654.

ON CERTAIN OCTIC SURFACES.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIV. (1877), pp. 249—264.]

I. CONSIDER the torse generated by the tangents of the quadriquadric curve, the intersection of the two quadric surfaces

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}w^2 = 0,$$
$$\mathrm{a}'x^2 + \mathrm{b}'y^2 + \mathrm{c}'z^2 + \mathrm{d}'w^2 = 0;$$

then, writing

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = a', \quad \mathrm{ad}' - \mathrm{a}'\mathrm{d} = f',$$
$$\mathrm{ca}' - \mathrm{c}'\mathrm{a} = b', \quad \mathrm{bd}' - \mathrm{b}'\mathrm{d} = g',$$
$$\mathrm{ab}' - \mathrm{a}'\mathrm{b} = c', \quad \mathrm{cd}' - \mathrm{c}'\mathrm{d} = h',$$

and therefore

$$a'f' + b'g' + c'h' = 0,$$

the equation of the torse, writing for greater convenience (a, b, c, f, g, h) in place of (a', b', c', f', g', h'), but understanding these letters as signifying the accented letters (a', b', c', f', g', h'), is

$$\begin{aligned}
& a^4f^2y^4z^4 + b^4g^2z^4x^4 + c^4h^2x^4y^4 \\
& + a^2f^4x^4w^4 + b^2g^4y^4w^4 + c^2h^4z^4w^4 \\
& + 2b^2c^2ghx^4y^2z^2 - 2c^2f^2ahx^4y^2w^2 + 2b^2f^2agx^4z^2w^2 \\
& + 2c^2a^2hfy^4z^2x^2 - 2a^2g^2bfy^4z^2w^2 + 2c^2g^2bhy^4x^2w^2 \\
& + 2a^2b^2fgz^4x^2y^2 - 2b^2h^2cgz^4x^2w^2 + 2a^2h^2cfz^4y^2w^2 \\
& - 2bcg^2h^2w^4y^2z^2 - 2cah^2f^2w^4z^2x^2 - 2abf^2g^2w^4x^2y^2 \\
& + 2(bg - ch)(ch - af)(af - bg)x^2y^2z^2w^2 = 0.
\end{aligned}$$

If in this equation we write

$$a'^2f' = a, \quad a'f'^2 = f,$$
$$b'^2g' = b, \quad b'g'^2 = g,$$
$$c'^2h' = c, \quad c'h'^2 = h;$$

and therefore

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = \frac{a}{\sqrt[3]{(af)}}, \quad \mathrm{ad}' - \mathrm{a}'\mathrm{d} = \frac{f}{\sqrt[3]{(af)}},$$

$$\mathrm{ca}' - \mathrm{c}'\mathrm{a} = \frac{b}{\sqrt[3]{(bg)}}, \quad \mathrm{bd}' - \mathrm{b}'\mathrm{d} = \frac{g}{\sqrt[3]{(bg)}},$$

$$\mathrm{ab}' - \mathrm{a}'\mathrm{b} = \frac{c}{\sqrt[3]{(ch)}}, \quad \mathrm{cd}' - \mathrm{c}'\mathrm{d} = \frac{h}{\sqrt[3]{(ch)}},$$

and consequently

$$(af)^{\frac{1}{3}} + (bg)^{\frac{1}{3}} + (ch)^{\frac{1}{3}} = 0;$$

then the equation becomes

$$\begin{aligned}
&a^2y^4z^4 + b^2z^4x^4 + c^2x^4y^4 \\
&+ f^2x^4w^4 + g^2y^4w^4 + h^2z^4w^4 \\
&+ 2bcx^4y^2z^2 - 2cfx^4y^2w^2 + 2bfx^4z^2w^2 \\
&+ 2cay^4z^2x^2 - 2agy^4z^2w^2 + 2cgy^4x^2w^2 \\
&+ 2abz^4x^2y^2 - 2bhz^4x^2w^2 + 2ahz^4y^2w^2 \\
&- 2ghw^4y^2z^2 - 2hfw^4z^2x^2 - 2fgw^4x^2y^2 \\
&+ 2\{(bg)^{\frac{1}{3}} - (ch)^{\frac{1}{3}}\}\{(ch)^{\frac{1}{3}} - (af)^{\frac{1}{3}}\}\{(af)^{\frac{1}{3}} - (ch)^{\frac{1}{3}}\}x^2y^2z^2w^2 = 0.
\end{aligned}$$

This same equation, without the relation

$$(af)^{\frac{1}{3}} + (bg)^{\frac{1}{3}} + (ch)^{\frac{1}{3}} = 0,$$

and with an arbitrary coefficient for $x^2y^2z^2w^2$; or say, the equation

$$\begin{aligned}
&a^2y^4z^4 + b^2z^4x^4 + c^2x^4y^4 \\
&+ f^2x^4w^4 + g^2y^4w^4 + h^2z^4w^4 \\
&+ 2bcx^4y^2z^2 - 2cfx^4y^2w^2 + 2bfx^4z^2w^2 \\
&+ 2cay^4z^2x^2 - 2agy^4z^2w^2 + 2cgy^4x^2w^2 \\
&+ 2abz^4x^2y^2 - 2bhz^4x^2w^2 + 2ahz^4y^2w^2 \\
&- 2ghw^4y^2z^2 - 2hfw^4z^2x^2 - 2fgw^4x^2y^2 \\
&+ 2kx^2y^2z^2w^2 = 0,
\end{aligned}$$

where a, b, c, f, g, h, k are arbitrary coefficients, is the general equation of an octic surface having the four nodal curves

$$\begin{array}{lllll}
x = 0, & . & hz^2w^2 & - gw^2y^2 & + ay^2z^2 = 0, \\
y = 0, & -hz^2w^2 & . & + fw^2x^2 & + bz^2x^2 = 0, \\
z = 0, & gy^2w^2 & - fw^2x^2 & . & + cx^2y^2 = 0, \\
w = 0, & -ay^2z^2 & - bz^2x^2 & - cx^2y^2 & . \quad = 0.
\end{array}$$

