655.

A MEMOIR ON DIFFERENTIAL EQUATIONS.

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WE have to do with a set of variables, which is either unipartite $(x, y, z, \ldots)$, or else bipartite $(x, y, z, \ldots; p, q, r, \ldots)$, the variables in the latter case corresponding in pairs x and p, y and q, &c.

A letter not otherwise explained denotes a function of the variables. Any such letter may be put $=$ const., viz. we thereby establish a relation between the variables; and when this is so, we use the *same* letter to denote the constant value of the function. Thus the set being $(x, y, z; p, q, r)$, H may denote a given function $pqr - xyz$; and then, if $H =$ const., we have $pqr - xyz = H$ (a constant). This notation, when once clearly understood, is I think a very convenient one.

The present memoir relates chiefly to the following subjects:

A. Unipartite set $(x, y, z, \ldots)$. The differential system

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \ldots,$$

and connected therewith the linear partial differential equation

$$X\frac{d\theta}{dx} + Y\frac{d\theta}{dy} + Z\frac{d\theta}{dz} + \ldots = 0:$$

also the lineo-differential

$$Xdx + Ydy + Zdz + \ldots.$$

B. Bipartite set $(x, y, z, \ldots; p, q, r, \ldots)$. The Hamiltonian system

$$\frac{dx}{\frac{dH}{dp}} = \frac{dy}{\frac{dH}{dq}} = \frac{dz}{\frac{dH}{dr}} = \ldots = \frac{dp}{-\frac{dH}{dx}} = \frac{dq}{-\frac{dH}{dy}} = \frac{dr}{-\frac{dH}{dz}} = \ldots,$$

and connected therewith the linear partial differential equation

$$\frac{dH}{dp}\frac{d\theta}{dx}-\frac{dH}{dx}\frac{d\theta}{dp}+\frac{dH}{dq}\frac{d\theta}{dy}-\frac{dH}{dy}\frac{d\theta}{dq}+\ldots=0,$$

otherwise written

$$(H,\ \theta),\ =\frac{d(H,\ \theta)}{d(p,\ x)}+\frac{d(H,\ \theta)}{d(q,\ y)}+\ldots,\ =0,$$

where H denotes a given function of the variables: also the Hamiltonian system as augmented by an equality $=dt$, and as augmented by this and another equality

$$=dV\div\left(p\frac{dH}{dp}+q\frac{dH}{dq}+r\frac{dH}{dr}\ldots\right).$$

C. Bipartite set $(x,\ y,\ z,\ldots;\ p,\ q,\ r,\ldots)$. The partial differential equation $H=\text{const.}$, where, as before, H is a given function of the variables, but $p, q, r, \ldots$ are now the differential coefficients in regard to $x, y, z, \ldots$ respectively of a function V of these variables, or, what is the same thing, there exists a function

$$V=\int(pdx+qdy+rdz+\ldots),$$

of the variables $x, y, z, \ldots$.

In what precedes, I have written $(x,\ y,\ z,\ldots)$ to denote a set of any number n of variables, and $(x,\ y,\ z,\ldots;\ p,\ q,\ r,\ldots)$ to denote a set of any even number $2n$ of variables, and the investigations are for the most part applicable to these general cases. But for greater clearness and facility of expression, I usually consider the case of a set $(x,\ y,\ z,\ w)$, or $(x,\ y,\ z;\ p,\ q,\ r)$, &c., as the case may be, consisting of a definite number of variables.

The greater part of the theory is not new, but I think that I have presented it in a more compact and intelligible form than has hitherto been done, and I have added some new results.

Introductory Remarks. Art. Nos. 1 to 3.

1. As already noticed, a letter not otherwise explained is considered as denoting a function of the variables of the set; but when necessary we indicate the variables by a notation such as $z=z(x,\ y)$; z is here a function (known or unknown as the case may be) of the variables x, y, the z on the right-hand side being in fact a functional symbol. And thus also $z=z(x,\ y)$, $=$ const. denotes that the function $z(x,\ y)$ of the variables x, y has a constant value, which constant value is $=z$, viz. we thus indicate a relation between the variables x, y.

2. The variables x, y, &c., may have infinitesimal increments dx, dy, &c.; and the equations of connexion between the variables then give rise to linear relations between these increments, the coefficients therein being differential coefficients and,

as such, represented in the usual notation; thus if $z=z(x, y)$, we have $dz=\frac{dz}{dx}dx+\frac{dz}{dy}dy$, where $\frac{dz}{dx}$, $\frac{dz}{dy}$ are the so-called partial differential coefficients of z in regard to x, y respectively. If we have $y=y(x)$, then also $dy=\frac{dy}{dx}dx$, and the foregoing equation becomes

$$dz=\left(\frac{dz}{dx}+\frac{dz}{dy}\frac{dy}{dx}\right)dx;$$

but considering the two equations $z=z(x, y)$ and $y=y(x)$ as determining z as a function of x, say $z=z(x)$, we have $dz=\frac{d(z)}{dx}dx$; whence comparing the two formulæ

$$\frac{d(z)}{dx}=\frac{dz}{dx}+\frac{dz}{dy}\frac{dy}{dx},$$

where $\frac{d(z)}{dx}$ is the so-called total differential coefficient of z in regard to x. The distinction is best made, not by any difference of notation $\frac{d(z)}{dx}$, $\frac{dz}{dx}$, but by appending in any case of doubt the equations or equation used in the differentiation. Thus we have $\frac{dz}{dx}$ where $z=z(x, y)$; or, as the case may be, $\frac{dz}{dx}$ where $z=z(x, y)$ and $y=y(x)$.

3. A relation between increments is always really a relation between differential coefficients: but we use the increments for symmetry and conciseness, as in the case of a differential system $\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}$, or in a question relating to the lineo-differential $Xdx+Ydy+Zdz$, for instance in the question whether this can be put $=du$.

Notations. Art. Nos. 4 to 6.

4. Functional determinants. If $a, b, c, \ldots$ are functions of the variables $x, y, z, w, \ldots$, then the determinants

$$\begin{vmatrix} \frac{da}{dx}, & \frac{da}{dy} \\ \frac{db}{dx}, & \frac{db}{dy} \end{vmatrix}, \quad \begin{vmatrix} \frac{da}{dx}, & \frac{da}{dy}, & \frac{da}{dz} \\ \frac{db}{dx}, & \frac{db}{dy}, & \frac{db}{dz} \\ \frac{dc}{dx}, & \frac{dc}{dy}, & \frac{dc}{dz} \end{vmatrix}, \text{ \&c.,}$$

are for shortness represented by

$$\frac{d(a,\ b)}{d(x,\ y)},\ \frac{d(a,\ b,\ c)}{d(x,\ y,\ z)},\ \text{\&c.,}$$

the notation being especially used in the first-mentioned case where the symbol is $\frac{d(a, b)}{d(x, y)}$. It is sometimes convenient to extend this notation, and for instance use $\frac{d(a, b)}{d(x, y, z)}$ to denote the series of determinants

$$\left| \begin{array}{ccc} \frac{da}{dx}, & \frac{da}{dy}, & \frac{da}{dz} \\ \frac{db}{dx}, & \frac{db}{dy}, & \frac{db}{dz} \end{array} \right|,$$

which can be formed by selecting in every way two columns to form thereout a determinant; the equation

$$\frac{d(a, b)}{d(x, y, z)} = 0$$

will then denote that each of these determinants is $= 0$.

The analogous notation
$$\frac{d(a, b, c)}{d(x, y)}$$

would denote non-existent determinants, viz. there are here not columns enough to form with them a determinant: and the notation is not required.

5. In the case of a bipartite set $(x, y, z, \ldots; p, q, r, \ldots)$, if a, b are any functions of these variables, we consider the derivative

$$(a, b) = \frac{d(a, b)}{d(p, x)} + \frac{d(a, b)}{d(q, y)} + \frac{d(a, b)}{d(r, z)} + \ldots,$$

viz. (a, b) is used to denote the sum of the functional determinants on the right hand.

6. Taking again $(x, y, z, w, \ldots)$ as the variables, then in the theory of the lineo-differential $Xdx + Ydy + Zdz + Wdw + \ldots$, we use certain derivative functions analogous to Pfaffians. They may be thus defined; viz. considering the numbers 1, 2, 3, 4, ... as corresponding to the variables $x, y, z, w, \ldots$ respectively, we have

$$1 = X,\ 2 = Y,\ 3 = Z,\ 4 = W, \text{ \&c.},$$

$$12 = \frac{dX}{dy} - \frac{dY}{dx},\ 13 = \frac{dX}{dz} - \frac{dZ}{dx}, \text{ \&c.},$$

$$\begin{aligned} 123 &= 1\,.\,23 + 2\,.\,31 + 3\,.\,12 \\ &= X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right), \end{aligned}$$

$$\begin{aligned} 1234 &= 12\,.\,34 + 13\,.\,42 + 14\,.\,23 \\ &= \left(\frac{dX}{dy} - \frac{dY}{dx}\right)\left(\frac{dZ}{dw} - \frac{dW}{dz}\right) + \left(\frac{dX}{dz} - \frac{dZ}{dx}\right)\left(\frac{dW}{dy} - \frac{dY}{dw}\right) + \left(\frac{dX}{dw} - \frac{dW}{dx}\right)\left(\frac{dY}{dz} - \frac{dZ}{dy}\right), \end{aligned}$$

and, adding for greater distinctness the next following cases,

$$12345 = 1 \,.\, 2345 + 2 \,.\, 3451 + 3 \,.\, 4512 + 4 \,.\, 5123 + 5 \,.\, 1234,$$

$$123456 = 12 \,.\, 3456 + 13 \,.\, 4561 + 14 \,.\, 5612 + 15 \,.\, 6123 + 16 \,.\, 2345,$$

where of course 2345, &c., have the significations mentioned above.

Dependency of Functions. Art. Nos. 7 and 8.

7. Two or more functions of the same variables may be independent, or else dependent or connected; viz. in the latter case any one of the functions is a function of the others $a = a(x)$, $b = b(x)$, the functions a, b are dependent, but if

$$a = a(x, y), \quad b = b(x, y),$$

then the condition of dependency is

$$\frac{d(a, b)}{d(x, y)} = 0,$$

and, similarly, if $a = a(x, y, z)$, $b = b(x, y, z)$, then the conditions of dependency are

$$\frac{d(a, b)}{d(x, y, z)} = 0,$$

viz. if the equations thus represented are all of them satisfied, the functions are dependent, but if not, then they are independent.

Observe that, when $a = a(x, y, z)$, $b = b(x, y, z)$ as above, if we choose to attend only to the variables x, y, treating z as a mere constant, there is then a single condition of dependency $\dfrac{d(a, b)}{d(x, y)} = 0$, and so if we attend only to the variable x, treating y, z as mere constants, then a and b are dependent. Thus when $a = x$, $b = x^2 + y$, the functions a, b are independent if we attend to both the variables x, y; dependent if y be regarded as a constant.

8. Further when $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$, the functions a, b, c are dependent; but when $a = a(x, y, z)$, $b = b(x, y, z)$, $c = c(x, y, z)$, the condition of dependency is

$$\frac{d(a, b, c)}{d(x, y, z)} = 0:$$

and so when $a = a(x, y, z, w)$, $b = b(x, y, z, w)$, $c = c(x, y, z, w)$, the conditions of dependency are

$$\frac{d(a, b, c)}{d(x, y, z, w)} = 0;$$

viz. if all the equations thus represented are satisfied, the functions are dependent; but if not, then they are independent. And so in other cases.

