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THE paper was as follows:

1. Show (independently of the theory of roots) how, if x satisfies an equation of the order n, a given rational function of x can in general be expressed as a rational and integral function of the order $n-1$. State the theorem in a more precise form, so as to make it true universally.

2. Investigate the form of the factors of $1 \pm \sin(2n+1)x$ considered as a function of $\sin x$; and give the formulæ in the two cases, $2n+1=3$ and 5 respectively.

3. Write down the substitutions which do not alter the function $ab+cd$; and explain the constitution of the group.

4. Find in a form adapted for calculation an approximate value for the sum of the middle $2\alpha+1$ terms of the expansion of $(1+1)^{2n}$, n being a large number, and α small in comparison therewith.

Obtain thence a complete and precise statement of the theorem that in a large number of tosses the numbers of heads and tails will probably be nearly equal.

5. A point in space is represented on a given plane by its projections from two fixed points. Show how a problem relating to points, lines, and planes, is thereby reduced to a problem *in plano;* and apply the method to construct the line of intersection of two planes each passing through three given points.

6. A weight is supported on a tripod of three unequal legs resting on a smooth horizontal plane, their feet connected in pairs by strings of given lengths. Show how to determine the tensions of the several strings.

7. Explain the ordinary configuration of a system of isoparametric lines on a spherical surface; for instance, what is the configuration when there are two points of minimum value, and one point of maximum value, of the parameter?

8. Find the attraction of an infinite circular cylinder, of uniform density, on a given exterior or interior point.

9. Determine the number of arbitrary constants contained in the equation of a surface of the order r which passes through the curve of intersection of two given surfaces of the orders m and n respectively.

10. Find, for the several values of p, the number of the conics passing through p given points and touching $5-p$ given lines; and, in each case, show how to obtain (in point-coordinates or line-coordinates, as may be most simple) the equations of the conics satisfying the conditions in question.

11. Investigate the theory of the linear transformation of a ternary quadric function into itself.

12. Explain the theory of the solution of a partial differential equation, given function of x, y, z, p, q, $r =$ arbitrary constant H; where p, q, r are the differential coefficients of the dependent variable u in regard to the independent variables x, y, z respectively.

I propose, not (as in former years) to give complete solutions, but only to notice in more or less detail the leading points in the several questions.

1. The expression is of course required, not only for a given integral function of x, but for a given fractional function. The case where the given function is integral presents no difficulty; when the given function is fractional, the most simple case is where it is $=\dfrac{1}{x-a}$; supposing the equation to be $f(x)=0$, here dividing $f(x)$ by $x-a$, we have a quotient $R(x)$ which is a rational and integral function of an order not exceeding $n-1$, and a remainder which is $=f(a)$; that is,

$$\frac{f(x)}{x-a} = R(x) + \frac{f(a)}{x-a};$$

or, in virtue of the given equation $\dfrac{f(a)}{x-a} = -R(x)$, viz. we have thus $\dfrac{1}{x-a}$ in the required form. But if $f(a)=0$, then we do not obtain such an expression of $\dfrac{1}{x-a}$. It has to be shown that the like considerations apply to any fractional function, and the precise form of the theorem is, that any rational function of x which does not become infinite for any value of x satisfying the given equation, can be expressed as a rational and integral function of an order not exceeding $n-1$.

2. The function $1-\sin(2n+1)x$ is a rational and integral function of $\sin x$, of the order $2n+1$; which if n is even (or $2n+1=4p+1$) contains, as is at once seen, the factor $1-\sin x$, but if n is odd (or $2n+1=4p-1$) the factor $(1+\sin x)$. Suppose that any other factor is $1-\dfrac{\sin x}{\sin\alpha}$, where $\sin\alpha$ not $=\pm 1$; then this will be a double factor if only $\sin x = \sin\alpha$ satisfies the condition

$$0 = \frac{d}{d\,.\sin x}\{1-\sin(2n+1)x\},$$

that is, $0 = \dfrac{\cos(2n+1)x}{\cos x}$; the value in question gives $\sin(2n+1)x = 1$, and therefore $\cos(2n+1)x = 0$; and it does not give $\cos x = 0$; hence every such factor $1 - \dfrac{\sin x}{\sin \alpha}$ is a double factor, or we have

$$1 - \sin(2n+1)x = (1 \pm \sin x)\,\Pi\left(1 - \frac{\sin x}{\sin \alpha}\right)^2.$$

Or the like result might be obtained by considering instead of $1 - \sin(2n+1)x$, the more general function

$$\sin(2n+1)a \pm \sin(2n+1)x,$$

and finally assuming $a = \tfrac{1}{2}\pi$.

