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ON A TORSE DEPENDING ON THE ELLIPTIC FUNCTIONS.

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On attempting to cover with paper one half-sheet of the foregoing sextic torse, [652], I found that the paper assumed approximately the form of a circular annulus of an angle exceeding 360°, and this led me to consider the general theory of the construction of a torse in paper, and, in particular, to consider the torses such that when developed into a plane the edge of regression becomes a circular arc. It is scarcely necessary to remark that, to construct in paper a circular annulus of an angle exceeding 360°, we have only to take a complete annulus, cut it along a radius, and then insert (gumming it on to the two terminal radii) a portion of an equal circular annulus; drawing from each point of the inner circular boundary a half-tangent, and considering these half-tangents as rigid lines, the paper will bend round them so as to form the half-sheet of a torse having for its edge of regression this inner boundary, which will assume the form of a closed curve with two equal and opposite maxima and two equal and opposite minima, described on a cylinder, and being *approximately* such as the curve given by the equations

$$x = \cos\theta,\ y = \sin\theta,\ z = m\cos 2\theta.$$

Considering, in general, an arc PQ (without inflexions) of any curve, and drawing at the consecutive points P, P', P'', &c. the several half-tangents PT, $P'T'$, $P''T''$, ..., then, considering these as rigid lines and bending the paper round them, we have the half-sheet of a torse, having for its edge of regression the curve in question now bent into a curve of double curvature. It is, moreover, clear that the edge of regression has at each point thereof the same radius of absolute curvature as the original plane curve; in fact, if in the plane curve $PP' = ds$, and the angle $T'PT$ between the consecutive half-tangents PT and $P'T'$ be $= d\phi$, these quantities ds and

$d\phi$ remain unaltered in the curve of double curvature; and the radius of absolute curvature is given by the equation $\rho d\phi = ds$. In particular when, as above, the arc is a circular one, say of radius $=\alpha$, then, however the paper is bent, the edge of regression has at each point thereof the radius of absolute curvature $=\alpha$.

Consider on any given surface, at a given point P thereof, and in a given direction, an element of length PP', then (under the restrictions presently mentioned) we can determine the consecutive element $P'P''$, such that the curve $PP'P''\ldots$ shall have at P a radius of absolute curvature $=\alpha$; in fact, r being the radius of curvature of the normal section of the surface through the element PP', the radius of curvature of the section inclined at an angle θ to the normal section is $=r\cos\theta$; so that we have only to take the section at the inclination $\theta, =\cos^{-1}\dfrac{\alpha}{r}$ to the normal section, and we have the consecutive element $P'P''$ such that the radius of absolute curvature of the curve $PP'P''$ is $=\alpha$. The necessary restriction, of course, is that $r > \alpha$; thus, if at the given point P the two principal radii of curvature are of the same sign (to fix the ideas, let the two principal radii and also α be each of them positive), then we may on the surface determine a direction PQ, for which the radius of curvature of the normal section is $=\alpha$; and then the direction of the element PP' may be any direction between PQ and the direction PR, corresponding to the greatest of the two principal radii.

Having obtained the element $P'P''$, we may, if the radius of absolute curvature at P' be given, construct the next element $P'P''$, and so on; that is to say, on a given surface starting from a given point P and given initial direction PP', we can (under a restriction, as above, as to the curvature at the different points of the surface) construct a curve having at the successive points thereof given values of the radius of absolute curvature; viz., the value may be given either as a function of the coordinates of the point on the surface, or as a function of the length of the curve measured say from the initial point P; it is in this last manner that in what follows the value of the radius of absolute curvature is assumed to be given.

We may thus, taking on paper an arc PQ with its half-tangents, apply it to a given surface, the point P to a given point, and the infinitesimal arc PP' to an element PP' in a given direction from the given point; and we thus obtain the half-sheet of a torse having for its edge of regression a determinate curve upon the surface. In particular, the arc PQ may be circular of the radius α, and the surface be a circular cylinder of radius a; and we thus obtain the torse having for edge of regression a curve on the cylinder radius α, and such that the radius of absolute curvature is at each point $=a$. There are three cases according as $a > \alpha$, $a = \alpha$, or $a < \alpha$; it is to be remarked that if $a > \alpha$, then the curve must at each point cut the generating line of the cylinder at an angle not exceeding $\cos^{-1}\dfrac{\alpha}{a}$, but that in the other two cases the angle may have any value whatever; and, further, that in every case when the angle is $=0$, viz. when the curve touches a generating line of the cylinder, then the osculating plane of the curve coincides with the tangent plane of the cylinder.

