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NOTE ON THE DEMONSTRATION OF CLAIRAUT'S THEOREM.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 166, 167.]

It seems worth while to indicate what the leading steps of the demonstration are.

The potential of the Earth's mass upon an external or superficial point is taken to be

$$V, \quad = \frac{V_0}{r} + \frac{V_1}{r^2} + \frac{V_2}{r^3} + \text{\&c.},$$

where V_1, V_2, V_3, ... are Laplace's functions of the angular coordinates.

The surface is assumed to be a nearly spherical surface $r = a(1+u)$, where $u = u_1 + u_2 + \text{\&c.}$, and u_1, u_2, ... are Laplace's functions of the angular coordinates. To be a surface of equilibrium, with an equation $V + \frac{1}{2}\omega^2 r^2 \sin^2\theta = C$, the latter must be equivalent to the equation $r = a(1+u)$, and *it follows* that we have

$$\begin{aligned} V_1 &= V_0 a u_1, \\ V_2 &= V_0 a^2 u_2 - \tfrac{1}{2}\omega^2 a^5 (\tfrac{1}{3} - \cos^2\theta), \\ V_3 &= V_0 a^3 u_3, \\ &\text{\&c.}, \end{aligned}$$

which values are to be substituted in the expression for V.

The whole force of gravity (due to the attraction and the centrifugal force) is taken to be $g, = -\frac{d}{dr}(V + \frac{1}{2}\omega^2 r^2 \sin^2\theta)$, and *it follows* that

$$g = \frac{V_0}{a^2}(1 + u_2 + 2u_3 + \dots) - \tfrac{2}{3}\omega^2 a - \tfrac{5}{2}\omega^2 a (\tfrac{1}{3} - \cos^2\theta),$$

which is of the form

$$g = G\left\{1 + u_2 - \tfrac{5}{2}\frac{\omega^2 a}{G}(\tfrac{1}{3} - \cos^2\theta) + 2u_3 + \ldots\right\}.$$

Taking the Earth to be the spheroid of revolution

$$r = a\{1 + \epsilon(\tfrac{1}{3} - \cos^2\theta)\},$$

then

$$u_2 = \epsilon(\tfrac{1}{3} - \cos^2\theta), \quad u_3 = 0, \ \&c.,$$

and the equation is

$$g = G\left\{1 - \left(\tfrac{5}{2}\frac{\omega^2 a}{G} - \epsilon\right)(\tfrac{1}{3} - \cos^2\theta)\right\},$$

or say

$$g = G\{1 - (\tfrac{5}{2}m - \epsilon)(\tfrac{1}{3} - \cos^2\theta)\},$$

where $m, = \dfrac{\omega^2 a}{G}$, is the ratio of the centrifugal force at the equator to the force of gravity, which is the theorem in question. The expression "it follows" has been twice used as meaning it follows as a mere analytical consequence, in the proper degree of approximation, the steps of the deduction being purposely omitted.