3. Relates to a theory which is not, but ought to be, treated of in the text books of the University. See Serret's *Algèbre Supérieure*, t. II., Sect. IV.

The substitutions which leave $ab + cd$ unaltered are

1	1, that is, the letters remain unchanged,
α	(ab), that is, a and b are interchanged,
β	(cd), that is, c and d are interchanged,
γ	$(ab)\ (cd)$, that is, a and b and also c and d are simultaneously interchanged,
δ	$(ac)\ (bd)$, same with a and c, b and d,
ϵ	$(ad)\ (bc)$, same with a and d, b and c,
ζ	$(acbd)$, that is, we cyclically change a into c, c into b, b into d, and d into a,
θ	$(adbc)$, that is, we cyclically change a into d, d into b, b into c, and c into a,

viz. we have eight substitutions $1, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta$ forming a group; that is, the product of any two of them, in either order, is a substitution of the group (or, what is the same thing, the effect of the successive performance of the two upon any arrangement $abcd$ is the same as that of the performance thereon of some other substitution of the group); thus we have $\alpha^2 = 1$, $\beta^2 = 1$, $\gamma^2 = 1$, $\alpha\beta = \beta\alpha = \gamma$, &c.; the system of these equations, which verify that the set of substitutions form a group, defines the constitution of the group—thus to take a more simple instance, a group of 4 may be $1, \alpha, \alpha^2, \alpha^3$ $(\alpha^4 = 1)$ or $1, \alpha, \beta, \alpha\beta$, $(\alpha^2 = 1,\ \beta^2 = 1,\ \alpha\beta = \beta\alpha)$.

4. The expression of the general coefficient is

$$= \frac{1\,.\,2 \ldots 2n}{1\,.\,2 \ldots n-\alpha\,.\,1\,.\,2 \ldots n+\alpha},$$

which can be transformed by the well-known formula

$$1\,.\,2 \ldots n = n^{n+\frac{1}{2}}\sqrt{(\pi)}\,e^{-n},$$

viz. the coefficient thus becomes

$$= \frac{2^{2n}}{\sqrt{(n\pi)}} \frac{1}{\left(1-\frac{\alpha}{n}\right)^{n-\alpha+\frac{1}{2}} \left(1+\frac{\alpha}{n}\right)^{n+\alpha+\frac{1}{2}}}.$$

Now α is supposed small in comparison with n, and the factors in the denominator have the logarithms

$$(n-\alpha+\tfrac{1}{2}) \log \left(1-\frac{\alpha}{n}\right), = (n-\alpha+\tfrac{1}{2})\left(-\frac{\alpha}{n}+\tfrac{1}{2}\frac{\alpha^2}{n^2}\right), = -\alpha + \tfrac{1}{2}\frac{\alpha^2}{n},$$

and

$$(n+\alpha+\tfrac{1}{2}) \log \left(1+\frac{\alpha}{n}\right), \qquad = \alpha + \tfrac{1}{2}\frac{\alpha^2}{n};$$

hence the denominator is $= e^{\frac{\alpha^2}{n}}$, and the final approximate value of the coefficient is

$$= \frac{2^{2n}}{\sqrt{(n\pi)}} e^{-\frac{\alpha^2}{n}}.$$

Hence, converting as usual the sum into a definite integral, we have the sum of the $2\alpha+1$ coefficients

$$= \frac{2^{2n}}{\sqrt{(n\pi)}} \int_{-\alpha}^{\alpha} e^{-\frac{\alpha^2}{n}} d\alpha,$$

or, what is the same thing,

$$= \frac{2^{2n}}{\sqrt{(\pi)}} \int_{-\frac{\alpha}{\sqrt{n}}}^{\frac{\alpha}{\sqrt{n}}} e^{-x^2} dx.$$

For the chance that the number of tosses lies between $n+\alpha$ and $n-\alpha$, this has merely to be divided by 2^{2n}; hence writing $\alpha = kn$, the chance that the number may be between $n(1+k)$ and $n(1-k)$ is

$$= \frac{1}{\sqrt{(\pi)}} \int_{-k\sqrt{n}}^{k\sqrt{n}} e^{-x^2} dx,$$

where observe that the integral, taken with the limits ∞, $-\infty$ has the value $\sqrt{(\pi)}$.