The analytical theory is very simple. Taking x, y, z as functions of the length s, we have

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1\,;$$

the condition, which expresses that the radius of absolute curvature is $= a$, then is

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = \frac{1}{a^2}.$$

By what precedes, the point (x, y, z) may be taken to be upon a given surface, say upon the cylinder $x^2 + y^2 = \alpha^2$; and we may then write $x = \alpha\cos\theta$, $y = \alpha\sin\theta$. Taking instead of s any independent variable u whatever, and using accents to denote the derived functions in regard to u, the equations become

$$x'^2 + y'^2 + z'^2 = s'^2,$$

$$x''^2 + y''^2 + z''^2 - s''^2 = \frac{1}{a^2} s'^4,$$

$$x = \alpha\cos\theta,\ y = \alpha\sin\theta.$$

From the last two equations we obtain

$$x'^2 + y'^2 = \alpha^2\theta'^2,\ x''^2 + y''^2 = \alpha^2(\theta''^2 + \theta'^4),$$

and the first two equations thus become

$$\alpha^2\theta'^2 + z'^2 = s'^2,$$

$$\alpha^2(\theta''^2 + \theta'^4) + z''^2 - s''^2 = \frac{1}{a^2} s'^4,$$

and from the first of these we find

$$s'' = \frac{\alpha^2\theta'\theta'' + z'z''}{(\alpha^2\theta'^2 + z'^2)^{\frac{1}{2}}},$$

whence the second equation is

$$\alpha^2(\theta''^2 + \theta'^4) + z''^2 - \frac{(\alpha^2\theta'\theta'' + z'z'')^2}{(\alpha^2\theta'^2 + z'^2)} = \frac{(\alpha^2\theta'^2 + z'^2)^2}{a^2},$$

or reducing, this is

$$(\alpha^2\theta'^2 + z'^2)(\theta''^2 + \theta'^4) + (\theta'^2 z''^2 - 2\theta'\theta'' z'z'' - \alpha^2\theta'^2\theta''^2) = \frac{1}{a^2\alpha^2}(\alpha^2\theta'^2 + z'^2)^3.$$

Taking here θ as the independent variable, we have $\theta' = 1$, $\theta'' = 0$, and the equation becomes

$$(\alpha^2 + z'^2) + z''^2 = \frac{1}{a^2\alpha^2}(\alpha^2 + z'^2)^3;$$

or, what is the same thing,

$$z''^2 = \frac{1}{a^2\alpha^2}(\alpha^2 + z'^2)^3 - (\alpha^2 + z'^2)^2.$$

Write here

$$\alpha^2 + z'^2 = \Omega^2,$$

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then

$$z'' = \frac{\Omega\Omega'}{\sqrt{(\Omega^2 - \alpha^2)}},$$

and the equation becomes

$$\frac{\Omega'^2}{\Omega^2 - \alpha^2} = \frac{\Omega^4}{a^2\alpha^2} - 1,$$

or say

$$\frac{a\alpha d\Omega}{\sqrt{(\Omega^2 - \alpha^2 . \Omega^4 - a^2\alpha^2)}} = d\theta,$$

viz. this equation determines Ω as a function of θ, and we then have

$$\begin{cases} ds = \Omega d\theta, \\ dz = \sqrt{(\Omega^2 - \alpha^2)}\, d\theta, \\ x = \alpha \cos\theta, \\ y = \alpha \sin\theta, \end{cases}$$

equations which determine x, y, z, s as functions of the parameter θ, and give thus the edge of regression of the torse in question.

It is clear that the formulæ are very much simplified in the case $a = \alpha$, where the radius of absolute curvature a is equal to the radius α of the cylinder; but it is worth while to develope the general case somewhat further.

Considering the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, to the modulus $k\,(=k') = \frac{1}{\sqrt{(2)}}$, assume

$$\Omega = -\frac{\sqrt{(a\alpha)}}{k}\frac{\operatorname{dn} u}{\operatorname{sn} u},$$

then

$$d\Omega = -\frac{\sqrt{(a\alpha)}}{k}\frac{\operatorname{cn} u\, du}{\operatorname{sn}^2 u},$$

$$\Omega^2 - \alpha^2 \;= \frac{a\alpha}{k^2 \operatorname{sn}^2 u}\left(\operatorname{dn}^2 u - \frac{\alpha}{a} k^2 \operatorname{sn}^2 u\right),$$

$$= \frac{a\alpha}{k^2 \operatorname{sn}^2 u}\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\},$$

$$\Omega^4 - a^2\alpha^4 = \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u}(\operatorname{dn}^4 u - k^4 \operatorname{sn}^4 u),$$

$$= \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u}(1 - 2k^2 \operatorname{sn}^2 u), \;= \frac{a^2\alpha^2}{k^4 \operatorname{sn}^4 u}\operatorname{cn}^2 u,$$

and hence

$$d\theta = \frac{k^2 \operatorname{sn} u\, du}{\sqrt{\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\}}},$$

