generalised idea of distance thus finds its definition without any antecedent use of the conception in its ordinary form. Cayley's view is summed up in his sentence†:—"......the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connexion with another figure, viz. the conic termed the absolute."

The metrical formulæ obtained when the absolute is real are identical with those of Lobatchewsky's and Bolyai's "hyperbolic" geometry: when the absolute is imaginary the formulæ are identical with those of Riemann's "elliptic" geometry; the limiting case between the two being that of ordinary Euclidian ("parabolic") geometry.

Cayley's memoir leads inevitably to the question, as to how far projective geometry can be defined in terms of space perception without the introduction of distance. This has been discussed by von Staudt‡ (in 1847, previous to Cayley's memoir), by Klein§ and by Lindemann||. The memoir thus points to a division of our space intuitions into two distinct parts: one, the more fundamental as not involving the idea of distance, the other, the more artificial as adding the idea of distance to the former. The consideration of the relation of these ideas to the philosophical account of space has not yet been brought to its ultimate issue.

* See the note which he added, *C. M. P.* vol. II. p. 604, to the Sixth Memoir; it contains some interesting historical and critical remarks.

† *Loc. cit.* § 230.

‡ *Geometrie der Lage*; also in his later *Beiträge zur Geometrie der Lage*, 1857.

§ *Math. Ann.* vol. VI. (1873), pp. 112—145.

|| *Vorlesungen über Geometrie* (Clebsch-Lindemann), vol. II. part I.; the third section is devoted to the subject.

It is in analytical geometry, both of curves and of surfaces, that the greatest variety of Cayley's contributions is to be found. There is hardly an important question in the whole range of either subject in the solution of which he has not had some share; and there are many properties our acquaintance with which is due chiefly, if not entirely, to him. How widely he has advanced the boundaries of knowledge in analytical geometry can be inferred even from the amount of his researches already incorporated in treatises such as those by Salmon, Clebsch and Frost; and yet they represent only a portion of what he has done. In these circumstances only a selection among his contributions can be indicated: it must be understood that, here as elsewhere, the statement does not pretend to be a complete account.

It is an old-established property that two curves of degrees $m$ and $n$ cut in $mn$ points, but that it is not possible to draw a curve of degree $n$ through any $mn$ arbitrarily selected points on a curve of degree $m$. As early as 1843, Cayley extended the property and showed that when a curve of degree $r$ higher than either $m$ or $n$ is to be drawn through the $mn$ points common to the two curves, they do not count for $mn$ conditions in its determination, but only for a number of conditions smaller than $mn$ by $\frac{1}{2}(m+n-r-1)(m+n-r-2)$. A single addition was made to the theorem by Bacharach* in 1886—taking account of the case when the undetermining points lie on a curve of degree $m+n-r-3$; with this exception the algebraical problem was completely solved by Cayley in his original paper†. The result is often called Cayley's intersection-theorem.

Another geometrical research of fundamental importance was embodied by him in a memoir‡, "On the higher singularities of a plane curve," published in 1866: it is there proved that any singularity whatever on a plane algebraical curve can be reckoned as equivalent to a definite number of the simple singularities constituted by the node, the ordinary cusp, the double tangent and the ordinary inflexional tangent. The theory has, since that date, been developed on lines different from Cayley's—owing to its importance in other theories, such as Abelian functions, variety in its development has proved both necessary and useful; but it was Cayley's investigations in continuation of Plücker's theory that have cleared the path for the later work of others.

The classification of cubic curves had been effected by Newton in his tract "Enumeratio linearum tertii ordinis," published in 1704: and six species had been added by Stirling and Cramer, the total then being 78. Plücker effected a new classification in his "System der analytischen Geometrie," published in 1835: his total number of species is 219, the division into species being more detailed than Newton's. Cayley re-examined the subject in his memoir§, "On the classification of cubic curves," expounding the principles of the two classifications and bringing them into comparison with one another; and entering into the discussion with full minuteness, he obtains the exact relation of the two classifications to one another—a result of great value in the theory.