The General Differential System. Art. Nos. 9 to 22.

9. Taking the set of variables to be (x, y, z, w), the system is

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{dw}{W},$$

and we associate with this the linear partial differential equation

$$X\frac{d\theta}{dx}+Y\frac{d\theta}{dy}+Z\frac{d\theta}{dz}+W\frac{d\theta}{dw}=0.$$

10. It is tolerably evident that the differential equations establish between x, y, z, w a threefold relation depending upon three arbitrary constants; in fact, regarding (x, y, z, w) as the coordinates of a point in four-dimensional space, and starting from any given point, the differential equations determine the ratios of the increments dx, dy, dz, dw, that is, the direction of passage to a consecutive point; and then again taking for x, y, z, w the coordinates of this point, the same equations give the direction of passage to the next consecutive point, and so on. The locus of the point is therefore a curve, or we have between the coordinates a threefold relation, and (the initial point being arbitrary) we have a curve of the system through each point of the four-dimensional space, viz. the relation must involve three arbitrary constants. But this being so, the constants will be expressible as functions of the coordinates, viz. the threefold relation involving the three constants will be expressible in the form $a=\text{const.}$, $b=\text{const.}$, $c=\text{const.}$, where a, b, c denote respectively functions of the coordinates (x, y, z, w).

11. Supposing that one of the relations is $a=\text{const.}$, it is clear that the increment

$$da, =\frac{da}{dx}dx+\frac{da}{dy}dy+\frac{da}{dz}dz+\frac{da}{dw}dw,$$

must become $=0$, on substituting therein for dx, dy, dz, dw, the values X, Y, Z, W to which by virtue of the differential equations they are proportional, viz. that we must have identically

$$X\frac{da}{dx}+Y\frac{da}{dy}+Z\frac{da}{dz}+W\frac{da}{dw}=0.$$

Conversely, when this is so, we have $da=0$, by virtue of the differential equation.

We say that a is a solution of the partial differential equation, and an integral of the differential equations, viz. any solution of the partial differential equation is an integral of the differential equations, and any integral of the differential equations is a solution of the partial differential equation, or, this being so, we may in general without risk of ambiguity, say simply a is an integral*; similarly b and c are integrals, and, by what precedes, there are three integrals a, b, c.

* Viz. we use indifferently, in regard to the differential equations and to the partial differential equation, the term integral, which is appropriate to the differential equations; the appropriate term in regard to the partial differential equation would be solution.

Observe that, in speaking of an integral a, we mean a function of the variables; the differential equations give between the variables the relation $a = \text{const.}$, and when this is so, we use the same letter a to denote the constant value of this function.

12. In speaking of the three integrals a, b, c we mean independent integrals; any function whatever ϕa of an integral a, or any function whatever $\phi(a, b)$ of two integrals a, b, is an integral (viz. it is an integral of the differential equations, and also a solution of the partial differential equation), but such dependent integrals give nothing new, and we require a third independent integral c, viz. we need this to express the threefold relation between the variables, given by the differential equations, and also to express the general solution $\phi(a, b, c)$ of the partial differential equation.

13. By what precedes the analytical condition, in order that the integrals a, b, c may be independent, is that they are such as not to satisfy the relations

$$\frac{d(a, b, c)}{d(x, y, z, w)} = 0.$$

14. We moreover see *à posteriori*, that there cannot be more than three independent integrals; in fact, if a, b, c, d are integrals, then, considering them as solutions of the partial differential equation, we have four equations which by the elimination therefrom of X, Y, Z, W, give

$$\frac{d(a, b, c, d)}{d(x, y, z, w)} = 0,$$

and this is the very equation which expresses that a, b, c, d are not independent.

15. The notion of the integrals may be arrived at somewhat differently thus: take a, b, c, d any functions of the variables, and write

$$A = X\frac{da}{dx} + Y\frac{da}{dy} + Z\frac{da}{dz} + W\frac{da}{dw},$$

and the like for B, C, D; then replacing the original variables x, y, z, w by the new variables a, b, c, d, the differential equations become

$$\frac{da}{A} = \frac{db}{B} = \frac{dc}{C} = \frac{dd}{D},$$

where A, B, C, D are to be (by means of the given values of a, b, c, d as functions of x, y, z, w) expressed as functions of a, b, c, d. If then $A = 0$, $B = 0$, $C = 0$, the differential equations become

$$\frac{da}{0} = \frac{db}{0} = \frac{dc}{0} = \frac{dd}{D};$$

viz. we have $da = 0$, $db = 0$, $dc = 0$, and therefore $a = \text{const.}$, $b = \text{const.}$, $c = \text{const.}$, that is, we have the integrals a, b, c as before.

16. There is no general process for obtaining an integral a of the differential equations. Supposing such integral known, we can introduce it as a variable, in place of one of the original variables, say w, viz. we thus reduce the system to

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{da}{0},$$

where X, Y, Z now denote the values assumed by these functions upon expressing therein w as a function of x, y, z, a, viz. they are now functions of x, y, z, a. The system thus breaks up into $da=0$ and the system

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

in which last (by virtue of the first equation, or $a=\text{const.}$) a is to be regarded as a constant; the original system of three equations between four variables is thus reduced to a system of two equations between three variables. Supposing b to be an integral of this reduced system, b is given as a function of x, y, z, a, but upon substituting herein for a its value as a function of x, y, z, w, we have b a function of the original variables x, y, z, w, and b is then a second integral of the original system.

17. In like manner supposing a and b to be known, we reduce the system to the single equation

$$\frac{dx}{X}=\frac{dy}{Y},$$

where X, Y are now functions of x, y, a, b; supposing an integral hereof to be c, we have c a function of x, y, a, b; but upon substituting herein for a, b their values as functions of x, y, z, w, we have c a function of x, y, z, w, and as such it is the third integral of the original system.

18. It may be remarked that if, to the original system, we join on an equality $=dt$, viz. if we consider the system

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}=\frac{dw}{W}(=dt),$$

where X, Y, Z, W are as before functions of the variables (x, y, z, w), then the integrals a, b, c of the original system being known, we can by means of them express for instance X as a function of x, a, b, c, and we have then, $\text{const.}=t-\int\frac{dx}{X}$, where the integration is to be performed regarding a, b, c as constants; writing $\int\frac{dx}{X}=\tau$, but after the integration replacing a, b, c by their values as functions of x, y, z, w, we have τ a function of x, y, z, w; and we say that $t-\tau$ is an integral; putting it $=\text{const.}$ we use also τ to denote the constant value of the integral $t-\int\frac{dx}{X}$ in question. Observe that here, the integrals a, b, c being known, the last integral $t-\tau$ is obtained by a quadrature.

19. The result would have been similar, if the adjoined equality had been $=\frac{dt}{T}$ (T a function of x, y, z, w), but in reference to subsequent matter, I retain the equality $=dt$, and adjoin a second equality $=\frac{dV}{\Omega}$ (Ω a function of x, y, z, w); we have then the integral $t-\tau$ as before, and another integral $V-\int\frac{\Omega dx}{X}$, where Ω, X are first expressed as functions of x, a, b, c, but after the integration a, b, c are replaced by their values as functions of (x, y, z, w), say this is the integral $V-\lambda$; this, when the integrals a, b, c are known, is (like $t-\tau$) obtained by a quadrature.

20. Attending only to the adjoined equality $=dt$, we can by means of the four integrals express each of the variables x, y, z, w as a function of a, b, c, $t-\tau$; viz. these four equations, regarding therein $t-\tau$ as a variable parameter, are in fact equivalent to the equations $a=$ const., $b=$ const., $c=$ const., which connect the variables x, y, z, w with the integrals a, b, c regarded as constants.

21. All that precedes is of course applicable to a system of $n-1$ equations between n variables, the number of independent integrals being $=n-1$.

22. I take an example with the three variables x, y, z; the differential equations being

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)},$$

and therefore the partial differential equation

$$x(y-z)\frac{d\theta}{dx}+y(z-x)\frac{d\theta}{dy}+z(x-y)\frac{d\theta}{dz}=0.$$

The integrals are $a=x+y+z$, $b=xyz$; and it will be shown how either of these integrals being known, the system is reduced to a single equation between two variables, say x, y.

First, a being known, $=x+y+z$ as before, we have

$$x(y-z)=x(x+2y-a),\quad y(z-x)=y(a-2x-y),$$

and the system is

$$\frac{dx}{x(x+2y-a)}=\frac{dy}{y(a-2x-y)},$$

which has the integral $b=xy(a-x-y)$; observe that this is a solution of the partial differential equation

$$x(x+2y-a)\frac{d\theta}{dx}+y(a-2x-y)\frac{d\theta}{dy}=0.$$

For a putting its value we find $b=xyz$.

Secondly, b being known, $= xyz$ as before, we have

$$x(y-z) = xy - \frac{y}{b},\ y(z-x) = \frac{b}{x} - xy,$$

and the system is

$$\frac{dx}{xy - \frac{b}{y}} = \frac{dy}{\frac{b}{x} - xy},$$

which has the integral $a = x + y + \frac{b}{xy}$; observe that this is a solution of the partial differential equation

$$\left(xy - \frac{b}{y}\right)\frac{d\theta}{dx} + \left(\frac{b}{x} - xy\right)\frac{d\theta}{dy} = 0.$$

For b putting its value, we find $a = x + y + z$.

The Multiplier. Art. Nos. 23 to 29.

23. First, if there are only two variables (x, y), the system consists of the single equation

$$\frac{dx}{X} = \frac{dy}{Y},$$

which may be written

$$Ydx - Xdy = 0.$$

Hence, if a be an integral, we have

$$\frac{da}{dx}dx + \frac{da}{dy}dy = 0;$$

the two will agree if there exists a function M such that

$$\frac{da}{dx} = MY,\ \frac{da}{dy} = -MX,$$

and thence, in virtue of the identity

$$\frac{d}{dy}\frac{da}{dx} = \frac{d}{dx}\frac{da}{dy},$$

we find

$$\frac{dMX}{dx} + \frac{dMY}{dy} = 0;$$

or, as this may also be written,

$$X\frac{dM}{dx} + Y\frac{dM}{dy} + M\left(\frac{dX}{dx} + \frac{dY}{dy}\right) = 0,$$

as the condition to determine the multiplier M. Supposing M known, we have $M(Ydx - Xdy) = da$, or say $a = \int M(Ydx - Xdy)$, viz. the integral a is determined by a quadrature.

24. In the case of three variables (x, y, z), the system is

$$\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z},$$

or, writing these in the form

$$Ydz-Zdy=0,\quad Zdx-Xdz=0,\quad Xdy-Ydx=0,$$

the course which immediately suggests itself is to seek for factors L, M, N, such that, a being an integral, we may have

$$L(Ydz-Zdy)+M(Zdx-Xdz)+N(Xdy-Ydx)=da,$$

but this does not lead to any result. The course taken by Jacobi is quite a different one: he, in fact, determines a multiplier M connected with *two* integrals a, b.