In fact, the equation of the surface may be written in the form

$$\begin{aligned} &w^4\{f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgx^2y^2\} \\ +\,2w^2 &\begin{Bmatrix}-cfx^4y^2-agy^4z^2-bhz^4x^2+2kx^2y^2z^2 \\ +bfx^4z^2+cgy^4x^2+ahx^4y^2\end{Bmatrix} \\ +\,&\{ay^2z^2+bz^2x^2+cx^2y^2\}^2=0,\end{aligned}$$

which puts in evidence the nodal curve

$$w=0,\ -ay^2z^2-bz^2x^2-cx^2y^2=0:$$

there are three similar forms which put in evidence the other three nodal curves.

The four curves are so related to each other that every line which meets three of them meets also the fourth curve; there is consequently a singly infinite series of lines meeting each of the four curves; these break up into four series of lines each forming an octic scroll, and each scroll has the four curves for nodal curves respectively; that is, each scroll is a surface included under the foregoing general equation, and derived from it by assigning a proper value to the constant k. To determine these values, write

$$\begin{cases}\lambda+\mu+\nu=0, \\ \dfrac{af}{\lambda^2}+\dfrac{bg}{\mu^2}+\dfrac{ch}{\nu^2}=0,\end{cases}$$

equations which give four systems of values for the ratios $(\lambda : \mu : \nu)$. We have then

$$k=af\frac{\nu-\mu}{\lambda}+bg\,\frac{\lambda-\nu}{\mu}+ch\frac{\mu-\nu}{\lambda},$$

viz. k has four values corresponding to the several values of $(\lambda : \mu : \nu)$.

The scroll in question is M. De La Gournerie's scroll Σ_1; the equation of the scroll Σ_1 is consequently obtained from the octic equation by writing therein the last-mentioned value of k.

It is to be noticed that k is, in effect, determined by a quartic equation; and, that, for a certain relation between the coefficients, this equation will have a twofold root. Assuming that this relation is satisfied, and assigning to k its twofold value, the resulting scroll becomes a torse; that is, two of the four scrolls coincide together and degenerate into a torse; corresponding to the remaining two values of k we have two scrolls, *companions* of the torse. In order to a twofold value of k, we must have

$$\frac{af}{\lambda^3}=\frac{bg}{\mu^3}=\frac{ch}{\nu^3};$$

and thence

$$(af)^{\frac{1}{3}}+(bg)^{\frac{1}{3}}+(ch)^{\frac{1}{3}}=0;$$

or, what is the same thing,

$$(af+bg+ch)^3-27abcfgh=0.$$

If for a moment we write $af=\alpha^3$, $bg=\beta^3$, $ch=\gamma^3$, and, therefore, $\alpha+\beta+\gamma=0$; then for the twofold root, we have $\lambda : \mu : \nu = \alpha : \beta : \gamma$, and consequently

$$\begin{aligned}k &= \alpha^2(\gamma-\beta)+\beta^2(\alpha-\gamma)+\gamma^2(\beta-\alpha)\\ &= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha),\end{aligned}$$

that is,

$$k=\{(af)^{\frac{1}{3}}-(bg)^{\frac{1}{3}}\}\{(bg)^{\frac{1}{3}}-(ch)^{\frac{1}{3}}\}\{(ch)^{\frac{1}{3}}-(af)^{\frac{1}{3}}\},$$

which agrees with the result in regard to the octic torse.

If in the octic equation we write (x, y, z, w) in place of (x^2, y^2, z^2, w^2), then we have the quartic equation

$$\begin{aligned}&a^2y^2z^2+b^2z^2x^2+c^2x^2y^2\\ &+f^2x^2w^2+g^2y^2w^2+h^2z^2w^2\\ &+2bcx^2yz-2cfx^2yw+2bfx^2zw\\ &+2cay^2zx-2agy^2zw+2cgy^2xw\\ &+2abz^2xy-2bhz^2xw+2ahz^2yw\\ &-2ghw^2yz-2hfw^2zx-2fgw^2xy\\ &+2kxyzw=0,\end{aligned}$$

which is the equation of a quartic surface touched by the planes $x=0$, $y=0$, $z=0$, $w=0$, in the four conics

$$\begin{array}{llllll} x=0, & . & hzw & -gwy & +ayz & =0,\\ y=0, & -hzw & . & +fwx & +bzx & =0,\\ z=0, & gyw & -fwx & . & +cxy & =0,\\ w=0, & -ayz & -bzx & -cxy & . & =0,\end{array}$$

respectively.

