638.

ON A q-FORMULA LEADING TO AN EXPRESSION FOR E_1.

[From the *Messenger of Mathematics,* vol. VI. (1877), pp. 63—66.]

It is to be shown that we have identically

$$(1+2q+2q^4+2q^9+\dots)^4-16\left(\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\frac{3q^3}{1-q^6}+\dots\right)$$

$$=\frac{1-9q-25q^3+49q^6+81q^{10}-\dots}{1-q-q^3+q^6+q^{10}-\dots} \quad\dots\dots\dots\dots\dots (A);$$

or, what is the same thing,

$$(1-2q+2q^4-2q^9+\dots)^4-16\left(\frac{-q}{1-q^2}+\frac{2q^2}{1-q^4}-\frac{3q^3}{1-q^6}+\dots\right)$$

$$=\frac{1+9q+25q^3+49q^6+81q^{10}+\dots}{1+q+q^3+q^6+q^{10}+\dots} \quad\dots\dots\dots\dots\dots (B),$$

where the form (A) is that intended to be made use of, but the form (B) is rather more convenient for the demonstration.

We have

$$(1-2q+2q^4-2q^9+\dots)^4=1+8\left\{\frac{-q}{1+q}+\frac{2q^2}{1+q^2}-\frac{3q^3}{1+q^3}+\dots\right\},$$

(Jacobi, *Fund. Nova,* p. 188, *Ges. Werke,* t. I., p. 239), taking the formula as there written down, and changing q into $-q$.

Also, if for a moment

$$X=1+q+q^3+q^6+q^{10}+\&c.,$$

and

$$X'=\frac{dX}{dq},$$

so that

$$qX' = q + 3q^3 + 6q^6 + 10q^{10} + \&c.,$$

then

$$X + 8qX' = 1 + 9q + 25q^3 + 49q^6 + 81q^{10} + \&c.,$$

so that the right-hand side of (B) is

$$\frac{X + 8qX'}{X}, \quad = 1 + 8q\frac{X'}{X}.$$

But (*Fund. Nova,* p. 185, *Ges. Werke,* t. I., p. 237),

$$X = \frac{1-q^2 \,.\, 1-q^4 \,.\, 1-q^6 \ldots}{1-q \;.\, 1-q^3 \,.\, 1-q^5 \ldots},$$

so that

$$\frac{X'}{X} = \frac{-2q}{1-q^2} - \frac{4q^3}{1-q^4} - \frac{6q^5}{1-q^6} - \ldots$$
$$+ \frac{1}{1-q} + \frac{3q^2}{1-q^3} + \frac{5q^4}{1-q^5} + \ldots.$$

And the equation (B) intended to be proved thus becomes

$$1 + \;\; 8\left\{\frac{-q}{1+q} + \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \ldots\right\}$$
$$-16\left\{\frac{-q}{1-q^2} + \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \ldots\right\}$$
$$= 1 + 8q\left\{\frac{-2q}{1-q^2} - \frac{4q^3}{1-q^4} - \frac{6q^5}{1-q^6} - \ldots\right.$$
$$\left. + \frac{1}{1-q} + \frac{3q^2}{1-q^3} + \frac{5q^4}{1-q^5} + \ldots\right\};$$

viz. omitting the terms unity, dividing by $8q$, and transposing, this is

$$-\frac{1}{1+q} + \frac{2q}{1+q^2} - \frac{3q^2}{1+q^3} + \ldots$$
$$+\frac{2}{1-q^2} - \frac{4q}{1-q^4} + \frac{6q^2}{1-q^6} - \ldots$$
$$+\frac{2q}{1-q^2} + \frac{4q^3}{1-q^4} + \frac{6q^5}{1-q^6} + \ldots$$
$$-\frac{1}{1-q} - \frac{3q^2}{1-q^3} - \frac{5q^4}{1-q^5} - \ldots = 0.$$

The second and third lines unite together, and the equation becomes

$$-\frac{1}{1+q} + \frac{2q}{1+q^2} - \frac{3q^2}{1+q^3} + \frac{4q^3}{1+q^4} - \ldots$$
$$+\frac{2}{1-q} - \frac{4q}{1+q^2} + \frac{6q^2}{1-q^3} - \frac{8q^3}{1+q^4} + \ldots$$
$$-\frac{1}{1-q} - \frac{3q^2}{1-q^3} - \frac{5q^4}{1-q^5} - \frac{7q^6}{1-q^7} - \ldots = 0;$$

or, collecting and arranging,

$$-\frac{1}{1+q}-\frac{2q}{1+q^2}-\frac{3q^2}{1+q^3}-\frac{4q^3}{1+q^4}-\frac{5q^4}{1+q^5}-\dots$$
$$+\frac{1}{1-q}\qquad+\frac{3q^2}{1-q^3}\qquad+\frac{5q^4}{1-q^5}+\dots=0,$$

an identity which it is easy to verify to any number of terms. But to prove it directly, we have only to add the pairs of terms in the alternate columns; calling the left-hand side Fq, we thus obtain

$$Fq=2q\left\{-\frac{1}{1+q^2}-\frac{2q^2}{1+q^4}-\frac{3q^4}{1+q^6}-\dots\right.$$
$$\left.+\frac{1}{1-q^2}\qquad+\frac{3q^4}{1-q^6}+\dots\right\};$$

viz. this equation is $Fq=2qF(q^2)$; and thence

$$Fq=2^2q^{1+2}F(q^4)=2^3q^{1+2+4}F(q^8)=\&\text{c.};$$

we thus have $Fq=0$.

The equation (B), or, what is the same thing, the equation (A) is thus proved.

Reverting to the equation (A), we have

$$(1+2q+2q^4+\dots)^4=\frac{4K^2}{\pi^2},$$

(Jacobi, *Fund. Nova*, p. 188, *Ges. Werke*, t. I., p. 239),

$$\left(\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}+\dots\right)=\frac{K^2}{2\pi^2}\left(1-\frac{E_1}{K}\right),$$

(*ib.*, p. 135; *ib.*, p. 189),

if $q=e^{-\frac{\pi K'}{K}}$, and K, E_1 are the complete functions F_1k, E_1k.

The left-hand side of the equation is thus

$$\frac{4K^2}{\pi^2}-\frac{8K^2}{\pi^2}\left(1-\frac{E_1}{K}\right),\ =\frac{4K^2}{\pi^2}\left(-1+\frac{2E_1}{K}\right),$$

and we have

$$\left(-1+\frac{2E_1}{K}\right)=\frac{\pi^2}{4K^2}\cdot\frac{1-9q^1-25q^3+49q^6+81q^{10}-\dots}{1-q^1-q^3+q^6+q^{10}-\dots},$$

which is a new expression for E_1 as a q-function. The expression on the right-hand side presents itself, Clebsch, *Theorie der Elasticität* (Leipzig, 1862), p. 162, and must have been obtained by him as a value for $\left(-1+\frac{2E_1}{K}\right)$; but there is no statement that this is so, nor anything to show how this form of q-function was arrived at. Mr Todhunter called my attention to the passage in Clebsch.