25. Supposing that a, b are independent integrals, we have

$$X\frac{da}{dx}+Y\frac{da}{dy}+Z\frac{da}{dz}=0,$$

$$X\frac{db}{dx}+Y\frac{db}{dy}+Z\frac{db}{dz}=0;$$

and determining from these equations the ratio of the quantities X, Y, Z, we may, it is clear, write

$$MX,\ MY,\ MZ=\frac{d(a,\ b)}{d(y,\ z)},\ \frac{d(a,\ b)}{d(z,\ x)},\ \frac{d(a,\ b)}{d(x,\ y)}.$$

It may be shown that we have identically

$$\frac{d}{dx}\frac{d(a,\ b)}{d(y,\ z)}+\frac{d}{dy}\frac{d(a,\ b)}{d(z,\ x)}+\frac{d}{dz}\frac{d(a,\ b)}{d(x,\ y)}=0,$$

and we thence deduce

$$\frac{d(MX)}{dx}+\frac{d(MY)}{dy}+\frac{d(MZ)}{dz}=0;$$

or, what is the same thing,

$$X\frac{dM}{dx}+Y\frac{dM}{dy}+Z\frac{dM}{dz}+M\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)=0,$$

as the condition for determining the multiplier M.

26. The use is as follows: supposing that M is known, and supposing also that one integral a of the system is known, we can then by a quadrature determine the other integral b. Thus, supposing that we know the integral $a, =a(x, y, z)$, we can by means of this integral express z in terms of x, y, a; and hence we may regard the unknown integral b as expressed in the like form, $b=b(x, y, a)$. The original values of $\frac{db}{dx}, \frac{db}{dy}, \frac{db}{dz}$ become on this supposition

$$\frac{db}{dx}+\frac{db}{da}\frac{da}{dx},\quad \frac{db}{dy}+\frac{db}{da}\frac{da}{dy},\quad \frac{db}{da}\frac{da}{dz},$$

and we thence find

$$\frac{d(a,\ b)}{d(y,\ z)},\quad \frac{d(a,\ b)}{d(z,\ x)},\quad \frac{d(a,\ b)}{d(x,\ y)} = -\frac{da}{dz}\frac{db}{dy},\quad \frac{da}{dz}\frac{db}{dx},\quad \frac{d(a,\ b)}{d(x,\ y)}.$$

We have therefore

$$MX,\ MY = -\frac{da}{dz}\frac{db}{dy},\quad \frac{da}{dz}\frac{db}{dx},$$

and, consequently,

$$db = \frac{db}{dx}dx + \frac{db}{dy}dy,\quad = \frac{M}{\frac{da}{dz}}(Ydx - Xdy);$$

viz. M, $\frac{da}{dz}$, Y, X being all of them expressed as functions of x, y, a, the expression on the right-hand is a complete differential, and we have

$$b = \int \frac{M}{\frac{da}{dz}}(Ydx - Xdy);$$

that is, the integral b is determined by a quadrature.

27. Thus, in the example No. 22,

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

and a value of the multiplier is $=1$. Supposing that the given integral is $a = x + y + z$, then $\frac{da}{dz} = 1$, and we have accordingly 1 as the multiplier of the equation

$$y(a - 2x - y)\,dx + x(a - x - 2y)\,dy = 0,$$

viz. this equation is integrable *per se*. Supposing the given integral to be $b = xyz$, then $\frac{db}{dz} = xy$, viz. we have $\frac{1}{xy}$ as the multiplier of the equation

$$\left(\frac{b}{x} - xy\right)dx + \left(\frac{b}{y} - xy\right)dy = 0,$$

and we thus in each case obtain the other integral as before.

28. The foregoing result may be presented in a more symmetrical form by taking in place of x, y any two variables $u = u(x,\ y,\ z)$, $v = v(x,\ y,\ z)$.

Supposing the integral a known as before, the system then is

$$\frac{du}{U} = \frac{dv}{V} = \frac{da}{0},$$

where $U,\ V = X\frac{du}{dx} + Y\frac{du}{dy} + Z\frac{du}{dz},\ X\frac{dv}{dx} + Y\frac{dv}{dy} + Z\frac{dv}{dz}$, these being expressed as functions of u, v, a; or, what is the same thing, we have $Vdu - Udv = 0$, a being in this equation regarded as a constant.

From the foregoing values of MX, MY, MZ, we deduce

$$MU,\ MV = \frac{d(u,\ a,\ b)}{d(x,\ y,\ z)},\quad \frac{d(v,\ a,\ b)}{d(x,\ y,\ z)}.$$

But forming the values of du, dv, da, db, we have an equation, determinant $= 0$, which equation may be written

$$du\frac{d(v,\ a,\ b)}{d(x,\ y,\ z)} - dv\frac{d(a,\ b,\ u)}{d(x,\ y,\ z)} + da\frac{d(b,\ u,\ v)}{d(x,\ y,\ z)} - db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)} = 0\,;$$

or, writing herein $da = 0$, this is

$$du\frac{d(v,\ a,\ b)}{d(x,\ y,\ z)} - dv\frac{d(u,\ a,\ b)}{d(x,\ y,\ z)} - db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)} = 0,$$

viz. this is

$$M(Vdu - Udv) = db\frac{d(u,\ v,\ a)}{d(x,\ y,\ z)},$$

or say

$$b = \int\left\{M \div \frac{d(u,\ v,\ a)}{d(x,\ y,\ z)}\right\}(Vdu - Udv),$$

where, on the right-hand side, everything must be expressed in terms of u, v, a. It thus appears that on expressing the final equation as a relation $Vdu - Udv = 0$ between the variables u and v, the multiplier hereof is $M \div \dfrac{d(u,\ v,\ a)}{d(x,\ y,\ z)}$. If $u,\ v = x,\ y$, this agrees with a foregoing result.

29. The theory is precisely the same for any number of variables. Thus, if there are four variables x, y, z, w, we have

$$MX,\ MY,\ MZ,\ MW = \frac{d(a,\ b,\ c)}{d(y,\ z,\ w)},\quad -\frac{d(a,\ b,\ c)}{d(z,\ w,\ x)},\quad \frac{d(a,\ b,\ c)}{d(w,\ x,\ y)},\quad -\frac{d(a,\ b,\ c)}{d(x,\ y,\ z)};$$

and, we have between the functions on the right-hand an identical relation, in virtue of which

$$\frac{d(MX)}{dx} + \frac{d(MY)}{dy} + \frac{d(MZ)}{dz} + \frac{d(MW)}{dw} = 0\,;$$

then, supposing that a value of M is known, and also any two integrals a, b, and that by means of these the equation to be finally integrated is expressed as a relation $Vdu - Udv = 0$ between any two variables u and v, the multiplier of this is

$$= M \div \frac{d(u,\ v,\ a,\ b)}{d(x,\ y,\ z,\ w)},$$

where U, V and this multiplier are to be expressed in terms of u, v, a, b.

The general result is that, given a value of the multiplier, and also all but one of the integrals, the final integral is expressible by a quadrature.

Pfaffian Theorem. Art. No. 30.

30. According as

the variables are	we have
x,	$Xdx = du$,
x, y,	$Xdx + Ydy = \lambda du$,
x, y, z,	$Xdx + Ydy + Zdz = \lambda du + dv$,
x, y, z, w,	$Xdx + Ydy + Zdz + Wdw = \lambda du + \mu dv$,

and so on; viz. the theorem is that, taking for instance two variables, a given lineo-differential $Xdx + Ydy$ is $= \lambda du$, that is, there exist λ, u functions of x, y, which verify this identity, or, what is the same thing, such that we have

$$X,\ Y = \lambda \frac{du}{dx},\ \lambda \frac{du}{dy};$$

and so, in the case of three variables, there exist λ, u, v functions of x, y, z, such that

$$X,\ Y,\ Z = \lambda \frac{du}{dx} + \frac{dv}{dx},\quad \lambda \frac{du}{dy} + \frac{dv}{dy},\quad \lambda \frac{du}{dz} + \frac{dv}{dz}.$$

The problem of determining the functions on the right-hand side is known as the Pfaffian Problem; this I do not at present consider, but only assume that there exist such functions.

The Hamiltonian System, its derivation from the general System. Art. Nos. 31 to 34.

31. Considering a bipartite set $(x, y, z: p, q, r)$, the general system of differential equations may be written

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{dp}{-X} = \frac{dq}{-Y} = \frac{dr}{-Z}.$$

But by the Pfaffian theorem we may write

$$Xdx + Ydy + Zdz + Pdp + Qdq + Rdr = \xi d\rho + \eta d\sigma + \zeta d\tau,$$

viz. there exist $\xi, \eta, \zeta, \rho, \sigma, \tau$ functions of the variables x, y, z, p, q, r, such that we have

$$X = \xi \frac{d\rho}{dx} + \eta \frac{d\sigma}{dx} + \zeta \frac{d\tau}{dx}, \ldots,\quad P = \xi \frac{d\rho}{dp} + \eta \frac{d\sigma}{dp} + \zeta \frac{d\tau}{dp}, \ldots,$$

and we have the foregoing general system expressed by means of these given functions $\xi, \eta, \zeta, \rho, \sigma, \tau$ of the variables.

32. But the lineo-differential

$$Xdx + Ydy + Zdz + Pdp + Qdq + Rdr$$

may be of a more special form; for instance, it may be a sum of two terms $=\xi d\rho+\eta d\sigma$: or, finally, it may be a single term $=\xi d\rho$, and in this case we have the Hamiltonian system, viz. writing H in place of ρ, if we have

$$Xdx+Ydy+Zdz+Pdp+Qdq+Rdr=\xi dH,$$

where H is a given function of the variables, then the system is

$$\frac{dx}{\frac{dH}{dp}}=\frac{dy}{\frac{dH}{dq}}=\frac{dz}{\frac{dH}{dr}}=\frac{dp}{-\frac{dH}{dx}}=\frac{dq}{-\frac{dH}{dy}}=\frac{dr}{-\frac{dH}{dz}},$$

which is the system in question.

33. Any integral a of the system is a solution of

$$\frac{dH}{dp}\frac{d\theta}{dx}+\frac{dH}{dq}\frac{d\theta}{dy}+\frac{dH}{dr}\frac{d\theta}{dz}-\frac{dH}{dx}\frac{d\theta}{dp}-\frac{dH}{dy}\frac{d\theta}{dq}-\frac{dH}{dz}\frac{d\theta}{dr}=0;$$

viz. writing, as above,

$$(H,\ \theta)=\frac{d(H,\ \theta)}{d(p,\ x)}+\frac{d(H,\ \theta)}{d(q,\ y)}+\frac{d(H,\ \theta)}{d(r,\ z)}$$

$$=\text{last-mentioned expression},$$

the partial differential equation is $(H,\ \theta)=0$; and, conversely, any solution of this equation is an integral of the differential equations.

34. It is obvious that a solution of $(H,\ \theta)=0$ is H; hence the entire system of independent solutions may be taken to be $H,\ a,\ b,\ c,\ d$; or, if we choose to consider a set of five independent solutions $a,\ b,\ c,\ d,\ e$, then we have $H=H(a,\ b,\ c,\ d,\ e)$ a function of these solutions.

An Identity in regard to the Functions $(H,\ \theta)$. Art. Nos. 35 and 36.