* *Math. Ann.* vol. XXVI. (1886), pp. 275—299.

† *C. M. P.* vol. I. No. 5; *Camb. Math. Jour.* vol. III. (1843), pp. 211—213.

‡ *C. M. P.* vol. V. No. 374; *Quart. Math. Jour.* vol. VII. (1866), pp. 212—223.

§ *C. M. P.* vol. V. No. 350; *Camb. Phil. Trans.* vol. XI. (1864), pp. 81—128.

To the theories of rational transformation and correspondence he made considerable additions. Two figures are said to be rationally transformable into one another when to a variable point of one of them corresponds reciprocally one (and only one) variable point of the other. The figure may be a space or it may be a locus in a space. Rational transformations between two spaces give rational transformations between loci in those spaces; but it is not in general true that rational transformations between two loci necessarily give rational transformations between the spaces in which those loci exist. There is thus a distinction between the theory of transformation of spaces and the theory of correspondence of loci. Both theories have occupied many investigators, the latter in particular; and Cayley's work may fairly be claimed to have added much to the knowledge of the theory as due* to Riemann, Cremona and others.

Further, there may be singled out for special mention, his investigations on the bitangents of plane curves, and, in particular, on the 28 bitangents of a non-singular quartic; his developments of Plücker's conception of foci; his discussion of the osculating conics of curves, and of the sextactic points on a plane curve (these are the places where a conic can be drawn through six consecutive points); his contributions to the geometrical theory of the invariants and covariants of plane curves; and his memoirs on systems of curves subjected to specified conditions. Moreover, he was fond of making models and of constructing apparatus intended for the mechanical description of curves. The latter finds record in various of his papers; even so lately as 1893 he exhibited, at a meeting of the Cambridge Philosophical Society, a curve-tracing mechanism connected with three-bar motion.

All the preceding results belong to plane geometry; no less important or less numerous were the results he contributed to solid geometry. The twenty-seven lines that lie upon a cubic surface were first announced in his memoir†, "On the triple tangent planes of surfaces of the third order," published in 1849, after a correspondence between Salmon and himself. Cayley devised a new method for the analytical expression of curves in space by introducing into the representation the cone passing through the curve and having its vertex at an arbitrary point. Again, by using Plücker's equations that connect the ordinary (simple) singularities of plane curves, he deduced equations connecting the ordinary (simple) singularities of the developable surface that is generated by the osculating plane of a given tortuous curve, and, therefore, also of any developable surface. He greatly extended Salmon's theory of reciprocal surfaces; and resuming a subject already discussed by Schläfli he produced‡ in 1869 his "Memoir on cubic surfaces," in which he dealt with their complete classification. Many of his memoirs are devoted to the theory of skew ruled surfaces, or scrolls as he called them. Our knowledge of geodesics, of orthogonal systems of surfaces, of the centro-surface of an ellipsoid, of the wave-surface, of the 16-nodal quartic surface, not to mention more,

* In this connexion a report by Brill and Noether, "Bericht über die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit" (*Jahresber. d. Deutschen Mathem.-Vereinigung*, vol. III. 1894) will be found—particularly the sixth and the tenth sections—to give a very valuable *résumé* of the theory and its history.

† *C. M. P.* vol. I. No. 76; *Camb. and Dubl. Math. Jour.* vol. IV. (1849), pp. 118—132. See also Salmon's *Solid Geometry* (third edition, 1874), p. 464, note.

‡ *C. M. P.* vol. VI. p. 412; *Phil. Trans.* (1869), pp. 231—326.

is due in part to the extensions he achieved. It is difficult to indicate parts of the general theory of surfaces and of twisted curves that do not owe at least something and frequently much to his labours; a mere reference to the index of a book like Salmon's *Solid Geometry* will show how vast has been his influence.