II. The octic surface

$$\begin{aligned}U=\;&b^2c^2f^2x^8+c^2a^2g^2y^8+a^2b^2h^2z^8+f^2g^2h^2w^8\\ &-2a^2cg\,(bg-ch)\,y^6z^2-2b^2ah\,(ch-af)\,z^6x^2-2c^2bf\,(af-bg)\,x^6y^2\\ &+2a^2bh\,(\;\;,,\;\;)\,y^2z^6+2b^2cf\,(\;\;,,\;\;)\,z^2x^6+2c^2ag\,(\;\;,,\;\;)\,x^2y^6\\ &-2f^2bc\,(\;\;,,\;\;)\,x^6w^2-2g^2ca\,(\;\;,,\;\;)\,y^6w^2-2h^2ab\,(\;\;,,\;\;)\,z^6w^2\\ &+2f^2gh\,(\;\;,,\;\;)\,x^2w^6+2g^2hf\,(\;\;,,\;\;)\,y^2w^6+2h^2fg\,(\;\;,,\;\;)\,z^2w^6\\ &+f^2(b^2g^2+c^2h^2-4bgch)\,w^4x^4+g^2(c^2h^2+a^2f^2-4chaf)\,w^4y^4+h^2(a^2f^2+b^2g^2-4abfg)\,w^4z^4\\ &+a^2(\;\;,,\;\;)\,y^4z^4+b^2(\;\;,,\;\;)\,z^4x^4+c^2(\;\;,,\;\;)\,x^4y^4\\ &-2gh\,(bcgh-a^2f^2-2afbg-2afch)\,w^4y^2z^2\\ &-2bh\,(\;\;,,\;\;)\,z^4x^2w^2\\ &+2cg\,(\;\;,,\;\;)\,y^4x^2w^2\\ &+2bc\,(\;\;,,\;\;)\,x^4y^2z^2\end{aligned}$$

$$\begin{aligned}
&- 2hf\,(cahf - b^2g^2 - 2bgaf - 2bgch)\, w^4z^2x^2 \\
&- 2cf\,(\quad\quad ,, \quad\quad)\, x^4y^2w^2 \\
&+ 2ah\,(\quad\quad ,, \quad\quad)\, z^4y^2w^2 \\
&+ 2ca\,(\quad\quad ,, \quad\quad)\, y^4x^2z^2 \\
&- 2fg\,(abfg - c^2h^2 - 2chaf - 2chbg)\, w^4x^2y^2 \\
&- 2ag\,(\quad\quad ,, \quad\quad)\, y^4z^2w^2 \\
&+ 2bf\,(\quad\quad ,, \quad\quad)\, x^4z^2w^2 \\
&+ 2ab\,(\quad\quad ,, \quad\quad)\, z^4x^2y^2 \\
&+ 2\Omega x^2y^2z^2w^2 = 0,
\end{aligned}$$

where the values of the coefficients indicated by (,,) are at once obtained by the proper interchanges of the letters, and where Ω is an arbitrary coefficient, is a surface having the four nodal conics

$$\begin{aligned}
x &= 0, \quad . \quad cy^2 - bz^2 + fw^2 = 0, \\
y &= 0, \; -cx^2 \quad . \quad + az^2 + gw^2 = 0, \\
z &= 0, \quad bx^2 - ay^2 \quad . \quad hw^2 = 0, \\
w &= 0, \; -fx^2 - gy^2 - hz^2 \quad . \quad = 0.
\end{aligned}$$

In fact, writing the equation under the form

$$w^2\Theta + (fx^2 + gy^2 + hz^2)^2 \times (b^2c^2x^4 + c^2a^2y^4 + a^2b^2z^4 - 2a^2bcy^2z^2 - 2b^2caz^2x^2 - 2c^2abx^2y^2) = 0,$$

we put in evidence the nodal conic $w = 0$, $fx^2 + gy^2 + hz^2 = 0$: and similarly for the other nodal conics.

It is to be observed, that the complete section by the plane $w = 0$ is the conic $fx^2 + gy^2 + hz^2 = 0$, twice repeated, and the quartic

$$b^2c^2x^4 + c^2a^2y^4 + a^2b^2z^4 - 2a^2bcy^2z^2 - 2ab^2cz^2x^2 - 2abc^2x^2y^2 = 0:$$

the latter being the system of four lines

$$\frac{x}{\sqrt{(a)}} + \frac{y}{\sqrt{(b)}} + \frac{z}{\sqrt{(c)}} = 0, \quad \frac{x}{\sqrt{(a)}} + \frac{y}{\sqrt{(b)}} - \frac{z}{\sqrt{(c)}} = 0,$$

$$\frac{x}{\sqrt{(a)}} - \frac{y}{\sqrt{(b)}} + \frac{z}{\sqrt{(c)}} = 0, \quad \frac{x}{\sqrt{(a)}} - \frac{y}{\sqrt{(b)}} - \frac{z}{\sqrt{(c)}} = 0.$$

The plane in question, $w = 0$, meets the other nodal conics in the six points

$$(x = 0,\ by^2 - cz^2 = 0), \quad (y = 0,\ cz^2 - ax^2 = 0), \quad (z = 0,\ ax^2 - by^2 = 0),$$

which six points are the angles of the quadrilateral formed by the above-mentioned four lines.

The four conics are such, that every line meeting three of these conics meets also the fourth conic. The lines in question form a double system: each of these systems has, in reference to any pair of nodal conics, a homographic property as follows; viz. considering for example the two conics in the planes $z=0$ and $w=0$ respectively, if a line meets these conics in the points P and Q respectively, and through these points respectively and the line $x=0$, $y=0$ we draw planes, then the system of the P planes and the system of the Q planes correspond homographically to each other, the coincident planes of the two systems being the planes $x=0$ and $y=0$ respectively.

Conversely, if through the line $(x=0,\ y=0)$ we draw the two homographically related planes meeting the two conics in the points P and Q respectively, then, for a proper value (determined by a quadratic equation) of the constant $k(=\Theta\div\theta)$ which determines the homographic relation, the line PQ will be a line meeting each of the four conics, and will belong to one or other of the above-mentioned two systems, as k is equal to one or the other of the two roots of the quadratic equation. The scroll generated by the lines meeting each of the four conics, or what is the same thing, any three of these conics, is *primâ facie* a scroll of the order 16; but by what precedes, it appears that this scroll breaks up into two scrolls, which will be each of the order 8. Moreover, each scroll has the four conics for nodal curves; and since the equation $U=0$ is the general equation of an octic surface having the four conics for nodal curves, it follows, that the equation of the scroll is derived from that of the octic surface $U=0$, by assigning a proper value to the indeterminate coefficient Ω; so that there are in fact two values of Ω, for each of which the surface $U=0$ becomes a scroll.