35. Taking the variables to be $(x,\ y,\ z,\ p,\ q,\ r)$, and $H,\ a,\ b$ to be any functions of these variables, we have the identity

$$(H,\ (a,\ b))+(a,\ (b,\ H))+(b,\ (H,\ a))=0,$$

which is now to be proved. For this purpose we write it in the slightly different form

$$((a,\ b),\ H)=(a,\ (b,\ H))-(b,\ (a,\ H)).$$

The first term on the right-hand side is

$$\left(\frac{da}{dp}\frac{d}{dx}+\frac{da}{dq}\frac{d}{dy}+\frac{da}{dr}\frac{d}{dz}-\frac{da}{dx}\frac{d}{dp}-\frac{da}{dy}\frac{d}{dq}-\frac{da}{dz}\frac{d}{dr}\right)$$

operating upon

$$\left(\frac{db}{dp}\frac{dH}{dx}+\frac{db}{dq}\frac{dH}{dy}+\frac{db}{dr}\frac{dH}{dz}-\frac{db}{dx}\frac{dH}{dp}-\frac{db}{dy}\frac{dH}{dq}-\frac{db}{dz}\frac{dH}{dr}\right);$$

and if we herein attend to the terms which contain the second differential coefficients of H, these are symmetrical functions of a, b. For instance,

$$\begin{array}{lll} \dfrac{d^2H}{dx^2}, & \text{coefficient is} & \dfrac{da}{dp}\dfrac{db}{dp}, \\ \dfrac{d^2H}{dx\,dy} & \text{"} \quad \text{"} & \dfrac{da}{dp}\dfrac{db}{dq}+\dfrac{da}{dq}\dfrac{db}{dp}, \\ \dfrac{d^2H}{dx\,dp} & \text{"} \quad \text{"} & -\dfrac{da}{dp}\dfrac{db}{dx}-\dfrac{da}{dx}\dfrac{db}{dp}, \\ \dfrac{d^2H}{dx\,dq} & \text{"} \quad \text{"} & -\dfrac{da}{dp}\dfrac{db}{dy}-\dfrac{da}{dy}\dfrac{db}{dp}. \end{array}$$

Hence, forming the like terms of the second terms $(b, (a, H))$ and subtracting, the terms in question all vanish: and we thus see that $(a, (b, H)) - (b, (a, H))$ is a linear function of the differential coefficients

$$\frac{dH}{dx},\ \frac{dH}{dy},\ \frac{dH}{dz},\ \frac{dH}{dp},\ \frac{dH}{dq},\ \frac{dH}{dr}.$$

36. Attending to any one of these, suppose $\dfrac{dH}{dx}$, the coefficient of this

$$\text{in } (a, (b, H)) \text{ is } = \left(a, \frac{db}{dp}\right)$$

$$\text{in } (b, (a, H)) \text{ "} \quad \left(b, \frac{da}{dp}\right), \quad = -\left(\frac{da}{dp}, b\right),$$

wherefore, in the difference of these, it is

$$\left(a, \frac{db}{dp}\right) + \left(\frac{da}{dp}, b\right), \quad = \frac{d}{dp}(a, b).$$

Hence, for the several terms

$$\frac{dH}{dx},\ \frac{dH}{dy},\ \frac{dH}{dz},\ \frac{dH}{dp},\ \frac{dH}{dq},\ \frac{dH}{dr},$$

the coefficients are

$$\left(\frac{d}{dp},\ \frac{d}{dq},\ \frac{d}{dr},\ -\frac{d}{dx},\ -\frac{d}{dy},\ -\frac{d}{dz}\right)(a, b);$$

or, what is the same thing, we have

$$(a, (b, H)) - (b, (a, H)) = ((a, b), H),$$

the identity in question.

The Poisson-Jacobi Theorem. Art. Nos. 37 to 39.

37. The foregoing identity shows that if $(H, a) = 0$, and $(H, b) = 0$, then also $(H, (a, b)) = 0$; or, what is the same thing, if a and b are solutions of the partial differential equation $(H, \theta) = 0$, then also (a, b) is a solution; or, say, if a, b are integrals, then also (a, b) is an integral.

Supposing that the set is (x, y, z, p, q, r), so that there are in all five integrals a, b, c, d, e, then the theorem may be otherwise stated, we have (a, b) a function of the integrals a, b, c, d, e.

Observe that, knowing only the integrals a and b, we find (a, b) as a function of x, y, z, p, q, r, this may be $=0$, or a determinate constant, or it may be such a function that by virtue of the given values of a and b it reduces itself to a function of a and b; in any of these cases the theorem does *not* determine a new integral. But if contrariwise the value of (a, b), obtained as above as a function of the variables, is not a function of a, b, then it is a new integral which may be called c.

38. To obtain in this way a new integral, we require two integrals a, b other than H; for knowing only the integrals a, H, the theorem gives only (a, H) an integral, and we have of course $(a, H)=0$, viz. we do not obtain a new integral.

But starting from two integrals a, b other than H, we *may* obtain as above a new integral c; and then again (a, c) and (b, c) will be integrals, one or both of which may be new. And it may therefore happen that in this way we obtain all the independent integrals a, b, c, d, e; or the process may on the other hand terminate, without giving all the independent integrals.

The theory is obviously applicable throughout to the case of a bipartite set $(x, y, z, \ldots, p, q, r, \ldots)$ of $2n$ variables.

39. It may be remarked here that, in the Hamiltonian system, a value of the multiplier is $M=1$; and consequently, if in any way all but one of the integrals, that is, $2n-2$ integrals, be known, the remaining integral can be found by a quadrature.

It is further to be noticed that, if we adjoin a new variable t and a term $=dt$ to the system of equations; then the $2n-1$ integrals of the original system being known, all the original variables can be expressed in terms of the $2n-1$ integrals regarded as constants and of one of the variables say x: we then have

$$dt = dx \div \frac{dH}{dp},$$

or

$$t - \epsilon = \int dx \div \frac{dH}{dp},$$

or say

$$\epsilon = t - \int dx \div \frac{dH}{dp},$$

viz. if after the integration we suppose the $2n-1$ integrals replaced each of them by its value, we have

$$\epsilon = t - \phi(x, y, z, \ldots, p, q, r, \ldots),$$

which is the remaining or $2n$th integral of the original system as augmented by the term $=dt$.

The Poisson-Jacobi theorem peculiar to the Hamiltonian Form. Art. Nos. 40 to 45.

40. Taking for greater simplicity the set (x, y, p, q), and writing

$$Xdx + Ydy + Pdp + Qdq = \xi d\rho + \eta d\sigma,$$

then the general system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dp}{-X} = \frac{dq}{-Y},$$

becomes

$$\frac{dx}{\xi \frac{d\rho}{dp} + \eta \frac{d\sigma}{dp}} = \frac{dy}{\xi \frac{d\rho}{dq} + \eta \frac{d\sigma}{dq}} = \frac{dp}{-\left(\xi \frac{d\rho}{dx} + \eta \frac{d\sigma}{dx}\right)} = \frac{dq}{-\left(\xi \frac{d\rho}{dy} + \eta \frac{d\sigma}{dy}\right)};$$

and the corresponding partial differential equation (θ the independent variable) is

$$\xi (\rho, \theta) + \eta (\sigma, \theta) = 0,$$

where

$$(\rho, \theta) = \frac{d(\rho, \theta)}{d(p, x)} + \frac{d(\rho, \theta)}{d(q, y)}, \quad (\sigma, \theta) = \frac{d(\sigma, \theta)}{d(p, x)} + \frac{d(\sigma, \theta)}{d(q, y)}.$$

It is to be shown that if a, b are solutions, viz. if we have

$$\xi (\rho, a) + \eta (\sigma, a) = 0,$$
$$\xi (\rho, b) + \eta (\sigma, b) = 0,$$

implying of course

$$(\rho, a)(\sigma, b) - (\rho, b)(\sigma, a) = 0,$$

then it is *not* in general true that we have (a, b) a solution; that is, *not* in general true that

$$\xi (\rho, (a, b)) + \eta (\sigma, (a, b)) = 0;$$

the condition for the truth of this equation is in fact $\frac{\eta}{\xi} =$ a function of ρ, σ, but when this is so, $\xi d\rho + \eta d\sigma$ is λdH, viz. there exist λ, H functions of ρ, σ (and therefore ultimately of x, y, p, q) satisfying this equation, and the system is really Hamiltonian.

41. We consider whether it is true that

$$\xi (\rho, (a, b)) + \eta (\sigma, (a, b)) = 0.$$

We have identically

$$((a, b), \rho) + ((b, \rho), a) + ((\rho, a), b) = 0,$$
$$((a, b), \sigma) + ((b, \sigma), a) + ((\sigma, a), b) = 0,$$

so that multiplying by ξ, η, and adding, the equation in question is

$$\xi [((b, \rho), a) + ((\rho, a), b)] + [((b, \sigma), a) + ((\sigma, a), b)] = 0.$$

But in virtue of the equations satisfied by a, b, we may write

$$(\rho, a) = \quad l\eta, \ (b, \rho) = -(\rho, b) = -m\eta,$$
$$(\sigma, a) = -l\xi, \ (b, \sigma) = -(\sigma, b) = \quad m\xi,$$

where l, m are indeterminate functions of x, y, p, q; and the equation in question now becomes

$$\xi[-(m\eta,\ a)+(l\eta,\ b)]+\eta[(m\xi,\ a)-(l\xi,\ b)]=0;$$

that is,

$$\begin{aligned}&\xi[-m(\eta,\ a)-\eta(m,\ a)+l(\eta,\ b)+\eta(l,\ b)]\\+&\eta[\ \ m(\xi,\ a)+\xi(m,\ a)-l(\xi,\ b)-\xi(l,\ b)]=0;\end{aligned}$$

viz. omitting the terms which destroy each other, this is

$$-m\xi(\eta,\ a)+l\xi(\eta,\ b)+m\eta(\xi,\ a)-l\eta(\xi,\ b)=0.$$

Substituting for $m\xi$, &c., their values, we have

$$(\sigma,\ b)(\eta,\ a)-(\sigma,\ a)(\eta,\ b)+(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)=0;$$

and the question is whether this is implied in the equations

$$\xi(\rho,\ a)+\eta(\sigma,\ a)=0,$$
$$\xi(\rho,\ b)+\eta(\sigma,\ b)=0.$$

42. Write $\eta=\kappa\xi$, the equation in question is

$$(\sigma,\ b)(\kappa\xi,\ a)-(\sigma,\ a)(\kappa\xi,\ b)+(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)=0;$$

that is,

$$\left\{\begin{matrix}(\sigma,\ b)\,\kappa(\xi,\ a)+\xi(\sigma,\ b)(\kappa,\ a)\\-(\sigma,\ a)\,\kappa(\xi,\ b)-\xi(\sigma,\ a)(\kappa,\ b)\end{matrix}\right\}+\{(\rho,\ b)(\xi,\ a)-(\rho,\ a)(\xi,\ b)\}=0;$$

viz.

$$(\xi,\ a)[(\rho,\ b)+\kappa(\sigma,\ b)]-(\xi,\ b)[(\rho,\ a)+\kappa(\sigma,\ a)]+\xi[(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)]=0;$$

and we wish to see whether this is implied in

$$(\rho,\ a)+\kappa(\sigma,\ a)=0,$$
$$(\rho,\ b)+\kappa(\sigma,\ b)=0,$$

which give

$$(\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)=0;$$

or, what is the same thing, whether these last equations imply

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=0.$$

Suppose κ is a function of ρ, σ, then, as is at once seen,

$$(\kappa,\ a)=\frac{d\kappa}{d\rho}(\rho,\ a)+\frac{d\kappa}{d\sigma}(\sigma,\ a),$$

$$(\kappa,\ b)=\frac{d\kappa}{d\rho}(\rho,\ b)+\frac{d\kappa}{d\sigma}(\sigma,\ b),$$

and thence

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=\frac{d\kappa}{d\rho}[(\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)];$$

viz. κ being a function of ρ and σ, the two equations imply the third.