Considering k as a given fraction however small, by increasing n we make $k\sqrt{(n)}$ as large as we please, and therefore the integral, as nearly as we please $= \sqrt{(\pi)}$, or the chance as nearly as we please $= 1$; and hence the complete and precise statement of the theorem, viz. by sufficiently increasing the number of tosses, the probability that the deviation from equality shall be any given percentage (as small as we please) of the whole number of tosses, can be made as nearly as we please equal to certainty.

Further, restoring α in place of kn, the chance of a number between $n+\alpha$ and $n-\alpha$ is

$$= \frac{1}{\sqrt{(\pi)}} \int_{-\frac{\alpha}{\sqrt{n}}}^{\frac{\alpha}{\sqrt{n}}} e^{-x^2} dx,$$

which when $\frac{\alpha}{\sqrt{(n)}}$ is small is $=\frac{2\alpha}{\sqrt{(n\pi)}}$, $\left(\text{more accurately } \frac{2\alpha+1}{\sqrt{(n\pi)}}, \text{ when } \alpha \text{ is small}\right)$; hence, however large α is, the chance of a deviation from equality not exceeding $\pm\alpha$, continually *diminishes* with n, and by making n sufficiently large becomes as small as we please.

5. The point is represented in the given plane by two points which lie *in lineâ* with a fixed point (say O) of that plane, viz. O is the intersection of the given plane by the line which joins the two projecting points.

A line is represented on the given plane by two lines, viz. these are the projections of the line from the two given points; each point of the line is represented by the points of intersection of the two lines by any line through O.

A plane may be represented on the given plane by means of its trace thereon, and of the two points (*in lineâ* with O) which represent any point of the plane.

Thus any problem relating to points, lines, and planes, in space is converted into a problem of plane geometry. For instance, to find the trace on the given plane of a plane through three given points A, B, C, the three points are represented by means of the pairs of points A_1, A_2; B_1, B_2; C_1, C_2, the points of each pair lying *in lineâ* with O; the required trace passes through the intersections with the given plane of the lines BC, CA, AB respectively, and we hence find it as the line through the three points which are the intersections of B_1C_1, B_2C_2, of C_1A_1, C_2A_2, and of A_1B_1, A_2B_2 respectively; that these points are in a line is a theorem of plane geometry, which, if not previously known, would have at once been given by the construction.

6. The solution ought obviously to be obtained from the principle of virtual velocities; taking a, b, c for the lengths of the legs, f, g, h for the lengths of the strings, and z for the height of the summit, z is a known function of a, b, c, f, g, h, $\left(z \text{ is in fact } =\frac{3V}{\Delta}\right.$, where V, the volume of the tetrahedron, is a given function of a, b, c, f, g, h; and Δ, the area of the base, is a given function of $\left.f, g, h\right)$. Writing then F, G, H for the tensions, and W for the weight, and regarding z, f, g, h as variable, the principle gives

$$Wdz + Fdf + Gdg + Hdh = 0,$$

that is,

$$F, G, H, = -W\frac{dz}{df}, -W\frac{dz}{dg}, -W\frac{dz}{dh},$$

respectively.

7. The ordinary case is when an isoparametric line has on one side of it larger values, on the other side of it smaller values of the parameter; the case where the isoparametric line is a line of maximum, or of minimum, parameter is excluded.