$$ds = \frac{k\sqrt{(a\alpha)} \operatorname{dn} u\, du}{\sqrt{\left\{1 - \left(1 + \frac{\alpha}{a}\right) k^2 \operatorname{sn}^2 u\right\}}},$$

$$dz = k\sqrt{(a\alpha)}\, du.$$

We have thus $z = k\sqrt{(a\alpha)}\, u$, no constant of integration being required, viz. u is a mere constant multiple of z: and the first and second equations then give s and θ as functions of u, that is, of z; but it is obviously convenient to retain u instead of expressing it in terms of z. As regards the form of these integrals observe that, writing $\operatorname{sn} u = \lambda$, we have

$$du = \frac{d\lambda}{\sqrt{\{1-\lambda^2 . 1-k^2\lambda^2\}}},$$

and thence

$$d\theta = \frac{k^2\lambda d\lambda}{\sqrt{\left\{1-\lambda^2 . 1-k^2\lambda^2 . 1-\left(1+\frac{a}{\alpha}\right)k^2\lambda^2\right\}}},$$

$$ds = \frac{k\sqrt{(a\alpha)}\, d\lambda}{\sqrt{\left\{1-\lambda^2 . 1-\left(1+\frac{\alpha}{a}\right)k^2\lambda^2\right\}}},$$

each of which is in fact reducible to elliptic integrals, but I do not further pursue this general case.

In the particular case $a = \alpha$, we have

$$1-\left(1+\frac{\alpha}{a}\right)k^2 \operatorname{sn}^2 u = \operatorname{cn}^2 u,$$

and the equations become

$$d\theta = \frac{k^2 \operatorname{sn} u\, du}{\operatorname{cn} u},\quad ds = \frac{k\alpha \operatorname{dn} u\, du}{\operatorname{cn} u},$$

which admit of immediate integration; viz. we have

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'},$$

or determining the constant so that θ may vanish for $u = 0$, say

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'} . \frac{1-k'}{1+k'}\right);$$

and

$$s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right);$$

viz. to verify these results we have

$$\frac{d\theta}{du} = \tfrac{1}{2}\frac{k^2}{k'} . k^2 \operatorname{sn} u \operatorname{cn} u\left\{\frac{1}{\operatorname{dn} u + k'} - \frac{1}{\operatorname{dn} u - k'}\right\},$$

$$= \frac{k^4 \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u - k'^2},\ = k^2\frac{\operatorname{sn} u}{\operatorname{cn} u},$$

and

$$\frac{ds}{du} = \tfrac{1}{2}k\alpha . \operatorname{cn} u \operatorname{dn} u\left\{\frac{1}{1+\operatorname{sn} u} + \frac{1}{1-\operatorname{sn} u}\right\},$$

$$= \frac{k\alpha \operatorname{cn} u \operatorname{dn} u}{1-\operatorname{sn}^2 u},\ = \frac{k\alpha \operatorname{dn} u}{\operatorname{cn} u}.$$

Hence, recurring to the original equations, and writing for convenience $a = \alpha = 1$, we see that a solution of the simultaneous equations

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

$$\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 = 1,$$

is

$$x = \cos\theta,\ y = \sin\theta,\ z = ku,$$

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),\ s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right),$$

where, as before, $k = k' = \dfrac{1}{\sqrt{(2)}}$.

Restoring the radius α, and writing the system in the form

$$x = \alpha\cos\theta,\ y = \alpha\sin\theta,\ z = k\alpha u,$$

$$\theta = \tfrac{1}{2}\frac{k^2}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),\ s = \tfrac{1}{2}k\alpha\log\left(\frac{1+\operatorname{sn} u}{1-\operatorname{sn} u}\right),$$

we see that, as u passes from $u = 0$ to $u = K$, and therefore z from $z = 0$ to $z = k\alpha K$ $\left(K \text{ the complete function } F_1\left\{\frac{1}{\sqrt{(2)}}\right\}\right)$, then θ and s each pass from 0 to ∞; and, similarly, as u passes from $u = 0$ to $u = -K$, that is, as z passes from 0 to $-k\alpha K$, then θ passes from 0 to ∞, and s from $s = 0$ to $s = -\infty$; viz. the curve makes in each direction an infinity of revolutions about the cylinder. Developing the cylinder, $\alpha\theta$ becomes an x-coordinate; viz. we have thus the plane curve

$$z = k\alpha u,$$

$$x = \tfrac{1}{2}\frac{k^2\alpha}{k'}\log\left(\frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}\cdot\frac{1-k'}{1+k'}\right),$$

which is a curve extending from the origin in the direction x positive, to touch at infinity the two parallel asymptotes $z = \pm k\alpha K$; and conversely, when such a plane curve is wound about the cylinder, there will be in each direction an infinity of revolutions round the cylinder.