43. But we wish to prove the converse, viz. that, if the two equations imply the third, then κ is a function of ρ, σ.

Now the equations

$$(\sigma,\ b)(\kappa,\ a)-(\sigma,\ a)(\kappa,\ b)=0,\quad (\sigma,\ b)(\rho,\ a)-(\sigma,\ a)(\rho,\ b)=0,$$

are transformable into

$$\begin{array}{ll}\dfrac{d(\sigma,\ \kappa)}{d(p,\ x)}\dfrac{d(b,\ a)}{d(p,\ x)}=0, & \dfrac{d(\sigma,\ \rho)}{d(p,\ x)}\dfrac{d(b,\ a)}{d(p,\ x)}=0\\ +q,\ y, & +q,\ y,\\ +p,\ q, & +p,\ q,\\ +p,\ y, & +p,\ y,\\ +x,\ q, & +x,\ q,\\ +x,\ y, & +x,\ y,\end{array}$$

the lines after the first being the corresponding terms with q, y, &c. instead of p, x. And if independently of the values of a, b, one of these equations implies the other, we must have

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)},\ \frac{d(\sigma,\ \kappa)}{d(q,\ y)},\ \frac{d(\sigma,\ \kappa)}{d(p,\ q)},\ \frac{d(\sigma,\ \kappa)}{d(p,\ y)},\ \frac{d(\sigma,\ \kappa)}{d(x,\ q)},\ \frac{d(\sigma,\ \kappa)}{d(x,\ y)},$$

proportional to the like expressions with σ, ρ instead of σ, κ; say these are

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ x)},\ \text{\&c.}$$

44. Assume κ a function of ρ, σ, x, y; we have

$$\frac{d(\sigma,\ \kappa)}{d(p,\ x)}=\frac{d\sigma}{dp}\left(\frac{d\kappa}{d\rho}\frac{d\rho}{dx}+\frac{d\kappa}{d\sigma}\frac{d\sigma}{dx}+\frac{d\kappa}{dx}\right)-\frac{d\sigma}{dx}\left(\frac{d\kappa}{d\rho}\frac{d\rho}{dp}+\frac{d\kappa}{d\sigma}\frac{d\sigma}{dp}\right)=\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dp},\ \text{\&c.};$$

the equations thus become

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dp}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ x)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(q,\ y)}+\frac{d\kappa}{dy}\frac{d\sigma}{dq}=\Lambda\frac{d(\sigma,\ \rho)}{d(q,\ y)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ q)}+\quad 0\quad=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ q)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(p,\ y)}+\frac{d\kappa}{dy}\frac{d\sigma}{dp}=\Lambda\frac{d(\sigma,\ \rho)}{d(p,\ y)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(q,\ x)}+\frac{d\kappa}{dx}\frac{d\sigma}{dq}=\Lambda\frac{d(\sigma,\ \rho)}{d(q,\ x)},$$

$$\frac{d\kappa}{d\rho}\frac{d(\sigma,\ \rho)}{d(x,\ y)}-\frac{d(\sigma,\kappa)}{d(x,y)}=\Lambda\frac{d(\sigma,\ \rho)}{d(x,\ y)}.$$

Hence, unless $\frac{d(\sigma, \rho)}{d(p, q)} = 0$, we have $\Lambda = \frac{d\kappa}{d\rho}$. The remaining five equations then are

$$\frac{d\kappa}{dx}\frac{d\sigma}{dp} = 0, \quad \frac{d\kappa}{dy}\frac{d\sigma}{dq} = 0,$$

$$\frac{d\kappa}{dy}\frac{d\sigma}{dp} = 0, \quad \frac{d\kappa}{dx}\frac{d\sigma}{dq} = 0, \quad \frac{d\sigma}{dx}\frac{d\kappa}{dy} - \frac{d\sigma}{dy}\frac{d\kappa}{dx} = 0,$$

which give, and are satisfied by $\frac{d\kappa}{dx} = 0$, $\frac{d\kappa}{dy} = 0$, viz. we then have κ a function of ρ, σ without x, y which is the theorem in question.

45. The proof fails if $\frac{d(\sigma, \rho)}{d(p, q)} = 0$. But here, unless also $\frac{d(\sigma, \rho)}{d(x, y)} = 0$, we can, by assuming in the first instance κ a function of ρ, σ, p, q, prove in like manner that κ is a function of only ρ and σ; if however we have as well $\frac{d(\sigma, \rho)}{d(p, q)} = 0$ and $\frac{d(\sigma, \rho)}{d(x, y)} = 0$, the last-mentioned process would also fail, but it can be shown the conclusion holds good in this case also; hence the conclusion that the Poisson-Jacobi theorem holds good only for a Hamiltonian system.

Conjugate Integrals of the Hamiltonian System. Art. Nos. 46 to 51.

46. For greater clearness, let $n = 4$, or let the variables be x, y, z, w, p, q, r, s; the system of differential equations therefore is

$$\frac{dx}{\frac{dH}{dp}} = \frac{dy}{\frac{dH}{dq}} = \frac{dz}{\frac{dH}{dr}} = \frac{dw}{\frac{dH}{ds}} = \frac{dp}{-\frac{dH}{dx}} = \frac{dq}{-\frac{dH}{dy}} = \frac{dr}{-\frac{dH}{dz}} = \frac{ds}{-\frac{dH}{dw}},$$

and any integral hereof is as before a solution of $(H, \theta) = 0$. Assume that the integrals are H, a, b, c, d, e, f, so that

$$(H, a) = 0, \quad (H, b) = 0, \quad (H, c) = 0, \quad (H, d) = 0, \quad (H, e) = 0, \quad (H, f) = 0.$$

Considering here a as denoting any integral whatever, that is, any solution whatever of the partial differential equation $(H, \theta) = 0$, it is to be shown that it is possible to determine θ so as to satisfy as well this equation $(H, \theta) = 0$, as also the new equation $(a, \theta) = 0$.

47. We, in fact, satisfy the first equation by taking

$$\theta, = \theta(H, a, b, c, d, e, f),$$

any function whatever of the seven integrals. But, θ having this value, we find

$$(a, \theta) = (a, H)\frac{d\theta}{dH} + (a, a)\frac{d\theta}{da} + (a, b)\frac{d\theta}{db} + (a, c)\frac{d\theta}{dc} + (a, d)\frac{d\theta}{dd} + (a, e)\frac{d\theta}{de} + (a, f)\frac{d\theta}{df};$$

or, since the first two terms on the right-hand vanish, the equation $(a, \theta)=0$ thus becomes

$$(a, b)\frac{d\theta}{db}+(a, c)\frac{d\theta}{dc}+(a, d)\frac{d\theta}{dd}+(a, e)\frac{d\theta}{de}+(a, f)\frac{d\theta}{df}=0.$$

But by the Poisson-Jacobi theorem (a, b), &c., are each of them a solution of $(H, \theta)=0$, viz. they are each of them a function of H, a, b, c, d, e, f. This is then a linear partial differential equation wherein the variables are H, a, b, c, d, e, f; or, since there are no terms in $\frac{d\theta}{dH}, \frac{d\theta}{da}$, we may regard a, H as constants, and treat it as a linear partial differential equation in b, c, d, e, f, the solutions of the equation being in fact the integrals, or any functions of the integrals, of

$$\frac{db}{(a, b)}=\frac{dc}{(a, c)}=\frac{dd}{(a, d)}=\frac{de}{(a, e)}=\frac{df}{(a, f)}.$$

48. Suppose any four integrals are b', c', d', e', so that a general integral is $\phi(H, a, b', c', d', e')$, then b', c', d', e' quà functions of H, a, b, c, d, e, f are integrals of the original equation $(H, \theta)=0$; hence *changing the notation* and writing b, c, d, e in place of these accented letters we have (H, a, b, c, d, e) as solutions of the two equations $(H, \theta)=0$, $(a, \theta)=0$; viz. a being any integral of the first of these equations, we see how to find four other integrals (b, c, d, e) which are such that

$$\begin{gathered}(H, a)=0,\quad (H, b)=0,\quad (H, c)=0,\quad (H, d)=0,\quad (H, e)=0,\\ (a, b)=0,\quad (a, c)=0,\quad (a, d)=0,\quad (a, e)=0.\end{gathered}$$

49. We proceed in the same course and endeavour to find θ, so that not only $(H, \theta)=0$, $(a, \theta)=0$, but also $(b, \theta)=0$. Assuming here $\theta=\theta(H, a, b, c, d, e)$ an arbitrary function of the integrals, the first and second equations are satisfied; for the third equation, we have

$$(b, \theta)=(b, H)\frac{d\theta}{dH}+(b, a)\frac{d\theta}{da}+(b, b)\frac{d\theta}{db}+(b, c)\frac{d\theta}{dc}+(b, d)\frac{d\theta}{dd}+(b, e)\frac{d\theta}{de};$$

viz. the first three terms here vanish, and the equation $(b, \theta)=0$ becomes

$$(b, c)\frac{d\theta}{dc}+(b, d)\frac{d\theta}{dd}+(b, e)\frac{d\theta}{de}=0,$$

where, b, c, d, e being solutions as well of $(H, \theta)=0$ as of $(a, \theta)=0$, we have (b, c) a solution of these two equations, and as such a function of H, a, b, c, d, e; and so (b, d) and (b, e) are each of them a function of the same variables. The above is therefore a linear partial differential equation wherein the variables are H, a, b, c, d, e, but as the equation does not contain $\frac{d\theta}{dH}, \frac{d\theta}{da}$, or $\frac{d\theta}{db}$, we may regard H, a, b as constants; and the solutions of the equation are, in fact, the integrals of

$$\frac{dc}{(b, c)}=\frac{dd}{(b, d)}=\frac{de}{(b, e)}.$$

50. Supposing that any two integrals are c', d', so that a general integral is $\phi(H, a, b, c', d')$, then c', d' quà functions of H, a, b, c, d, e are integrals of the former equations $(H, \theta)=0$, $(a, \theta)=0$, so that again *changing the notation*, and writing c, d instead of the accented letters, we have (H, a, b, c, d) as solutions of the three equations $(H, \theta)=0$, $(a, \theta)=0$, $(b, \theta)=0$, viz. a being any solution of the first equation, and b any solution of the first and second equations, we see how to find two others c, d, of the same two equations, which are such that

$$\begin{aligned}(H, a)=0,\quad (H, b)=0,\quad &(H, c)=0,\quad (H, d)=0,\\ (a, b)=0,\quad &(a, c)=0,\quad (a, d)=0,\\ &(b, c)=0,\quad (b, d)=0;\end{aligned}$$

or, attending only to the integrals H, a, b, c, these are integrals of the equations $(H, \theta)=0$, $(a, \theta)=0$, $(b, \theta)=0$, such that

$$(H, a)=0,\quad (H, b)=0,\quad (H, c)=0,\quad (a, b)=0,\quad (a, c)=0,\quad (b, c)=0.$$

We here say that H, a, b, c are a system of conjugate solutions. Attempting to continue the process, it would appear that there is not any new independent integral d, such that $(H, d)=0$, $(a, d)=0$, $(b, d)=0$, $(c, d)=0$ (the first three of these are satisfied by the integral d found above, but the last of them is not); we may, however, taking d an arbitrary function of H, a, b, c, replace H by d; viz. we thus have the four integrals a, b, c, d, such that

$$(a, b)=0,\quad (a, c)=0,\quad (a, d)=0,\quad (b, c)=0,\quad (b, d)=0,\quad (c, d)=0,$$

and which are consequently said to form a conjugate system.