The lines in the neighbourhood of a point of maximum, or of minimum, parameter are ovals surrounding the point in question, each oval being itself surrounded by the consecutive oval. Supposing that there are two points of minimum parameter, we have round each of them a series of ovals, until at length an oval belonging to the one of them comes to unite itself with an oval belonging to the other, the two ovals altering themselves into a figure of eight. Surrounding this we have a closed curve (in the first instance a deeply twice-indented oval) which (in the case supposed of there being, besides the two points of minimum parameter, a single point of maximum parameter) is in fact an oval surrounding the point of maximum parameter, and the remaining curves are the series of ovals surrounding that point. If we project stereographically from the point of maximum parameter (so that this point is represented by the points at infinity) we have a figure of eight, each loop containing within it a series of continually diminishing closed curves, and the figure of eight itself surrounded by a series of continually increasing closed curves.

8. The investigation by means of the Potential presents the difficulty that the Potential of the infinite cylinder has no determinate value, as at once appears from the limiting case where the cylinder is reduced to a right line; the difficulty is perhaps rather apparent than real, inasmuch as the partial differential equations contain only differential coefficients $\frac{dV}{dr}$, $\frac{d^2V}{dr^2}$, where $\frac{dV}{dr}$ as representing an attraction, and therefore also $\frac{d^2V}{dr^2}$, are determinate. But it is safer to work directly with the Attraction; the Attraction of an infinite line acts in the perpendicular plane through the attracted point, and is inversely proportional to the distance; the problem is thus reduced to the plane problem of a circle of uniform density, force varying as $(\text{distance})^{-1}$, attracting a point in its own plane. This is precisely similar to the case of a sphere with the ordinary law of attraction; dividing the circle into rings, each ring exerts an attraction $=0$ upon an interior point, and an attraction as if collected at the centre upon an exterior point. Hence, writing a for the radius of the cylinder, and r for the distance of the attracted point, the attraction is $=\pi r$ for an interior point, and $=\frac{\pi a^2}{r}$ for an exterior point.

9. The theory is precisely the same as for curves; taking the surfaces to be $U=0$ of the order m, and $V=0$ of the order n, the general form of the equation of a surface of the order r (r not less than m or n) is $LU+MV=0$, where L is the general function of the order $r-m$, and M the general function of the order $r-n$; and so long as r is less than $m+n$, we obtain the required number of arbitrary constants as the sum of the numbers of the coefficients of L and of M, less unity. But as soon as r is $=m+n$ a modification arises, viz. we obtain here an identity by assuming $L=V$, $M=-U$, and so for any larger value of r, we have an identity by assuming $L=V\phi$, $M=-U\phi$, where ϕ is the general function of the order $r-m-n$.

10. The numbers are known to be 1, 2, 4, 4, 2, 1, which values are obtained most easily (though not in the way which is theoretically most interesting) by finding for

the first three cases the equation of the required conic in point-coordinates; and then, by changing these into line-coordinates, we have the equations for the remaining three cases.

$p=5$: 5 points. The equation of the conic is

$$(a, b, c, f, g, h \between x, y, z)^2 = 0,$$

and we have 5 linear equations to determine the ratios of the coefficients; the number is therefore $=1$.

$p=4$: 4 points and 1 line. Taking $U=0$ and $V=0$, the equations of any two conics each passing through the four points, the equation of the required conic will be $U+\lambda V=0$, and the condition of touching a given line gives a quadric equation for λ; the number is therefore $=2$.

$p=3$: 3 points and 2 lines. In the same manner, by taking $U=0$, $V=0$, $W=0$, for the equations of any three conics through the three points; or if the equations of the lines containing the three points in pairs are $x=0$, $y=0$, $z=0$, then the equations of the three conics are $yz=0$, $zx=0$, $xy=0$, and the equation of any conic through these points is $fyz+gzx+hxy=0$; the conditions of touching two given lines $\xi x+\eta y+\zeta z=0$ and $\xi' x+\eta' y+\zeta' z=0$, are

$$\sqrt{f}\sqrt{\xi}+\sqrt{g}\sqrt{\eta}+\sqrt{h}\sqrt{\zeta}=0, \quad \sqrt{f}\sqrt{\xi'}+\sqrt{g}\sqrt{\eta'}+\sqrt{h}\sqrt{\zeta'}=0;$$

we have thus the ratios $\sqrt{f} : \sqrt{g} : \sqrt{h}$ linearly determined in terms of $\sqrt{\xi}$, $\sqrt{\eta}$, &c.; there is no loss of generality in taking $\sqrt{\xi}$, $\sqrt{\xi'}$ each with a determinate sign, the signs of $\sqrt{\eta}$, &c. being then arbitrary, we have $2^4, =16$ values of $\sqrt{f} : \sqrt{g} : \sqrt{h}$, and therefore one-fourth of this $=4$, for the number of values of $f : g : h$; that is, the number is $=4$.