To sustain the foregoing conclusions, take $x=\theta' y$, $x=\theta y$ for the equations of the two planes through the line $(x=0,\ y=0)$, which meet the z-conic and w-conic in the points P and Q respectively; then the equations of the line PQ are

$$\sqrt{(f\theta^2+g)}\,(x-\theta' y)+\sqrt{(-h)}\,(\theta'-\theta)\,z=0,$$
$$-\sqrt{(b\theta'^2-a)}\,(x-\theta y)^2+\sqrt{(-h)}\,(\theta'-\theta)\,w=0,$$

or, writing therein $\theta'=k\theta$, the equations are

$$\sqrt{(f\theta^2+g)}\,(x-k\theta y)+\sqrt{(-h)}\,(k-1)\,\theta z=0,$$
$$-\sqrt{(bk^2\theta^2-a)}\,(x-\theta y)+\sqrt{(-h)}\,(k-1)\,\theta w=0.$$

To find where the line in question meets the plane $y=0$, we have

$$\sqrt{(f\theta^2+g)}\,x+\sqrt{(-h)}\,(k-1)\,\theta z=0,$$
$$-\sqrt{(bk^2\theta^2-a)}\,x+\sqrt{(-h)}\,(k-1)\,\theta w=0,$$

and thence

$$(f\theta^2+g)\,x^2+h\,(k-1)^2\,\theta^2 z^2=0,$$
$$(bk^2\theta^2-a)\,x^2+h\,(k-1)^2\,\theta^2 w^2=0,$$

or multiplying a, g and adding

$$(af+bgk^2)\,x^2+h\,(k-1)^2\,(az^2+gw^2)=0,$$

or assuming

$$af+bgk^2+ch\,(k-1)^2=0,$$

the equation is

$$-cx^2+az^2+gw^2=0.$$

That is, k being determined by the quadric equation $af+bgk^2+ch(k-1)^2=0$, the line PQ meets the y-conic $y=0$, $-cx^2+az^2+gw^2=0$; and, in a similar manner, it appears that the line PQ also meets the x-conic $x=0$, $cy^2-bz^2+fw^2=0$.

Writing for greater symmetry $1 : -k : k-1 = \lambda : \mu : \nu$, we have

$$\lambda + \mu + \nu = 0,$$
$$af\lambda^2+bg\mu^2+ch\nu^2=0,$$

so that there are two systems of values of (λ, μ, ν) corresponding to, and which may be used in place of, the two values of k respectively.

Starting now from the equations

$$(f\theta^2+g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2=0,$$
$$(bk^2\theta^2-a)(\theta y-x)^2+h(k-1)^2\theta^2w^2=0,$$

the elimination of θ from these equations leads to an equation $U=0$, of the above mentioned form but with a determinate value of the coefficient.

The process, although a long one, is interesting and I give it in some detail.

Elimination of θ from the foregoing equations.

We have

$$U=M\Pi\,[(f\theta^2+g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2],$$

where Π denotes the product of the expressions corresponding to the four roots of the equation

$$(bk^2\theta^2-a)(\theta y-x)^2+h(k-1)^2\theta^2w^2=0.$$

Observing that this equation does not contain z, and that the expression under the sign Π does not contain w, it is at once seen that the product Π is in regard to (z, w) a rational and integral function of the form $(z^2, w^2)^4$; and since, in regard to (z, w), U is also a rational and integral function of the same form $(z^2, w^2)^4$, it is clear that the factor M does not contain z or w, but is a function of only (x, y). To determine it we may write $z=0$, $w=0$: this gives

$$c^2(bx^2-ay^2)^2(fx^2+gy^2)^2=M\Pi\,(f\theta^2+g)(k\theta y-x)^2,$$

where

$$(bk^2\theta^2-a)(\theta y-x)^2=0,$$

and the values of θ are therefore $+\dfrac{\sqrt{(a)}}{k\sqrt{(b)}}$, $-\dfrac{\sqrt{(a)}}{k\sqrt{(b)}}$, $\dfrac{x}{y}$, $\dfrac{x}{y}$. Hence substituting and observing that

$$c^2h^2(k-1)^4=(af+bgk^2)^2,$$

it is easy to find

$$M=\frac{y^4}{x^4}\,\frac{b^4k^4}{h^2(k-1)^8},$$

that is, we have

$$\frac{x^4}{y^4}\,\frac{h^2(k-1)^8}{b^4k^4}\,U=\Pi\,[(f\theta^2+g)(k\theta y-x)^2+h(k-1)^2\,\theta^2z^2],$$

where

$$(bk^2\theta^2-a)(\theta y-x)^2+h(k-1)^2\,\theta^2w^2=0.$$

If for greater convenience we write $\theta=\frac{x}{y}\phi$, then this formula becomes

$$\frac{y^4}{x^4}\,\frac{h^2(k-1)^8}{b^4k^4}\,U=\Pi\,[(fx^2\phi^2+gy^2)(k\phi-1)^2+h(k-1)^2\,z^2\phi^2],$$

where

$$(bk^2x^2\phi^2-ay^2)(\phi-1)^2+h(k-1)^2\,w^2\phi^2=0,$$

or, what is the same thing,

$$\left(\phi^2-\frac{ay^2}{bk^2x^2}\right)(\phi-1)^2+\frac{h(k-1)^2w^2}{bk^2x^2}\,\phi^2=0.$$