51. The process is of course general, and it shows how, in the case of a Hamiltonian system of $2n$ variables, it is possible to find a system $H, a, b, \dots, f$ consisting of H and $n-1$ other integrals, or, if we please, a system of n integrals $a, b, \dots, f, g$, such that the derivative of any two integrals whatever of the system is $=0$; any such system is termed a conjugate system.

Hamiltonian System—the function V. Art. Nos. 52 to 58.

52. Taking a Hamiltonian system with the original variables x, y, z, p, q, r, we adjoin the two new variables t, V, forming the extended system

$$\frac{dx}{\frac{dH}{dp}}=\frac{dy}{\frac{dH}{dq}}=\frac{dz}{\frac{dH}{dr}}=\frac{dp}{-\frac{dH}{dx}}=\frac{dq}{-\frac{dH}{dy}}=\frac{dr}{-\frac{dH}{dz}}=dt=\frac{dV}{p\frac{dH}{dp}+q\frac{dH}{dq}+r\frac{dH}{dr}}.$$

Supposing the integrals of the original system to be a, b, c, d, e, we have $H=H(a, b, c, d, e)$ a determinate function of these integrals; also an integral $\tau=t-\phi(x, y, z, p, q, r)$ and an integral $\lambda=V-\psi(x, y, z, p, q, r)$; these integrals, exclusive of the last of them, serve to express x, y, z, p, q, r as functions of $a, b, c, d, e, t-\tau$; and the last integral then gives $V=\lambda+$ a function of the last-mentioned quantities.

53. We consider the differential expression

$$dV - p\,dx - q\,dy - r\,dz,$$

which, treating the integrals as constants, that is, in the expressions of V, x, y, z, regarding t as the only variable, is at once seen to be $=0$; hence, if we regard all the integrals as variables, the value is

$$= d\lambda + A\,da + B\,db + C\,dc + D\,dd + E\,de,$$

without any term in $d\tau$, since this enters originally in the form $dt - d\tau$, and therefore disappears with dt.

The coefficients A, B, C, D, E are of course functions of a, b, c, d, e, $t-\tau$; it is to be shown that they contain $t-\tau$ linearly, viz. that in these coefficients respectively the coefficients of $t-\tau$ are

$$\frac{dH}{da},\ \frac{dH}{db},\ \frac{dH}{dc},\ \frac{dH}{dd},\ \frac{dH}{de},$$

where H is expressed as above in the form $H(a, b, c, d, e)$; this being so, the entire term in $t-\tau$ will be $(t-\tau)\,dH$; each coefficient, for instance A, has besides a part A', which is a function of a, b, c, d, e without $t-\tau$, or *changing the notation* and writing the unaccented letters to denote these parts of the original coefficients, the final result is

$$dV - p\,dx - q\,dy - r\,dz = (t-\tau)\,dH + d\lambda + A\,da + B\,db + C\,dc + D\,dd + E\,de,$$

where H stands for its value $H(a, b, c, d, e)$, and A, B, C, D, E are functions of a, b, c, d, e without $t-\tau$.

54. To prove the theorem, we have

$$A = \frac{dV}{da} - p\frac{dx}{da} - q\frac{dy}{da} - r\frac{dz}{da},$$

and thence

$$\frac{dA}{dt} = \frac{d^2V}{da\,dt} - \frac{dp}{dt}\frac{dx}{da} - \frac{dq}{dt}\frac{dy}{da} - \frac{dr}{dt}\frac{dz}{da} - p\frac{d^2x}{da\,dt} - q\frac{d^2y}{da\,dt} - r\frac{d^2z}{da\,dt}$$

$$= \frac{d}{da}\left\{\frac{dV}{dt} - p\frac{dx}{dt} - q\frac{dy}{dt} - r\frac{dz}{dt}\right\}$$

$$+ \frac{dp}{da}\frac{dx}{dt} + \frac{dq}{da}\frac{dy}{dt} + \frac{dr}{da}\frac{dz}{dt} - \frac{dp}{dt}\frac{dx}{da} - \frac{dq}{dt}\frac{dy}{da} - \frac{dr}{dt}\frac{dz}{da};$$

and then substituting for $\frac{dV}{dt}$, &c., their values from the system of differential equations, the first line vanishes, and the second line becomes

$$= \frac{dH}{dp}\frac{dp}{da} + \frac{dH}{dq}\frac{dq}{da} + \frac{dH}{dr}\frac{dr}{da} + \frac{dH}{dx}\frac{dx}{da} + \frac{dH}{dy}\frac{dy}{da} + \frac{dH}{dz}\frac{dz}{da},$$

$$= \frac{dH}{da};$$

and hence $A = (t-\tau)\dfrac{dH}{da} + A'$, and the like for the other coefficients B, C, D, E, which is the theorem in question.

55. We may have between two coefficients of the formula, for instance, D and E, a relation $\dfrac{dD}{de} = \dfrac{dE}{dd}$, and I will for the present assume, without proving it, the theorem that if a, b, c are conjugate integrals, then this relation $\dfrac{dD}{de} - \dfrac{dE}{dd} = 0$, holds good, merely mentioning that the proof depends on the consideration of certain symbols $[a, b]$, which are the converses, so to speak, of the symbols (a, b), viz. considering the variables x, y, z, p, q, r as given functions of $a, b, c, d, e, t-\tau$, then we have

$$[a, b] = \frac{d(p, x)}{d(a, b)} + \frac{d(q, y)}{d(a, b)} + \frac{d(r, z)}{d(a, b)}.$$

The assumption is used only in the two following Nos. 56 and 57.

56. Supposing then that a, b, c are conjugate integrals, we have $\dfrac{dD}{de} - \dfrac{dE}{dd} = 0$, and there exists therefore ϕ, a function of a, b, c, d, e, such that

$$d\phi = A'\,da + B'\,db + C'\,dc + D\,dd + E\,de,$$

(A', B', C' functions of the same quantities a, b, c, d, e), we have therefore

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + d\phi + (A - A')\,da + (B - B')\,db + (C - C')\,dc.$$

Taking as above a, b conjugate integrals $(a, b) = 0$, and c any function whatever of a, b, H, then a, b, c are conjugate integrals, and the formula holds good. Suppose further that a, b, H are absolute constants, then $dH = 0$, $da = 0$, $db = 0$, $dc = 0$, and the formula becomes

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + d\phi;$$

or, writing this under the form,

$$p\,dx + q\,dy + r\,dz = dV - d\lambda - d\phi,$$

it follows that $p\,dx + q\,dy + r\,dz$ is an exact differential, a theorem which may be stated as follows: viz. if a, b are conjugate integrals of the Hamiltonian system, and if from the equations $H = \text{const.}$, $a = \text{const.}$, $b = \text{const.}$, we express p, q, r as functions of x, y, z, then $p\,dx + q\,dy + r\,dz$ is an exact differential; or, what is the same thing, p, q, r are the differential coefficients $\dfrac{dU}{dx}, \dfrac{dU}{dy}, \dfrac{dU}{dz}$ of U a function of x, y, z. This is, in fact, a fundamental theorem in regard to the partial differential equation $H = \text{const.}$, and it will presently be proved in a different manner.

57. If, as before, a and b are conjugate integrals, then, writing as we may do λ in place of $\lambda + \phi$, and finding V as a function of x, y, z, a, b, H from the equation

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz),$$

and again treating a, b, H as variable, we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + A da + B db,$$

where A, B are functions of the integrals a, b, c, d, e, that is, they are themselves integrals, which may be taken for the integrals d, e, or we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + d\,da + e\,db;$$

we have therefore

$$\frac{dV}{da} = d, \quad \frac{dV}{db} = e,$$

equations which, on substituting therein for a, b, H their values as functions of x, y, z, p, q, r, determine the integrals d, e, which with a, b, H or a, b, c, are the remaining integrals of the Hamiltonian system; and further

$$\frac{dV}{dH} = t - \tau,$$

which, when in like manner, we substitute therein for a, b, H, their values as functions of x, y, z, p, q, r, determines τ, the remaining integral of the system as increased by the equality $= dt$.

58. Reverting to the general theorem No. 52, let x_0, y_0, z_0, p_0, q_0, r_0, t_0 be corresponding values of the variables x, y, z, p, q, r, t; and let a_0, &c., ..., V_0 be the same functions of x_0, y_0, z_0, p_0, q_0, r_0, t_0 that a, &c., ..., V are of the variables; we have $a = a_0, \ldots, e = e_0$, and corresponding to the equation

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + A da + \ldots + E de,$$

the like equation

$$dV_0 - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0 = d\lambda + (t_0 - \tau)\,dH + A da + \ldots + E de.$$

Hence, subtracting

$$dV - dV_0 = (t - t_0)\,dH + p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0,$$

or, considering only H as an absolute constant,

$$dV - dV_0 = \qquad p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0;$$

viz. if from the equations $H = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$, we express p, q, r, p_0, q_0, r_0 as functions of x, y, z, x_0, y_0, z_0, H, then

$$p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0$$

will be an exact differential. And in particular regarding x_0, y_0, z_0 as constants, then $p\,dx + q\,dy + r\,dz$ is an exact differential, viz. there exists a function

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz).$$

We have thus again arrived at a solution of the partial differential equation $H = \text{const.}$

The Partial Differential Equation $H = const.$ Art. Nos. 59 to 70.

59. In what just precedes we have, in fact, brought the theory of the Hamiltonian system into connexion with a partial differential equation, viz. we have determined the variables p, q, r as functions of x, y, z such that $p\,dx + q\,dy + r\,dz$ is an exact differential $= dV$; but we now consider the subject in a more regular manner.

The partial differential equation is $H = \text{const.}$ viz. here H denotes, in the first instance, a given function of p, q, r, x, y, z, where p, q, r are the differential coefficients of a function V of x, y, z, or, what is the same thing, there exists a function V of x, y, z such that $p\,dx + q\,dy + r\,dz = dV$; and then, this function H being constant, we use the same letter H to denote the constant value of the function. The equation $H = \text{const.}$ is the most general form of a partial differential equation of the first order which contains the independent variable only through its differential coefficients p, q, r, and it is for convenience put in a form containing the arbitrary constant H, which constant might without loss of generality be put $= 0$ or $=$ any other determinate value.

60. We seek to determine p, q, r as functions of x, y, z, satisfying the given equation $H = \text{const.}$, and such that we have $p\,dx + q\,dy + r\,dz$ an exact differential $= dV$; this would be done if we can find two other equations $K = \text{const.}$ and $L = \text{const.}$, such that the values of p, q, r obtained from the three equations give p, q, r functions having the property in question. Attending to only two of the equations, say $H = \text{const.}$ and $K = \text{const.}$, we have here p, q, r functions of x, y, z, such that $p\,dx + q\,dy + r\,dz$ is an exact differential, and two of the equations which serve to determine p, q, r as functions of x, y, z are $H = \text{const.}$, $K = \text{const.}$ We have to prove the following fundamental theorem, viz. that $(H, K) = 0$.