One group of subjects interested him throughout his life, the theory of periodic functions, in particular, of elliptic functions: it was to the latter that his only book was devoted. But in a subject, the main lines of which were established so definitely before he began to write*, it is impossible, without entering into great detail, to mark out the contributions that are directly due to him. When a theory is in such a stage as was that of elliptic functions about 1842, the work of one writer sometimes helps to fill the gaps left by that of another, sometimes develops another writer's results from a different point of view; the composite theory depends, in part, upon the coordination of complementary results.

Abel's famous paper†, "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendantes," presented to the French Academy of Sciences in 1826, and unfortunately delayed in publication‡ for nearly fifteen years, attracted Cayley's attention quite early in his scientific career. In 1845 Cayley published his "Mémoire sur les fonctions doublement périodiques,"§ in which he considered Abel's doubly-infinite products of the form

$$u(x) = x\Pi\Pi\left(1+\frac{x}{w}\right),$$

where $w=(m, n)=m\Omega+n\Upsilon$, the ratio $\Omega : \Upsilon$ is not real, and the product is taken for all positive and all negative integer values of $m$ and of $n$ between positive and negative infinity, except simultaneous zero values. He showed that such products can be used to obtain Jacobi's elliptic functions by constructing fractions such as

$$u(x+\tfrac{1}{2}\Omega) \div u(x);$$

and he also showed that the actual value of any product involves an exponential factor $e^{\frac{1}{2}Bx^2}$, where the value of the constant $B$ depends upon the relation|| between the infinities of $m$ and of $n$. The results were of definite importance at the time of their discovery, and they still hold their place. But the form of the doubly-infinite product has been modified¶ by Weierstrass, who takes

$$\sigma(x) = x\Pi\Pi\left\{\left(1+\frac{x}{w}\right)e^{-\frac{x}{w}+\frac{1}{2}\frac{x^2}{w^2}}\right\},$$

* The history will be found in Casorati, *Teorica delle funzioni di variabili complesse*, 1868, and in Enneper, *Elliptische Functionen, Theorie und Geschichte*, second edition, 1890, where other references are given.

† *Œuvres complètes d'Abel* (Christiania, 1881), vol. I. pp. 145—211.

‡ The circumstances are recited in § 9 of the appendix to the volume, by Bjerknes, *Niels Henrik Abel, Tableau de sa vie et de son action scientifique* (Gauthier-Villars, Paris, 1885).

§ *C. M. P.* vol. I. No. 25; *Liouville*, vol. x. (1845), pp. 385—420.

|| This is sometimes expressed differently, as follows. Points are taken having $m$ and $n$ for their Cartesian co-ordinates; those which occur for infinite values of $m$ and of $n$ lie at infinity, and may be considered to lie upon a curve altogether at infinity, the shape of which is determined by the relation between the infinities of $m$ and of $n$.

The value of the constant $B$ is said to depend upon the shape of this bounding curve.

¶ Weierstrass's investigations on infinite products are contained in his memoir "Zur Theorie der eindeutigen analytischen Functionen" (*Abh. d. K. Akad. d. Wiss. zu Berlin*, 1876); also in his book *Abhandlungen aus der Functionenlehre*, 1886.

a function the value of which is independent of any particular form of relation between the infinities of $m$ and of $n$. Owing to the latter simplification, Cayley's results are, as he himself remarked*, partly superseded by those of Weierstrass.

Cayley had great admiration for the works of both Abel and Jacobi; he had begun to read the latter's *Fundamenta Nova* immediately after his degree. The prominent position occupied in that work by the theory of transformation naturally attracted his interest; and, even as early as 1844 and 1846, he wrote short memoirs upon the subject, obtaining in one of them a function, due to Abel and now known as the octahedral function. Further memoirs of a similar tenor appeared occasionally; they deal chiefly with transformation as concerned with the known differential relation of the form

$$\{(1-x^2)(1-k^2x^2)\}^{-\frac{1}{2}}\,dx = M\{(1-y^2)(1-\lambda^2y^2)\}^{-\frac{1}{2}}\,dy.$$