Suppose that the terms in U which contain z^2 are $=\Theta z^2$; then we have

$$\frac{y^4}{x^4}\,\frac{h(k-1)^6}{b^4k^4}\,\Theta=\Sigma\phi_1{}^2\Pi'\,(fx^2\phi^2+gy^2)(k\phi-1)^2,$$

or, what is the same thing,

$$\Theta=\frac{b^4k^4}{h(k-1)^6}\,\frac{x^4}{y^4}\,\Sigma\phi_1{}^2\Pi'\,(fx^2\phi^2+gy^2)(k\phi-1)^2,$$

where Π' refers to the remaining three roots ϕ_2, ϕ_3, ϕ_4; this may also be written

$$\Theta=\frac{b^4k^4}{h(k-1)^6}\,\frac{x^4}{y^4}\,\Pi\,(fx^2\phi^2+gy^2)(k\phi-1)^2\,.\,\Sigma\,\frac{\phi^2}{(fx^2\phi^2+gy^2)(k\phi-1)^2}.$$

Hence, observing that we have identically

$$\left(\phi^2-\frac{ay^2}{bk^2x^2}\right)(\phi-1)^2+\frac{h(k-1)^2w^2}{bk^2x^2}\,\phi^2=(\phi-\phi_1)(\phi-\phi_2)(\phi-\phi_3)(\phi-\phi_4),$$

and writing $\phi=\pm\frac{iy\sqrt{(g)}}{x\sqrt{(f)}}$, $\phi=\frac{1}{k}$, $\{i=\sqrt{(-1)}$ as usual$\}$, we find

$$\Pi\,\{\phi x\sqrt{(f)}\pm iy\sqrt{(g)}\}=\frac{h(k-1)^2}{bk^2}\,[c\,\{x\sqrt{(f)}\pm iy\sqrt{(g)}\}^2 fgw^2]\,y^2,$$

$$\Pi\,(k\phi-1)\qquad=\frac{(k-1)^2}{bx^2}\,(bx^2-ay^2+hw^2);$$

whence, writing for shortness

$$\begin{aligned}\Delta&=[c\,\{x\sqrt{(f)}+iy\sqrt{(g)}^2\}-fgw^2]\,[c\,\{x\sqrt{(f)}-iy\sqrt{(g)}^2\}-fgw^2],\\&=c^2f^2x^4+c^2g^2y^4+f^2g^2w^4+2cfg^2y^2w^2-2cf^2gx^2w^2+2c^2fgx^2y^2,\end{aligned}$$

we find

$$\Pi\,(fx^2\phi^2+gy^2)=\frac{h^2\,(k-1)^4}{b^2k^4}\,\Delta y^4,$$

$$\Pi\,(k\phi-1)^2=\frac{(k-1)^4}{b^2}\,(bx^2-ay^2+hw^2)\,\frac{1}{x^4},$$

and thence

$$\Pi\,(fx^2\phi^2+gy^2)\,(k\phi-1)^2=\frac{h^2\,(k-1)^8}{b^4k^4}\,\Delta\,(bx^2-ay^2+hw^2)^2\,\frac{y^4}{x^4},$$

and consequently

$$\Theta=h\,(k-1)^2\,\Delta\,(bx^2-ay^2+hw^2)^2.\,\Sigma\,\frac{\phi^2}{(fx^2\phi^2+gy^2)\,(k\phi-1)^2}.$$

Hence, writing

$$\frac{\phi^2}{(fx^2\phi^2+gy^2)\,(k\phi-1)^2}=\frac{A}{(k\phi-1)^2}+\frac{B}{k\phi-1}+\frac{C}{x\phi\,\sqrt{(f)}+iy\,\sqrt{(g)}}+\frac{D}{x\phi\,\sqrt{(f)}-iy\,\sqrt{(g)}},$$

we may calculate separately the terms

$$\Sigma\left\{\frac{A}{(k\phi-1)^2}+\frac{B}{k\phi-1}\right\},$$

and

$$\Sigma\left\{\frac{C}{x\phi\,\sqrt{(f)}+iy\,\sqrt{(g)}}+\frac{D}{x\phi\,\sqrt{(f)}-iy\,\sqrt{(g)}}\right\}.$$

The first of these is

$$=\frac{1}{(k-1)^2\,(fx^2+k^2gy)^2\,(bx^2-ay^2+hw^2)^2}\,\{x,\ y,\ w\}^6,$$

if for shortness

$$\begin{aligned}\{x,\ y,\ w\}^6=(fx^2+k^2gy^2)\,&[4\,\{(2-k)\,bx^2-ay^2+h\,(1-k)\ w^2\}^2\\ &-2\,(bx^2-ay^2+hw^2)\,\{(6-6k+k^2)\,bx^2-ay^2+h\,(1-k)^2\,w^2\}]\\ &+4k^2\,(k-1)\,gy^2\,(bx^2-ay^2+hw^2)\,\{(2-k)\,bx^2-ay^2+h\,(1-k)\ w^2\}:\end{aligned}$$

the second is

$$=\frac{2}{(k-1)^2\,(fx^2+k^2gy^2)^2\,\Delta}\,(x,\ y,\ w)^6,$$

if for shortness

$$\begin{aligned}(x,\ y,\ w)^6=\{&(fx^2-k^2gy^2)\,(cfx^2-cgy^2-fgw^2)-4ckfgx^2y^2\}\\ &\times\{fgbk^2x^2+[2ch\,(k-1)^2+af\,]\,gy^2+fgh\,(k-1)^2\,w^2\}\\ &+2\,\{k^2bg-ch\,(k-1)^2\}\,fgx^2y^2\,\{c\,(k+1)\,(fx^2-kgy^2)-kfgw^2\};\end{aligned}$$

and hence

$$\Theta=\frac{1}{(fx^2+k^2gy^2)^2}\,[h\Delta\,\{x,\ y,\ w\}^6+2\,(bx^2-ay^2+hw^2)^2\,(x,\ y,\ w)^6],$$

which must be a rational and integral function of $(x,\ y,\ w)$.