61. In fact, from the equations $H = \text{const.}$, $K = \text{const.}$, treating x, y, z as independent variables, we have

$$\frac{dH}{dx} + \frac{dH}{dp}\frac{dp}{dx} + \frac{dH}{dq}\frac{dq}{dx} + \frac{dH}{dr}\frac{dr}{dx} = 0,$$

$$\frac{dK}{dx} + \frac{dK}{dp}\frac{dp}{dx} + \frac{dK}{dq}\frac{dq}{dx} + \frac{dK}{dr}\frac{dr}{dx} = 0;$$

and if from these equations in order to eliminate $\frac{dp}{dx}$ we multiply by $\frac{dK}{dp}$, $-\frac{dH}{dp}$, and add, we find

$$\frac{d(K, H)}{d(p, x)} + \quad . \quad + \frac{d(K, H)}{d(p, q)}\frac{dq}{dx} + \frac{d(K, H)}{d(p, r)}\frac{dr}{dx} = 0;$$

and, in precisely the same way,

$$\frac{d(K, H)}{d(q, y)} + \frac{d(K, H)}{d(q, p)}\frac{dp}{dy} + \quad . \quad + \frac{d(K, H)}{d(q, r)}\frac{dr}{dy} = 0,$$

$$\frac{d(K, H)}{d(r, z)} + \frac{d(K, H)}{d(r, p)}\frac{dp}{dz} + \frac{d(K, H)}{d(r, q)}\frac{dq}{dz} + \quad . \quad = 0.$$

Adding these together, we have

$$(K, H) + \frac{d(K, H)}{d(q, r)}\left(\frac{dr}{dy} - \frac{dq}{dz}\right) + \frac{d(K, H)}{d(r, p)}\left(\frac{dp}{dz} - \frac{dr}{dx}\right) + \frac{d(K, H)}{d(p, q)}\left(\frac{dq}{dx} - \frac{dp}{dy}\right) = 0;$$

viz. if $p\,dx + q\,dy + r\,dz$ be an exact differential, then $(H, K) = 0$, which is the theorem in question.

62. In the case where the variables are (x, y, p, q), we have simply

$$(K, H) + \frac{d(K, H)}{d(p, q)}\left(\frac{dq}{dx} - \frac{dp}{dy}\right) = 0;$$

viz. $p\,dx + q\,dy$ being a complete differential, $(K, H) = 0$. Conversely, if $(K, H) = 0$, then $\frac{dq}{dx} - \frac{dp}{dy} = 0$, and $p\,dx + q\,dy$ is an exact differential; viz. this is so unless $\frac{d(K, H)}{d(p, q)} = 0$; this equation would imply that K, H considered as functions of p, q, are functions one of the other: and, supposing it to hold good, we could not from the equations $H = 0$, $K = 0$ determine p, q as functions of x, y, for, eliminating one of the variables p, q, the other would disappear of itself. We hence obtain the complete statement of the converse theorem, viz. the functions H, K being such that it is possible from the equations $H = 0$, $K = 0$ to express p, q as functions of x, y, then, if $(H, K) = 0$, we have $p\,dx + q\,dy$ an exact differential.

63. Returning to the case of the variables (x, y, z, p, q, r), if p, q, r are determined as functions of x, y, z by the three equations $H = 0$, $K = 0$, $L = 0$, then, by what precedes, in order that $p\,dx + q\,dy + r\,dz$ may be a complete differential, we must have $(H, K) = 0$, $(H, L) = 0$, $(K, L) = 0$; and it further appears that if these equations are satisfied, then we have, conversely,

$$\frac{dr}{dy} - \frac{dq}{dz} = 0,\quad \frac{dp}{dz} - \frac{dr}{dx} = 0,\quad \frac{dq}{dx} - \frac{dp}{dy} = 0,$$

that is, $p\,dx + q\,dy + r\,dz$ is an exact differential; viz. this is the case unless we have between H, K, L the relation

$$\begin{vmatrix} \frac{d(H, K)}{d(q, r)}, & \frac{d(H, K)}{d(r, p)}, & \frac{d(H, K)}{d(p, q)} \\ H, L\ , & H, L\ , & H, L \\ K, L\ , & K, L\ , & K, L \end{vmatrix} = 0,$$

where in the determinant the second and third lines are the same functions of H, L and K, L respectively that the first line is of H, L.

The determinant is, in fact, equal to the square of

$$\frac{d(H, K, L)}{d(p, q, r)},$$

and, if it vanish, it is impossible, by means of the equations $H=0$, $K=0$, $L=0$, to determine p, q, r as functions of x, y, z. Hence, if the last-mentioned equations are such that by means of them it is possible to effect the determination, and if, moreover, $(H, K)=0$, $(H, L)=0$, $(K, L)=0$, then $p\,dx+q\,dy+r\,dz$ will be an exact differential.

64. Considering H as given, we have, by what precedes, K, L solutions of the linear partial differential equation $(H, \theta)=0$; and since also K, L must be such that $(K, L)=0$, they are conjugate solutions; or in conformity with what precedes, using the small letters a, b instead of K, L, we have the following theorem for the integration of the partial differential equation $H=\text{const.}$, where as before H is a given function of x, y, z, p, q, r.

Find a and b, such that H, a, b are a system of conjugate solutions of the linear partial differential equation $(H, \theta)=0$: then from the equations $H=\text{const.}$, $a=\text{const.}$, $b=\text{const.}$, determining p, q, r as functions of a, b, H, and in the result treating these quantities as constants, we have $p\,dx+q\,dy+r\,dz$ an exact differential $=dV$, and thence

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

an expression for V containing the three arbitrary constants λ, a, b, and therefore a complete solution of the given partial differential equation $H=\text{const.}$

The theorem applies to the case where n has any value whatever, viz. if there are n variables $x, y, z, \ldots$, then we have to find the $n-1$ integrals $a, b, c, \ldots$, constituting with H a system of conjugate integrals; and the theorem holds good.

In particular, if $n=2$, or the independent variables are x and y, then we find any solution a of the partial differential equation $(H, \theta)=0$; the values p, q derived from the equations $H=\text{const.}$, $a=\text{const.}$, give $V=\lambda+\int(p\,dx+q\,dy)$, a complete solution.

65. But there is a different solution depending on the consideration of corresponding values; viz. if the independent variables be as before x, y, z, p, q, r, and if $x_0, y_0, z_0, p_0, q_0, r_0$ are corresponding values of x, y, z, p, q, r, then, taking a, b, c, d, e to be integrals of $(H, \theta)=0$: so that H is here a given function of a, b, c, d, e, since the number of independent variables is $=5$: and representing by a_0, b_0, c_0, d_0, e_0 the like functions of $x_0, y_0, z_0, p_0, q_0, r_0$, we form the equations

$$H=\text{const.},\ a=a_0,\ b=b_0,\ c=c_0,\ d=d_0,\ e=e_0.$$

We have the theorem that, expressing by means of these equations p, q, r, as functions of $x, y, z, x_0, y_0, z_0, H$, and regarding therein x_0, y_0, z_0, H as constants, we have $p\,dx+q\,dy+r\,dz$ an exact differential, and therefore

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

a solution of the equation $H = \text{const.}$ involving the arbitrary constants λ, x_0, y_0, z_0 (one more than required for a complete solution).

The theorem is here stated in the form proper for the solution of the partial differential equation $H = \text{const.}$; a more general statement will be given further on.

66. I take first $n = 2$, or the independent variables to be x, y; here p, q are determined by the equations $a = a_0$, $b = b_0$, $c = c_0$, $H = \text{const.}$, and it is to be shown that $p\,dx + q\,dy = dV$.

Considering p, q, p_0, q_0 as functions of the independent variables x, y, then differentiating in regard to x, and eliminating $\frac{dp}{dx}$, $\frac{dp_0}{dx}$, $\frac{dq_0}{dx}$, we find

$$\begin{vmatrix} \frac{da}{dx} + \frac{da}{dq}\frac{dq}{dx}, & \frac{da}{dp}, & \frac{da_0}{dp_0}, & \frac{da_0}{dq_0} \\ \frac{db}{dx} + \frac{db}{dq}\frac{dq}{dx}, & \frac{db}{dp}, & \frac{db_0}{dp_0}, & \frac{db_0}{dq_0} \\ \frac{dc}{dx} + \frac{dc}{dq}\frac{dq}{dx}, & \frac{dc}{dp}, & \frac{dc_0}{dp_0}, & \frac{dc_0}{dq_0} \\ \frac{dH}{dx} + \frac{dH}{dq}\frac{dq}{dx}, & \frac{dH}{dp}, & 0, & 0 \end{vmatrix} = 0,$$

viz. this is

$$\frac{d(a_0, b_0)}{d(p_0, q_0)}\left\{\frac{d(H, c)}{d(p, x)} + \frac{d(H, c)}{d(p, q)}\frac{dq}{dx}\right\} + \&c. = 0.$$

But in the same way

$$\frac{d(a_0, b_0)}{d(p_0, q_0)}\left\{\frac{d(H, c)}{d(q, y)} + \frac{d(H, c)}{d(q, p)}\frac{dp}{dy}\right\} + \&c. = 0;$$

adding these two equations we have

$$\frac{d(a_0, b_0)}{d(p_0, q_0)}\left\{(H, c) + \frac{d(H, c)}{d(p, q)}\left(\frac{dq}{dx} - \frac{dp}{dy}\right)\right\} + \&c. = 0,$$

the terms denoted by the &c. being the like terms with b, c, a and c, a, b in place of a, b, c. We have $(H, a) = 0$, $(H, b) = 0$, $(H, c) = 0$, and the equation, in fact, is

$$\left\{\Sigma \frac{d(a_0, b_0)}{d(p_0, q_0)}\frac{d(H, c)}{d(p, q)}\right\}\left(\frac{dq}{dx} - \frac{dp}{dy}\right) = 0;$$

viz. we have $\frac{dq}{dx} - \frac{dp}{dy} = 0$, the condition for an exact differential.

67. Coming now to the case where the independent variables are x, y, z, we proceed in the same way with the equations $H = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$. Differentiating in regard to x, and eliminating

$$\frac{dp}{dx}, \frac{dq}{dx}, \frac{dp_0}{dx}, \frac{dq_0}{dx}, \frac{dr_0}{dx},$$

we find for $\frac{dr}{dx}$ the equation

$$\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}\left\{\frac{dr}{dx}\frac{d(a,\ b,\ H)}{d(r,\ p,\ q)}+\frac{d(a,\ b,\ H)}{d(x,\ p,\ q)}\right\}+\&\text{c.}=0.$$

We have in the same way for $\frac{dp}{dz}$ the equation

$$\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}\left\{\frac{dp}{dz}\frac{d(a,\ b,\ H)}{d(p,\ r,\ q)}+\frac{d(a,\ b,\ H)}{d(z,\ r,\ q)}\right\}+\&\text{c.}=0,$$

whence, adding, we obtain

$$\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{d(a,\ b,\ H)}{d(r,\ p,\ q)}+\frac{d(a,\ b,\ H)}{d(x,\ p,\ q)}+\frac{d(a,\ b,\ H)}{d(r,\ z,\ q)}\right\}+\&\text{c.}=0,$$

where the terms denoted by the &c. are the like terms corresponding to the different permutations of the letters $a,\ b,\ c,\ d,\ e$.