The contributions made to the transformation theory by Sohnke, Joubert, and Hermite, as well as Jacobi's original investigations, all depend upon the use of transcendental functions of the quantity $q\left(=e^{-\pi\frac{K'}{K}}\right)$: yet the results are such that they ought to be deducible by ordinary algebraical processes. It was Cayley's wish to deal with this theory by pure algebra; two simple cases had already thus been discussed by Jacobi, but the extension to the less simple cases proved difficult. Cayley's "Memoir on the transformation of elliptic functions†," carries on the algebraical theory and places it in a clearer light than before. But though he made a distinct advance in dealing with particular cases, he still found it necessary to use the $q$-transcendents for making any definite advance in the general case. And the same compulsion occurs in the chapters of his *Treatise on Elliptic Functions*, where transformation is discussed at considerable length.

He resumed his investigations in 1886, still dealing with the algebraical method, but applying it to a simplified form of elliptic integral due to Brioschi. Though the problem is not solved‡ completely for the general case, he has devised a method which is effective at least in part; it easily leads to new results connected with the modular equations in the known simpler cases previously solved.

The theta-functions are the subject of several of his papers. He began§ with a direct establishment of Jacobi's relation

$$\sqrt{k}\,snu = \mathrm{H}(u) \div \Theta(u),$$

obtained in the *Fundamenta Nova* by a long and cumbrous process; and he proceeded to the construction of the linear differential equations satisfied by the theta-functions. Except, however, in so far as they arise in the transformation theory, they do not appear to have occupied him until about 1877. In that year and in the succeeding

* *C. M. P.* vol. I. p. 586.

† *C. M. P.* vol. IX. No. 577; *Phil. Trans.* 1874, pp. 397—456.

‡ The memoirs of this period belonging to the transformation of elliptic functions were published in the *American Journal of Mathematics*, vol. IX. (1887), pp. 193—224; vol. X. (1888), pp. 71—93.

§ "On the Theory of Elliptic Functions," *C. M. P.* vol. I. No. 45; *Camb. and Dubl. Math. Jour.* vol. II. (1847), pp. 256—266.

years he wrote a number of papers dealing with the theta-functions as on an independent basis and not as a detail in elliptic functions. Though the investigations are concerned with $p$-tuple functions, yet, partly for simplicity, and partly in order to secure the greater detailed development of the theory, the papers deal chiefly with the cases $p=1$, $p=2$.

Previous to Cayley's investigations, the most valuable algebraical results in this subject were those of Rosenhain* and Göpel† which had connected the double theta-functions with the theory of the Abelian functions of two variables, and those of Weierstrass, developed by Königsberger‡ to give the "addition-theorem." Proceeding in his "Memoir on the single and double theta-functions"§ more by Göpel's method than by Rosenhain's, Cayley resumes the whole theory. He pays special attention to the relations among the squares of the functions and to the derivation of the biquadratic relation among four of the functions, which is the same as the equation of Kummer's sixteen-nodal quartic surface. To this relation and to the geometry of this associated surface he frequently recurred, both specifically in isolated papers and generally in researches upon quartic surfaces.

As connected, in part, with elliptic functions, his investigations on the porism of the in- and circumscribed polygon should be mentioned. The porismatic property of two conics, viz. that they may be related to each other so that one polygon (and, if one polygon, then an infinite number of polygons) can be inscribed in one and circumscribed about the other, is due to the geometrician Poncelet. The special case when the conics are two circles had been discussed analytically by Jacobi||, using elliptic functions for the purpose. Cayley undertook, first in 1853, the analytical discussion of the most general case of two conics, also using elliptic functions; and he obtained¶ the relations, necessary for the porism, for the several polygons as far as the enneagon. And it may be remarked, as a characteristic instance of Cayley's habit of proceeding to general cases, that he did not leave the matter at this stage. In a memoir** "On the problem of the in- and circumscribed triangle" he raises the question as to the number of polygons which are such that their angular points lie on a given curve or given curves of any order and their sides touch another given curve or given curves of any class. Using the theory of correspondence, he solves the question completely in the case of a triangle—taking account of the fifty-two cases that arise through the possibility of two curves, or more than two curves, being one and the same curve.