In partial verification of this, observe that, because U contains the terms

$$2b^2cf\,(ch-af\,)\,x^6z^2+2\Omega x^2y^2z^2w^2,$$

Θ should contain the terms

$$2b^2cf(ch - af)\,x^6 + 2\Omega x^2y^2w^2,$$

viz. in Θ the term in x^6 should be $= 2b^2cf(ch - af)\,x^6$.

Now writing $y = 0$, $w = 0$, we have

$$\begin{aligned}\Delta &= c^2f^2x^4,\\ \{x, y, w\}^6 &= b^2f\,\{4(2-k)^2 - 2(6 - 6k + k^2)\}\,x^6,\\ &= b^2f(4 - 4k + 2k^2)\,x^6,\\ (x, y, w)^6 &= bcf^3gk^2x^6;\end{aligned}$$

and hence the required term of Θ is x^6 multiplied by

$$b^2c^2fh(4 - 4k + 2k^2) + 2b^3cfgk^2:$$

viz. the coefficient is

$$\begin{aligned}&= 2b^2cf\,[ch(2 - 2k + k^2) + bgk^2],\\ &= 2b^2cf\,[ch + ch(1-k)^2 + bgk^2],\end{aligned}$$

which in virtue of the relation $af + bgk^2 + ch(1-k)^2$ becomes, as it should do,

$$= 2b^2cf(ch - af).$$

The actual division by $(fx^2 + k^2gy^2)^2$ would, however, be a very tedious process, and it is to be observed, that we only require to know the term $2\Omega x^2y^2w^2$ of Θ. We may therefore adopt a more simple course as follows: the terms of Θ which contain w^2 are $= (Ax^4 + 2\Omega x^2y^2 + By^4)\,w^2$, hence writing for a moment

$$\{x, y, w\}^6 = P + Qw^2, \quad (x, y, w)^6 = R + Sw^2,$$

and observing that we have

$$\begin{aligned}\Delta &= c^2(fx^2 + gy^2)^2 - 2c^2fg\,(fx^2 - gy^2)\,w^2 + \&c.,\\ (bx^2 - ay^2 + hw^2)^2 &= (bx^2 - ay^2)^2 + 2h\,(bx^2 - ay^2)\,w^2 + \&c.,\end{aligned}$$

we have

$$\begin{aligned}(fx^2 + k^2gy^2)^2\,(Ax^4 + 2\Omega x^2y^2 + By^4) = {}& c^2h\,(fx^2 + gy^2)^2\,Q - 2c^2fgh\,(fx^2 - gy^2)\,P\\ &+ (bx^2 - ay^2)^2\,S + 2h\,(bx^2 - ay^2)\,R.\end{aligned}$$

But in this identical equation we may write $x^2 = a$, $y^2 = b$, which gives

$$(af + k^2bg)^2\,(Aa^2 + 2\Omega ab + Bb^2) = c^2h\,(af + bg)^2\,Q - 2c^2fgh\,(af - bg)\,P;$$

and from the equation

$$\{x, y, w\}^6 = P + Qw^2,$$

we have

$$\begin{aligned}P + Qw^2 &= (af + bgk^2)\begin{bmatrix}4\,\{(1-k)\,ab + (1-k)\,hw^2\}^2\\ -\,2hw^2(5 - 6k + k^2)\,ab\end{bmatrix}\\ &\quad + 4k^2(k-1)\,bghw^2\,(1-k)\,ab,\\ &= -\,ch\,(k-1)^2\,\{4(k-1)^2(a^2b^2 + 2w^2abh) - 2(5 - 6k + k^2)\,hw^2\}\\ &\quad - 4k^2(k-1)^2\,ab^2ghw^2,\end{aligned}$$

that is,

$$P = -4ch(k-1)^4 a^2b^2,$$

$$Q = \quad (k-1)^2 abh\{ch(-6k^2+4k+2)-4k^2b\},$$

whence

$$(k-1)^2(Aa^2+2\Omega ab+Bb^2) = ab(af+bg)^2[ch(-6k^2+4k+2)-4k^2bg] \\ + 8fg(af-bg)(k-1)^2(ab)^2.$$

But we have

$$Aa^2+Bb^2 = -2ab(af-bg)(-afbg+c^2h^2+2chaf+2chbg),$$

and thence

$$2(k-1)^2\Omega = (af+bg)^2[ch(-6k^2+4k+2)-4k^2bg] \\ +(k-1)^2(af-bg)\begin{pmatrix}-2afbg+2c^2h^2+4chaf+4chbg \\ +8afbg\end{pmatrix},$$

or

$$(k-1)^2\Omega = (af+bg)^2[ch(-3k^2+2k+1)-2k^2bg] \\ +(k-1)^2(af-bg)[3afbg+c^2h^2+2chaf+2chbg].$$

