649.

ADDITION TO MR GLAISHER'S NOTE ON SYLVESTER'S PAPER, "DEVELOPMENT OF AN IDEA OF EISENSTEIN."

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THE formula (11) [in the Note], under a slightly different form, is demonstrated by me in an addition [263] to Sir J. F. W. Herschel's paper "On the formulæ investigated by Dr Brinkley, &c.," *Phil. Trans.* t. CL., 1860, pp. 321—323. The demonstration is in effect as follows: let u denote a series of the form $1+bx+cx^2+dx^3+\ldots$, and let u^i (where i is positive or negative, integer or fractional) denote the development of the i-th power of u, continued up to the term which involves x^n, the terms involving higher powers of x being rejected; $u^0, u^1, u^2, \ldots$, and generally u^s will denote in like manner the developments of these powers up to the terms involving x^n, or, what is the same thing, they will be the values of u^i corresponding to $i=0, 1, 2, \ldots, s$. By the formula $u^i = 1 + \frac{i}{1}(u-1) + \frac{i \cdot i-1}{1 \cdot 2}(u-1)^2 + \ldots$ as far as the term involving $(u-1)^n$, u^i is a rational and integral function of i of the degree n, and can therefore be expressed in terms of the values $u^0, u^1, u^2, \ldots, u^n$ which correspond to $i=0, 1, 2, \ldots, n$. Let s have any one of the last-mentioned values, then the expression

$$\frac{i \cdot i-1 \cdot i-2 \ldots i-n}{i-s} \; \frac{1}{s \cdot s-1 \ldots 2 \cdot 1 \cdot -1 \cdot -2 \ldots -(n-s)},$$

which as regards i is a rational and integral function of the degree n (the factor $i-s$ which occurs in the numerator and denominator being of course omitted), vanishes for each of the values $i=0, 1, 2, \ldots, n$, except only for the value $i=s$, in which case it becomes equal to unity. The required formula is thus seen to be

$$u^i = \Sigma \left\{ \frac{i \cdot i-1 \cdot i-2 \ldots i-n}{i-s} \; \frac{1}{s \cdot s-1 \ldots 2 \cdot 1 \cdot -1 \cdot -2 \ldots -(n-s)} u^s \right\},$$

where the summation extends to the several values $s=0, 1, 2,.., n$; or, what is the same thing, it is

$$u^i = \Sigma \left\{ \frac{i \,.\, i-1 \,.\, i-2 \ldots i-n}{i-s} \; \frac{(-)^{n-s} 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots (n-s)} \, u^s \right\},$$

or, changing the sign of i, it is

$$u^{-i} = \Sigma \left\{ \frac{i \,.\, i+1 \,.\, i+2 \ldots i+n}{i+s} \; \frac{(-)^s 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots n-s} \, u^s \right\},$$

where, as before, s has the values 0, 1, 2,.., n successively. Or, what is the same thing, we have

$$C_{-i,\, n} = \Sigma \left\{ \frac{i \,.\, i+1 \,.\, i+2 \ldots i+n}{i+s} \; \frac{(-)^s 1}{1 \,.\, 2 \ldots s \,.\, 1 \,.\, 2 \ldots n-s} \, C_{s,\, n} \right\},$$

where the term corresponding to $s=0$, as containing the factor $C_{0,\, n}$ vanishes except in the case $n=0$ (for which it is $=1$); and omitting this evanescent term, this is in fact the formula (11).