11. This is a known theory; taking x_1, y_1, z_1 for the linear functions of x, y, z, which are such that

$$(a, b, c, f, g, h \between x_1, y_1, z_1)^2 = (a, b, c, f, g, h \between x, y, z)^2,$$

then assuming $x_1, y_1, z_1 = 2\xi - x,\ 2\eta - y,\ 2\zeta - z$ respectively, we have

$$(a, \ldots \between 2\xi - x,\ 2\eta - y,\ 2\zeta - z)^2 = (a, \ldots \between x, y, z)^2,$$

or, omitting terms which destroy each other, and throwing out the factor 4, this is

$$(a, \ldots \between \xi, \eta, \zeta)^2 = (a, \ldots \between \xi, \eta, \zeta \between x, y, z),$$

an equation which is satisfied identically by assuming

$$\begin{aligned}
a\xi + h\eta + g\zeta &= ax + hy + gz \quad . \quad - \nu\eta + \mu\zeta, \\
h\xi + b\eta + f\zeta &= hx + by + fz + \nu\xi \quad . \quad - \lambda\zeta, \\
g\xi + f\eta + c\zeta &= gx + fy + cz - \mu\xi + \lambda\eta \quad . \quad ,
\end{aligned}$$

where λ, μ, ν are arbitrary; viz. multiplying by ξ, η, ζ, and adding we have the equation in question. The three equations determine ξ, η, ζ as linear functions of x, y, z; and we have thence x_1, y_1, z_1 as linear functions of x, y, z; viz. this is a solution containing three arbitrary constants λ, μ, ν.

12. The partial differential equation might equally well have been proposed in the form, given function of x, y, z, p, q, $r=0$, viz. the equation then is $\phi(x, y, z, p, q, r)=0$, the general partial differential equation involving the three independent variables x, y, z, and the derived functions p, q, r of the dependent variable u, *but not involving the dependent variable u.* The question is therefore in effect as follows: to find p, q, r functions of x, y, z connected by the foregoing equation, and, moreover, such that $pdx+qdy+rdz$ is an exact differential; for then writing $u=\int(pdx+qdy+rdz)$, we have the solution of the given partial differential equation.

Whatever be the method adopted, it comes out that the solution depends on the integration of the system of ordinary differential equations

$$\frac{dp}{-\frac{d\phi}{dx}}=\frac{dq}{-\frac{d\phi}{dy}}=\frac{dr}{-\frac{d\phi}{dz}}=\frac{dx}{\frac{d\phi}{dp}}=\frac{dy}{\frac{d\phi}{dq}}=\frac{dz}{\frac{d\phi}{dr}},$$

and the answer consists first in showing this, and secondly, in showing how from an integral or integrals of the system we pass to the solution of the partial differential equation.

Considering the partial differential equation in the form actually proposed, we may instead of ϕ write H, where H will stand for that given function of x, y, z, p, q, r which is the value of the arbitrary constant H; making this change, and putting the foregoing equal quantities equal to the differential dt of a new variable, the system of ordinary differential equations is

$$\frac{dp}{dt}=-\frac{dH}{dx},\quad \frac{dq}{dt}=-\frac{dH}{dy},\quad \frac{dr}{dt}=-\frac{dH}{dz},$$

$$\frac{dx}{dt}=\quad\frac{dH}{dp},\quad \frac{dy}{dt}=\quad\frac{dH}{dq},\quad \frac{dz}{dt}=\quad\frac{dH}{dr},$$

where H is a given function of x, y, z, p, q, r. This is, in fact, the Hamiltonian system of equations; and it was in view to the connexion that the partial differential equation was proposed in its actual form.