Writing $-3k^2+2k+1 = -3(k-1)^2-4(k-1)$, this is

$$\Omega = (af+bg)^2\left[ch\left(-3-\frac{4}{k-1}\right)-\frac{2k^2}{(k-1)^2}bg\right]+(af-bg)[3afbg+c^2h^2+2chaf+2chbg],$$

or since $1 : -k : k-1 = \lambda : \mu : \nu$; and writing for shortness $(af, bg, ch) = (\alpha, \beta, \gamma)$, this is

$$\Omega = (\alpha+\beta)^2\left\{\gamma\left(-3-\frac{4\lambda}{\nu}\right)-\frac{2\mu^2}{\nu^2}\beta\right\}+(\alpha-\beta)\{3\alpha\beta+\gamma^2+2\gamma\alpha+2\gamma\beta\},$$

which is the value of Ω: viz. the conclusion arrived at is that, eliminating θ from the equations

$$(f\theta^2 \quad +g)(k\theta y-x)^2+h(k-1)^2\theta^2z^2 = 0,$$

$$(bk^2\theta^2-a)(\quad \theta y-x)^2+h(k-1)^2\theta^2w^2 = 0:$$

where k denotes a determinate function of af, bg, ch, viz. writing $af, bg, ch = \alpha, \beta, \gamma$ and $1 : -k : k-1 = \lambda : \mu : \nu$, we have

$$\lambda + \mu + \nu = 0,$$

$$\alpha\lambda^2+\beta\mu^2+\gamma\nu^2 = 0,$$

equations which serve to determine k: the result of the elimination is the octic equation

$$b^2c^2f^2x^8 + \ldots + 2\Omega x^2y^2z^2w^2 = 0,$$

where Ω has the last-mentioned value.

The value of Ω is unsymmetrical in its form, and there are apparently six values; viz. writing

$$A = (\beta+\gamma)^2 \left\{\alpha\left(-3-\frac{4\mu}{\lambda}\right)-\frac{2\nu^2}{\lambda^2}\gamma\right\}+(\beta-\gamma)(S+\beta\gamma+\alpha^2),$$

$$B = (\gamma+\alpha)^2 \left\{\beta\left(-3-\frac{4\nu}{\mu}\right)-\frac{2\lambda^2}{\mu^2}\alpha\right\}+(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

$$C = (\alpha+\beta)^2 \left\{\gamma\left(-3-\frac{4\lambda}{\nu}\right)-\frac{2\mu^2}{\nu^2}\beta\right\}+(\alpha-\beta)(S+\alpha\beta+\gamma^2),$$

$$A_1 = (\beta+\gamma)^2 \left\{\alpha\left(-3-\frac{4\nu}{\lambda}\right)-\frac{2\mu^2}{\lambda^2}\beta\right\}-(\beta-\gamma)(S+\beta\gamma+\alpha^2),$$

$$B_1 = (\gamma+\alpha)^2 \left\{\beta\left(-3-\frac{4\lambda}{\mu}\right)-\frac{2\nu^2}{\mu^2}\gamma\right\}-(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

$$C_1 = (\alpha+\beta)^2 \left\{\gamma\left(-3-\frac{4\mu}{\nu}\right)-\frac{2\lambda^2}{\nu^2}\alpha\right\}-(\alpha-\beta)(S+\alpha\beta+\gamma^2),$$

where for shortness $S=2(\beta\gamma+\gamma\alpha+\alpha\beta)$, the six values would be A, B, C, A_1, B_1, C_1. But we have really

$$A=B=C=-A_1=-B_1=-C_1;$$

so that Ω has really only two values, equal and of opposite signs, or, what is the same thing, Ω^2 has a unique value. In fact, writing for shortness

$$\lambda+\mu+\nu=P,\quad \alpha\lambda^2+\beta\mu^2+\gamma\nu^2=X,$$

we find at once the identity

$$\lambda^2(A+A_1)=(\beta+\gamma)^2(-2X-4\lambda\alpha P),$$

so that $A=-A_1$, in value of $P=0$, $X=0$. And similarly $B=-B_1$, $C=-C_1$.

But the demonstration of the equation $A=B$ is more complicated. We have

$$A-B=-3\alpha(\beta+\gamma)^2-4\alpha(\beta+\gamma)^2\frac{\mu}{\lambda}-2\gamma(\beta+\gamma)^2\frac{\nu^2}{\lambda^2}+(\beta-\gamma)(S+\beta\gamma+\alpha^2)$$

$$+3\beta(\gamma+\alpha)^2+4\beta(\gamma+\alpha)^2\frac{\nu}{\mu}+2\alpha(\gamma+\alpha)^2\frac{\lambda^2}{\mu^2}-(\gamma-\alpha)(S+\gamma\alpha+\beta^2),$$

that is,

$$\begin{aligned}\lambda^2\mu^2(A-B)=\{&-3\alpha(\beta+\gamma)^2+3\beta(\gamma+\alpha)^2+(\beta-\gamma)(S+\beta\gamma+\alpha^2)-(\gamma-\alpha)(S+\gamma\alpha+\beta^2)\}\lambda^2\mu^2\\&-4\alpha(\beta+\gamma)^2\lambda\mu^3\\&-2\gamma(\beta+\gamma)^2\nu^2\mu^2\\&+4\beta(\gamma+\alpha)^2\nu\lambda^2\mu\\&+2\alpha(\gamma+\alpha)^2\lambda^4,\end{aligned}$$

or, denoting for a moment the coefficient of $\lambda^2\mu^2$ by K, and writing also $\gamma\nu^2 = X - \alpha\lambda^2 - \beta\mu^2$, $\nu = P - \lambda - \mu$, this is