From time to time Cayley turned his attention to questions in theoretical dynamics, choosing them as subjects of his lectures during his earlier years as professor. Among them may be mentioned his investigations on attractions, specially those on the attraction of ellipsoids, to which he devotes five memoirs††, discussing the methods of Legendre,

* *Mém. des Sav. Étr.* vol. XI. (1851), pp. 361—468; the paper is dated 1846.

† *Crelle*, vol. XXXV. (1847), pp. 277—312.

‡ *Crelle*, vol. LXIV. (1865), pp. 17—42.

§ *Phil. Trans.* 1880, pp. 897—1002.

|| *Ges. Werke*, vol. I. pp. 277—293; this paper was published first in *Crelle*, vol. III. (1828), pp. 376—389.

¶ In a set of five papers, *C. M. P.* vol. II. Nos. 113, 115, 116, 128; *ibid.*, vol. IV. No. 267.

** *C. M. P.* vol. VIII. No. 514; *Phil. Trans.* (1871), pp. 369—412.

†† *C. M. P.* vol. I. Nos. 75, 89; vol. II. Nos. 164, 173, 193.

Jacobi, Gauss, Laplace, and Rodrigues; and his evaluations or reductions of multiple definite integrals connected with attractions and potentials in general, particularly his "Memoir on Prepotentials*," in which he discusses the reduction of the most general integral of the type that can occur in dealing with the potential-problem related to hyperspace. He also frequently recurred at intervals, before drawing up his reports about to be quoted, to the consideration of the motion of rotation of a solid body about a fixed point under no forces. By introducing Rodrigues's co-ordinates into the equations of motion he was able to reduce the solution of the problem to quadratures; but the final solution of this case, in the most elegant form, is due to Jacobi himself; it involves single theta-functions. It may be remarked that the next substantial advance made in the theory of motion of a body under the action of forces is due to the late Madame Sophie Kovalewsky, who, in a memoir†, to which the Bordin Prize of 1888 was awarded by the Paris Academy of Sciences, has shown that the motion can, in a particular case, be determined in terms of double theta-functions when the body rotating round a fixed point is subject to the force of gravity.

Sometimes, after reading widely upon a subject, Cayley would draw up a report recounting the chief researches in it made by the great writers. It occasionally happens in the development of a theory that periods come when the incorporation and the marshalling of created ideas seem almost necessary preliminaries to further progress. Cayley was admirably fitted for work of this kind, owing not only to his faculty of clear and concise exposition, but also to his wide and accurate knowledge. Among such reports, two are of particular importance; his "Report on the recent progress of theoretical dynamics‡" and his "Report on the progress of the solution of certain special problems of dynamics§" have proved of signal service to other writers and to students. His knowledge and his power of summarising are shown also in some interesting articles on mathematical topics, written by him for the *Encyclopædia Britannica*.

Cayley also had a great enthusiasm for some of the branches of physical astronomy. Some idea of the value and importance of his labours in this subject, particularly in connexion with the development of the disturbing function in both the lunar theory and the planetary theory, and with the general developments of the functions that arise in elliptic motion, may be gathered by consulting the series of memoirs || which he communicated to the Royal Astronomical Society.

Special reference should be made to one of Cayley's astronomical papers. In 1853 Adams had made a new investigation of the value of the secular acceleration of the moon's mean motion, and, taking account of the variation in the eccentricity of the earth's orbit, had obtained a value which differed from that given by Laplace. Unfortunately, Adams's result was disputed by some of the great school of French physical

* *Phil. Trans.* 1875, pp. 675—774.

† *Mém. des Sav. Étr.*, vol. XXXI. (1894), No. 1.

‡ *C. M. P.* vol. III. No. 195; *Brit. Assoc. Report* (1857), pp. 1—42.

