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ON THE GENERAL EQUATION OF DIFFERENCES OF THE SECOND ORDER.

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CONSIDER the equation of differences

$$u_x = a_{x-1}\, u_{x-1} + b_{x-2}\, u_{x-2},$$

viz. we have

$$u_2 = a_1 u_1 + b_0 u_0,$$
$$u_3 = a_2 u_2 + b_1 u_1,$$
$$u_4 = a_3 u_3 + b_2 u_2,$$
$$u_5 = a_4 u_4 + b_3 u_3,$$
$$u_6 = a_5 u_5 + b_4 u_4,$$
&c.,

and thence

$$u_3 = \left|\begin{array}{l} a_2 a_1 \\ + b_1 \end{array}\right| u_1 + \quad a_2 \quad b_0 u_0,$$

$$u_4 = \left.\begin{array}{l} a_3 a_2 a_1 \\ + a_3 b_1 \\ + a_1 b_2 \end{array}\right| u_1 + \left.\begin{array}{l} a_3 a_2 \\ + b_2 \end{array}\right| b_0 u_0,$$

$$u_5 = \left.\begin{array}{l} a_4 a_3 a_2 a_1 \\ + a_4 a_3 b_1 \\ + a_4 a_1 b_2 \\ + a_2 a_1 b_3 \\ + b_1 b_3 \end{array}\right| u_1 + \left.\begin{array}{l} a_4 a_3 a_2 \\ + a_4 b_2 \\ + a_2 b_3 \end{array}\right| b_0 u_0,$$

$$u_6 = \left.\begin{array}{l} a_5a_4a_3a_2a_1 \\ +\,a_5a_4a_3b_1 \\ +\,a_5a_4a_1b_2 \\ +\,a_5a_2a_1b_3 \\ +\,a_3a_2a_1b_4 \\ +\,a_5b_3b_1 \\ +\,a_3b_4b_1 \\ +\,a_1b_4b_2 \end{array}\right| u_1 + \left.\begin{array}{l} a_5a_4a_3a_2 \\ +\,a_5a_4b_2 \\ +\,a_5a_2b_3 \\ +\,a_3a_2b_4 \\ +\,b_4b_2 \end{array}\right| b_0u_0,$$

&c.

It is now easy to see the law; viz. writing for instance

$$u_6 = 54321\,.\,u_1 + 5432\,.\,b_0u_0,$$

then 54321 has a leading term $a_5a_4a_3a_2a_1$: it has terms derived from this by changing any pair a_2a_1 into b_1, a_3a_2 into b_2, a_4a_3 into b_3, a_5a_4 into b_4: it has terms derived by changing any two pairs a_4a_3, a_2a_1 into b_3b_1; a_5a_4, a_2a_1 into b_4b_1; a_5a_4, a_3a_2 into b_4b_2, and so on; where observe that the expression a pair denotes the product of two consecutive a's.

And, similarly, 5432 has a leading term $a_5a_4a_3a_2$; the other terms being derived from this in the same manner precisely.

The solution of $u_x = lx\,(au_{x-1} - u_{x-2})$ is included in, and might be deduced from the foregoing, but it is convenient to obtain it separately. Supposing for greater simplicity that $u_{-1} = 0$, $u_0 = 1$ (or, what is the same thing, $u_0 = 1$, $u_1 = l_1a$), then we find

$$u_0 = 1,$$

$$u_1 = l_1a,$$

$$u_2 = l_2l_1a^2 - l_2,$$

$$u_3 = l_3l_2l_1a^3 - \left|\begin{array}{l} l_3l_2 \\ +\,l_3l_1 \end{array}\right| a.$$

$$u_4 = l_4l_3l_2l_1a^4 - \left|\begin{array}{l} l_4l_3l_2 \\ +\,l_4l_3l_1 \\ +\,l_4l_2l_1 \end{array}\right| a^2 + l_4l_2,$$

$$u_5 = l_5l_4l_3l_2l_1a^5 - \left|\begin{array}{l} l_5l_4l_3l_2 \\ +\,l_5l_4l_3l_1 \\ +\,l_5l_4l_2l_1 \\ +\,l_5l_3l_2l_1 \end{array}\right| a^3 + \left|\begin{array}{l} l_5l_4l_2 \\ +\,l_5l_3l_2 \\ +\,l_5l_3l_1 \end{array}\right| a,$$

&c.,

viz. we may for example write

$$u_5 = l_5\, 4321\,.\, a^5 - 4321\,(\cdot)\, a^3 + 4321\,(:)\, a\,;$$

where

$$4321 \text{ denotes } l_4 l_3 l_2 l_1:$$

in 4321 (·), we omit successively each number, viz. we thus obtain

$$432 + 431 + 421 + 321\,,$$
$$= l_4 l_3 l_2 + l_4 l_3 l_1 + l_4 l_2 l_1 + l_3 l_2 l_1:$$

in 4321 (:), we omit successively each two non-consecutive numbers, viz. the omitted numbers being 1, 3; 1, 4; 2, 4, we obtain

$$42 + 32 + 31,$$
$$= l_4 l_2 + l_3 l_2 + l_3 l_1:$$

and so on, the omissions being each three numbers, each four numbers, &c., no two of them being consecutive; thus in 654321 (∴), the omissions are 5, 3, 1, and 6, 4, 2; or the symbol is

$$642 + 531\,,$$
$$= l_6 l_4 l_2 + l_5 l_3 l_1.$$

As an application, a solution of the differential equation $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) + (x-a)\,y = 0$ is $y = u_0 + u_1 x + u_2 x^2 + \text{\&c.}$, where $n^2 u_n = a u_{n-1} - u_{n-2}$, and in particular $1^2 u_1 = a u_0$; the equation of differences is thus of the form in question, and retaining l_n in place of its value, $= n^2$, the solution is $u_0 = 1$, $u_1 = l_1 a$, $u_2 = l_2 l_1 a^2 - l_2$, &c. *ut suprà.* The differential equation was considered by the Rev. H. J. Sharpe, who mentioned it to Prof. Stokes.