$$\begin{aligned}
&= K\lambda^2\mu^2 \\
&\quad - 4\alpha\,(\beta+\gamma)^2\,\lambda\mu^2 \\
&\quad - 2\;\;(\beta+\gamma)^2\,\mu^2\,(X - \alpha\lambda^2 - \beta\mu^2) \\
&\quad + 4\beta\,(\gamma+\alpha)^2\,\lambda^2\mu\,(P - \lambda - \mu) \\
&\quad + 2\alpha\,(\gamma+\alpha)^2\,\lambda^4, \\
&= -2\;\;(\beta+\gamma)^2\,\mu^2 X + 4\beta\,(\gamma+\alpha)^2\,\lambda^2\mu P \\
&\quad + 2\alpha\,(\gamma+\alpha)^2\,\lambda^4 \\
&\quad - 4\beta\,(\gamma+\alpha)^2\,\lambda^3\mu \\
&\quad + \{-4\beta\,(\gamma+\alpha)^2 + K + 2\alpha\,(\beta+\gamma)^2\}\,\lambda^2\mu^2 \\
&\quad\quad - 4\alpha\,(\beta+\gamma)^2\,\lambda\mu^3 \\
&\quad\quad + 2\beta\,(\beta+\gamma)^2\,\mu^4,
\end{aligned}$$

and here the coefficient of $\lambda^2\mu^2$ is found to be

$$= 2\,\{\alpha\beta\,(\alpha+\beta) + \gamma\,(\alpha-\beta)^2 - 3\gamma^2\,(\alpha+\beta)\}.$$

Hence, the terms without X or P are $= 2\nabla$, where

$$\begin{aligned}
\nabla = \;\; & \alpha\,(\gamma+\alpha)^2\lambda^4 \\
& - 2\beta\,(\gamma+\alpha)^2\lambda^3\mu \\
& + \{\alpha\beta\,(\alpha+\beta) + \gamma\,(\alpha-\beta)^2 - 3\gamma^2\,(\alpha+\beta)\}\,\lambda^2\mu^2 \\
& - 2\alpha\,(\beta+\gamma)^2\,\lambda\mu^3 \\
& + \beta\;(\beta+\gamma)^2\,\mu^3,
\end{aligned}$$

and this is identically

$$= + \left.\begin{matrix} (\alpha+\gamma)\,\lambda^2 \\ 2\gamma\lambda\mu \\ +(\beta+\gamma)\,\mu^2 \end{matrix}\right\} \times \left\{\begin{matrix} \alpha\,(\gamma+\alpha)\,\lambda^2 \\ -2\,(\beta\gamma+\gamma\alpha+\alpha\beta)\,\lambda\mu \\ +\beta\,(\beta+\gamma)\,\mu^2, \end{matrix}\right.$$

where observing that

$$\alpha\lambda^2 + \beta\mu^2 + \gamma\,(P - \lambda - \mu)^2 = X,$$

we have the first factor

$$(\alpha+\gamma)\,\lambda^2 + (\beta+\gamma)\,\mu^2 + 2\gamma\lambda\mu = X - \gamma P^2 + 2\gamma P\,(\lambda+\mu),$$

and consequently

$$\begin{aligned}
\lambda^2\mu^2\,(A-B) = & -2\,(\beta+\gamma)^2\,\mu^2 X + 4\beta\,(\gamma+\alpha)^2\lambda^2\mu P \\
& + 2\,\{X - \gamma P^2 + 2\gamma P\,(\lambda+\mu)\}\,\{\alpha\,(\gamma+\alpha)\,\lambda^2 - 2\,(\beta\gamma+\gamma\alpha+\alpha\beta)\,\lambda\mu + \beta\,(\beta+\gamma)\,\mu^2\};
\end{aligned}$$

viz. in virtue of $P = 0$, $X = 0$, we have $A = B$. And thus

$$A = B = C = -A_1 = -B_1 = -C_1:$$

so that the only values of Ω are, say, A and $-A$.

Reverting to the original equations

$$(f\theta^2 \;\; + g)(k\theta y - x)^2 + h(k-1)^2\theta^2 z^2 = 0,$$
$$(bk^2\theta^2 - a)(\;\theta y - x)^2 + h(k-1)^2\theta^2 w^2 = 0,$$

say these are

$$(\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{d},\ \mathrm{e}\between \theta,\ 1)^4 = 0,$$
$$(\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{d}',\ \mathrm{e}'\between \theta,\ 1)^4 = 0,$$

then the coefficients in the two equations have the values

$$\begin{array}{ll} fk^2y^2, & bk^2y^2, \\ -2kfxy, & -2bk^2xy, \\ fx^2 + gky^2 + h(k-1)^2 z^2, & bk^2x^2 - ay^2 + h(k-1)^2 w^2, \\ -2gkxy, & 2axy, \\ gx^2, & -ax^2, \end{array}$$

where observe that only c contains z^2, and only c' contains w^2. The result of the elimination is

$$\begin{vmatrix} & & & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e} \\ & & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & \\ & \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & & \\ \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{d}, & \mathrm{e}, & & & \\ & & & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}' \\ & & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & \\ & \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & & \\ \mathrm{a}', & \mathrm{b}', & \mathrm{c}', & \mathrm{d}', & \mathrm{e}', & & & \end{vmatrix} = 0;$$

viz. here the only terms which contain z^8 and w^8 are

$$\mathrm{c}^2\mathrm{a}'^2\mathrm{e}'^2 + \mathrm{c}'^2\mathrm{a}^2\mathrm{e}^2,$$

and hence the terms in z^8 and w^8 are

$$h^4(k-1)^8 z^8 \,.\, a^2b^2k^4x^4y^4 + h^4(k-1)^8 w^8 \,.\, f^2g^2k^4x^4y^4,$$

viz. these are

$$= h^2k^4(k-1)^8 x^4y^4(a^2b^2h^2z^8 + f^2g^2h^2w^8),$$

or assuming that the determinant contains as a factor the function $b^2c^2f^2x^8 + \ldots + 2\Omega x^2y^2z^2w^2$, with a properly determined value of Ω, we see that the other factor is $= h^2k^4(k-1)^8 x^4y^4$, which agrees with a preceding result.