650.

ON A QUARTIC SURFACE WITH TWELVE NODES.

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WRITE for shortness

$$\begin{aligned} a&=\beta-\gamma, & f&=\alpha-\delta, & af&=p,\\ b&=\gamma-\alpha, & g&=\beta-\delta, & bg&=q,\\ c&=\alpha-\beta, & h&=\gamma-\delta, & ch&=r; \end{aligned}$$

then, θ being a variable parameter, the surface in question is the envelope of the quadric surface

$$(\alpha+\theta)^2 aghX^2+(\beta+\theta)^2 bhfY^2+(\gamma+\theta)^2 cfgZ^2+(\delta+\theta)^2 abcW^2=0;$$

viz. this is

$$\Sigma\alpha^2 aghX^2\,.\,\Sigma aghX^2-\Sigma\alpha aghX^2=0.$$

There are no terms in X^4, &c.; the coefficient of Y^2Z^2 is

$$\gamma^2 cfg\,.\,bfh+\beta^2 bfh\,.\,cfg-2\beta bfh\,.\,\gamma cfg,$$

which is

$$=bcf^2gh\,(\beta-\gamma)^2,\ =a^2bcf^2gh,\ =abcfgh\,.\,p.$$

Hence the whole equation divides by $abcfgh$, and throwing out this factor, the result is

$$p\,(Y^2Z^2+X^2W^2)+q\,(Z^2X^2+Y^2W^2)+r\,(X^2Y^2+Z^2W^2)=0,$$

or, observing that $p+q+r=0$, this may also be written

$$p\,(YZ+XW)^2+q\,(ZX+YW)^2+r\,(XY+ZW)^2=0,$$

and also

$$p(YZ-XW)^2+q(ZX-YW)^2+r(XY-ZW)^2=0.$$

The more general equation

$$(p,\ q,\ r,\ l,\ m,\ n\between YZ+XW,\ ZX+YW,\ XY+ZW)^2=0$$

represents a quartic surface (octadic) having the 8 nodes

$$\begin{array}{ll}(1,\ 0,\ 0,\ 0), & (\bar{1},\ 1,\ 1,\ 1),\\ (0,\ 1,\ 0,\ 0), & (1,\ \bar{1},\ 1,\ 1),\\ (0,\ 0,\ 1,\ 0), & (1,\ 1,\ \bar{1},\ 1),\\ (0,\ 0,\ 0,\ 1), & (1,\ 1,\ 1,\ \bar{1}).\end{array}$$

We have

	$d_X U=$		$d_Y U=$
$p.$	XW^2+YZW	$p.$	YZ^2+XZW
$q.$	YW^2+YZW	$q.$	YW^2+XZW
$r.$	ZW^2+YZW	$r.$	YX^2+XZW
$l.$	$2XYZ+W(Y^2+Z^2)$	$l.$	$2XYW+Z(W^2+X^2)$
$m.$	$2XYW+Z(W^2+Y^2)$	$m.$	$2YZX+W(Z^2+X^2)$
$n.$	$2XZW+Y(W^2+Z^2)$,	$n.$	$2YZW+X(W^2+Z^2)$,

	$d_Z U=$		$d_W U=$
$p.$	Y^2Z+XYW	$p.$	X^2W+XYZ
$q.$	X^2Z+XYW	$q.$	Y^2W+XYZ
$r.$	W^2Z+XYW	$r.$	Z^2W+XYZ
$l.$	$2ZXW+Y(W^2+X^2)$	$l.$	$2WYZ+X(Y^2+Z^2)$
$m.$	$2YZW+X(W^2+Y^2)$	$m.$	$2WZX+Y(Z^2+X^2)$
$n.$	$2ZXY+W(X^2+Y^2)$,	$n.$	$2WXY+Z(X^2+Y^2)$.

Hence there will be a node

$$\begin{array}{l}1,\ \bar{1},\ \bar{1},\ 1,\ \text{if}\ p+q+r+2l-2m-2n=0,\\ \bar{1},\ 1,\ \bar{1},\ 1,\ \ldots\ p+q+r-2l+2m-2n=0,\\ \bar{1},\ \bar{1},\ 1,\ 1,\ \ldots\ p+q+r-2l-2m+2n=0,\\ 1,\ 1,\ 1,\ 1,\ \ldots\ p+q+r+2l+2m+2n=0;\end{array}$$

or say there will be

1 of these nodes if $p+q+r+2l+2m+2n=0$,

2 .... $p+q+r+2l=0,\ m+n=0$,

3 .... $p+q+r=2l=-2m=-2n$,

4 .... $p+q+r=0,\ l=0,\ m=0,\ n=0$;

viz. the surface having the 12 nodes is the original surface

$$p(YZ+XW)^2+q(ZX+YW)^2+r(XY+ZW)^2,$$

where

$$p+q+r=0.$$

The Jacobian of the quadrics

$$YZ+XW=0,\ ZX+YW=0,\ XY+ZW=0,$$

is

$$\left\| \begin{matrix} W, & Z, & Y, & X \\ Z, & W, & X, & Y \\ Y, & X, & W, & Z \end{matrix} \right\| = 0;$$

viz. the equations are

$$\begin{aligned} X^3 - X(Y^2+Z^2+W^2) + 2YZW &= 0, \\ Y^3 - Y(Z^2+X^2+W^2) + 2ZXW &= 0, \\ Z^3 - Z(X^2+Y^2+W^2) + 2XYW &= 0, \\ W^3 - W(X^2+Y^2+Z^2) + 2XYW &= 0, \end{aligned}$$

each of which is satisfied in virtue of any one of the pairs of equations

$$\begin{array}{l|l} (Y-Z=0,\ X-W=0) & (Y+Z=0,\ X+W=0), \\ (Z-X=0,\ Y-W=0) & (Z+X=0,\ Y+W=0), \\ (X-Y=0,\ Z-W=0) & (X+Y=0,\ Z+W=0), \end{array}$$

so that the Jacobian curve is, in fact, the six lines represented by these equations.

Any two of the three tetrads form an octad, the 8 points of intersection of three quadric surfaces; a figure representing the relation of the 12 points to each other may be constructed without difficulty.

Each tetrad is a sibi-conjugate tetrad *quoad* the quadric $X^2+Y^2+Z^2+W^2=0$. The three tetrads are not on the same quadric surface.