§ *C. M. P.* vol. IV. No. 298; *Brit. Assoc. Report* (1862), pp. 184—252.

|| They are included, with very few exceptions, in the third and the seventh volumes of the *Collected Mathematical Papers.*

astronomers, notably by Pontécoulant, and, in consequence, some hesitation about acceptance was felt by some English astronomers, perhaps not unnaturally in view of the severe criticisms expressed. Cayley made an independent investigation of the necessary approximations, and devised a new method for introducing the variation of the eccentricity in question—a method effective perhaps chiefly owing to the instinct and power with which he carried out the laborious analysis required. The memoir, in which he embodied his results and which was entitled "On the secular acceleration of the moon's mean motion*," completely confirmed the value obtained by Adams, and was of substantial help in settling the controversy.

And, in the last place, the preceding sketch of Cayley's contributions to mathematical science seems to refer, for the most part, only to long memoirs. Yet it must not therefore be supposed that his shorter papers (which are very numerous) can safely be neglected. Sometimes he wrote a simple note not so much to convey new results as to set out his view of some particular theorem; these notes were always fresh and often suggestive. He was specially gratified when he had obtained a brief solution of some question, and his quite short papers frequently contain most important results. For instance, in the brief paper †, "On the theory of the singular solutions of differential equations of the first order," he was the first to give a clear exposition of the theory which in Boole's book had been left in an imperfect state. He there obtained the broad essential results of the theory, and it is particularly on his work, and on the work of Darboux published very soon after Cayley's, that ulterior researches are based.

What has been said may be sufficient to point out Cayley's place among the mathematicians of his time, and to indicate the services he rendered to the science which he loved so well. But he was more than a mathematician. With a singleness of aim, which Wordsworth could have chosen for his "Happy Warrior," he persevered to the last in his nobly lived ideal. His life had a significant influence on those who knew him: they admired his character as much as they respected his genius: and they felt that, at his death, a great man had passed from the world.

A. R. F.

1 *June*, 1895.

* *C. M. P.* vol. III. No. 221; *Monthly Not. R. A. S.* vol. XXII. (1862), pp. 171—231.

† *C. M. P.* vol. VIII. No. 545; *Messenger of Math.* vol. II. (1873), pp. 6—12.

# COURSES OF LECTURES DELIVERED BY PROFESSOR CAYLEY.

(M. denotes Michaelmas Term ; L. denotes Lent Term.)

1863. M. Analytical Geometry.
1864. M. Analytical Geometry.
1865. M. Analytical Geometry and Mechanics.
1866. M. Dynamics.
1867. M. Miscellaneous Analysis.
1868. M. Dynamics and Differential Equations.
1869. M. Analytical Geometry.
1870. M. Theories of correspondence and transformation in analytical geometry.
1871. M. Graphical Geometry.
1872. M. Elliptic Functions.
1873. M. Theory of Equations and Miscellaneous Analysis.
1874. M. Integral Calculus.
1875. M. On a course of pure mathematics.
1876. M. Differential Equations.
1877. M. Algebra.
1878. M. Solid Geometry.
1879. M. Differential Equations.
1880. M. Theory of Equations.
1881. M. Abel's Theorem and the Theta Functions.
1882. M. Abelian and Theta Functions.
1883. M. Higher Algebra and the Theory of Numbers.
1884. M. Some recent developments in Analysis and Geometry.
1885. M. Higher Algebra.
1886. M. Differential Equations and Analytical Geometry.

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| 1887. | L. | Differential Equations. |
| | M. | Quaternions and other non-commutative algebras. |
| 1888. | L. | Analytical Geometry. |
| | M. | Elliptic Functions. |
| 1889. | L. | Analytical Geometry. |
| | M. | Solid Geometry. |
| 1890. | L. | Theory of Equations. |
| | M. | Elliptic Functions, in particular Transformation and the modular equations. |
| 1891. | L. | Analytical Geometry. |
| | M. | Higher Algebra and Analytical Geometry. |
| 1892. | L. | Higher Algebra and Analytical Geometry. |
| | M. | On a course of pure mathematics. |
| 1893. | L. | On a course of pure mathematics. |
| | M. | Analytical Geometry. |
| 1894. | L. | The known transcendental functions. |
| | M. | Analytical Geometry. |
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