The equation may be simplified; we have identically

$$-\frac{da}{dq}(b,\ H)-\frac{db}{dq}(H,\ a)-\frac{dH}{dq}(a,\ b)=\frac{d(a,\ b,\ H)}{d(x,\ p,\ q)}+\frac{d(a,\ b,\ H)}{d(z,\ r,\ q)};$$

or, since $(H,\ a)=0$, $(H,\ b)=0$, the left-hand side is simply $-\frac{dH}{dq}(a,\ b)$, and the equation becomes

$$\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{d(a,\ b,\ H)}{d(r,\ p,\ q)}-\frac{dH}{dq}(a,\ b)\right\}+\&\text{c.}=0.$$

68. This ought to give $\frac{dr}{dx}-\frac{dp}{dz}=0$; it will, if only

$$\Sigma\left\{\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}(a,\ b)\right\}=0,$$

which is thus the condition which has to be proved. By the Poisson-Jacobi theorem, $(a,\ b)$ is a function of $a,\ b,\ c,\ d,\ e$: if we write

$$(a_0,\ b_0)=\frac{d(a_0,\ b_0)}{d(p_0,\ x_0)}+\frac{d(a_0,\ b_0)}{d(q_0,\ y_0)}+\frac{d(a_0,\ b_0)}{d(r_0,\ z_0)},$$

then $(a_0,\ b_0)$ is the same function of $a_0,\ b_0,\ c_0,\ d_0,\ e_0$; but these are equal to $a,\ b,\ c,\ d,\ e$ respectively, and we then have $(a,\ b)=(a_0,\ b_0)$, and the theorem to be proved is

$$\Sigma\left\{\frac{d(c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0)}(a_0,\ b_0)\right\}=0.$$

But, substituting for $(a_0,\ b_0)$ its value, the function on the left-hand side is, it is easy to see, the sum of the three functional determinants

$$\frac{d(a_0,\ b_0,\ c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0,\ p_0,\ x_0)},\quad\frac{d(a_0,\ b_0,\ c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0,\ q_0,\ y_0)},\quad\frac{d(a_0,\ b_0,\ c_0,\ d_0,\ e_0)}{d(p_0,\ q_0,\ r_0,\ r_0,\ z_0)},$$

and each of these, as containing the same letter twice in the denominator, that is, as having two identical columns, is $=0$; the theorem is thus proved. And in the same way $\frac{dp}{dy}-\frac{dq}{dx}$, $\frac{dq}{dz}-\frac{dr}{dy}$ are each $=0$; that is, $p\,dx+q\,dy+r\,dz=dV$.

69. The proof would fail if the factors multiplying $\frac{dr}{dx}-\frac{dp}{dy}$, &c., or any one of these factors, were $=0$. I have not particularly examined this, but the meaning must be that here the equations $a=a_0$, &c., $H=\text{const.}$, fail to give for p, q, r expressions as functions of x, y, z, x_0, y_0, z_0, H; whenever such expressions are obtainable, we have

$$p\,dx+q\,dy+r\,dz=dV.$$

The proof in the case of a greater number of variables, say in the next case where the independent variables are x, y, z, w, would probably present greater difficulty, but I have not examined this.

70. Taking the independent variables to be x and y, we may from the equations $a=a_0$, $b=b_0$, $c=c_0$, $H=\text{const.}$ (which last equation may also be written $H=H_0=\text{const.}$) find p, q, p_0, q_0 as functions of x, y, x_0, y_0, H; and we have then the theorem that, considering only H as a constant,

$$p\,dx+q\,dy-p_0\,dx_0-q_0\,dy_0=dV.$$

To show this, we have to prove the further equations $\frac{dp}{dx_0}+\frac{dp_0}{dx}=0$, &c.; we find

$$\frac{dp}{dx_0}\Sigma\left\{\frac{d(b_0,\ c_0)}{d(p_0,\ q_0)}\,\frac{d(a,\ H)}{d(p,\ q)}\right\}-\frac{dH}{dq}\,\frac{d(a_0,\ b_0,\ c_0)}{d(x_0,\ p_0,\ q_0)}=0,$$

$$\frac{dp_0}{dx}\Sigma\left\{\frac{d(b,\ c)}{d(p,\ q)}\,\frac{d(a_0,\ H_0)}{d(p_0,\ q_0)}\right\}-\frac{dH_0}{dq_0}\,\frac{d(a,\ b,\ c)}{d(x,\ p,\ q)}=0,$$

and it is to be shown that the coefficients of $\frac{dp}{dx_0}$, $\frac{dp_0}{dx}$ are equal and of opposite signs, and that the other two terms are equal; viz. this being so, subtracting the two equations, we have the required relation $\frac{dp}{dx_0}+\frac{dp_0}{dx}=0$. Now H, H_0 are the same functions of a, b, c and of a_0, b_0, c_0; and there is no real loss of generality in assuming $c=H$, $c_0=H_0$; but this being so, the first coefficient is

$$\frac{d(b_0,\ H_0)}{d(p_0,\ q_0)}\,\frac{d(a,\ H)}{d(p,\ q)}+\frac{d(H_0,\ a_0)}{d(p_0,\ q_0)}\,\frac{d(b,\ H)}{d(p,\ q)},$$

and the second is

$$\frac{d(b,\ H)}{d(p,\ q)}\,\frac{d(a_0,\ H_0)}{d(p_0,\ q_0)}+\frac{d(H,\ a)}{d(p,\ q)}\,\frac{d(b_0,\ H_0)}{d(p_0,\ q_0)},$$

which only differ by their signs. As regards the other two terms, we have identically

$$\frac{da}{dq}(b,\ H)+\frac{db}{dq}(H,\ a)+\frac{dH}{dq}(a,\ b)=\frac{d(a,\ b,\ H)}{d(x,\ p,\ q)},$$

which, in virtue of $(a, H)=0$, $(b, H)=0$, becomes

$$\frac{dH}{dq}(a\,,\,b\,)=\frac{d\,(a,\ b,\ H)}{d\,(x,\ p,\ q)};$$

similarly,

$$\frac{dH_0}{dq_0}(a_0,\ b_0)=\frac{d\,(a_0,\ b_0,\ H_0)}{d\,(x_0,\ p_0,\ q_0)}.$$

Hence the terms in question are

$$-\frac{dH}{dq}\frac{dH_0}{dq_0}(a_0,\ b_0),\quad -\frac{dH}{dq}\frac{dH_0}{dq_0}(a,\ b),$$

which are equal in virtue of $(a,\ b)=(a_0,\ b_0)$; and, similarly, the other conditions might be proved. But the proof would be more difficult in the case of a greater number of variables.

Examples. Art. Nos. 71 to 79.

71. The variables are taken to be $x,\ y,\ z,\ p,\ q,\ r$. As a first example, which will serve as an illustration of most of the preceding theorems, suppose $pqr-1=H$; the Hamiltonian system, with the adjoined equalities, is here

$$\frac{dx}{qr}=\frac{dy}{rp}=\frac{dz}{pq}=\frac{dp}{0}=\frac{dq}{0}=\frac{dr}{0}=dt=\frac{dV}{3pqr}.$$

The integrals of the original system may be taken to be

$$\begin{aligned} a &= p,\\ b &= q,\\ c &= r,\\ d &= qy-px,\\ e &= rz-px, \end{aligned}$$

and there is of course the integral $H=pqr-1$, which is connected with the foregoing five integrals by the relation $H=abc-1$.

We form at once the equations

$$\begin{array}{llll} (a,\ b)=0, & (a,\ c)=0, & (a,\ d)=-a, & (a,\ e)=-a,\\ & (b,\ c)=0, & (b,\ d)=\ \ b, & (b,\ e)=\ \ 0,\\ & & (c,\ d)=\ \ 0, & (c\,,\ e)=\ \ c,\\ & & & (d,\ e)=\ \ 0; \end{array}$$

hence it happens that no two of these integrals $a,\ b,\ c,\ d,\ e$ give by the Poisson-Jacobi theorem a new integral. To show how the theorem might have given a new integral, suppose that the known integrals had been $\alpha=p+q$, and $e=rz-px$, then $(\alpha,\ e)=-p$: or the theorem gives the new integral $a=p$.

We have as a conjugate system a, b, c; also the conjugate systems H, a, b; H, a, c; H, b, c; H, b, e; H, c, d; H, d, e; but the first three of these, considering therein H as standing for its value $abc-1$, are substantially equivalent to the first-mentioned system (a, b, c).

72. Postponing the consideration of the augmented system, we now consider the partial differential equation $pqr = 1 + H$, where H is a given constant and p, q, r denote the differential coefficients of a function V. The most simple solution is that given by the conjugate system H, a, b, viz. here p, q, r are determined by the equations $p = a$, $q = b$, $pqr = 1 + H$, that is, $r = \frac{1+H}{ab}$; or, introducing for symmetry the constant c, where $abc = 1 + H$ as before, then $r = c$, and we have

$$V = \lambda + \int (adx + bdy + cdz), \quad = \lambda + ax + by + cz,$$

where a, b, c are connected by the just-mentioned equation $abc = 1 + H$. This is therefore a solution containing say the arbitrary constants λ, a, b, and, as such, is a complete solution.

But any other conjugate system gives a complete solution, and a very elegant one is obtained from the system H, d, e. Writing for symmetry $\beta - \alpha$, $\gamma - \alpha$ in place of d, e, we have here to find p, q, r from the equations

$$H = pqr - 1, \quad qy - px = \beta - \alpha, \quad rz - px = \gamma - \alpha;$$

or, if we assume $\theta = px - \alpha$, then

$$H = pqr - 1; \ px, \ qy, \ rz = \theta + \alpha, \ \theta + \beta, \ \theta + \gamma$$

respectively, whence

$$(1 + H)\, xyz = (\theta + \alpha)(\theta + \beta)(\theta + \gamma),$$

which equation determines θ as a function of x, y, z (in fact, it is a function of the product xyz), and then

$$p, q, r = \frac{\theta+\alpha}{x}, \quad \frac{\theta+\beta}{y}, \quad \frac{\theta+\gamma}{z},$$

and we have

$$V = \lambda + \int \left(\frac{\theta+\alpha}{x} dx + \frac{\theta+\beta}{y} dy + \frac{\theta+\gamma}{z} dz \right).$$

There is no difficulty in effecting the integration directly by introducing θ as a new variable, and we find

$$V = \lambda + 3\theta - \alpha \log \frac{\theta+\alpha}{x} - \beta \log \frac{\theta+\beta}{y} - \gamma \log \frac{\theta+\gamma}{z}.$$

Or, starting from this form, we may verify it by differentiation; the value of dV is

$$d\theta \left(3 - \frac{\alpha}{\theta+\alpha} - \frac{\beta}{\theta+\beta} - \frac{\gamma}{\theta+\gamma} \right) + \frac{\alpha\, dx}{x} + \frac{\beta\, dy}{y} + \frac{\gamma\, dz}{z},$$

where the term in $d\theta$ is

$$=\theta\, d\theta\left(\frac{1}{\theta+\alpha}+\frac{1}{\theta+\beta}+\frac{1}{\theta+\gamma}\right),$$

which, from the equation which determines θ, is

$$=\theta\left(\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}\right),$$

and the value of dV is thus

$$=\frac{\theta+\alpha}{x}dx+\frac{\theta+\beta}{y}dy+\frac{\theta+\gamma}{z}dz.$$

The solution contains apparently the four constants λ, α, β, γ, but there is no loss of generality in writing, for instance, $\alpha=0$, and the number of constants really contained in the solution may be regarded as 3.

73. To show how the equations $H=\text{const.}$, $a=a_0$, $b=b_0$, $c=c_0$, $d=d_0$, $e=e_0$ give a solution; remarking that these equations are $pqr-1=H$, $p=p_0$, $q=q_0$, $r=r_0$, $qy-px=q_0y_0-p_0x_0$, $rz-px=r_0z_0-p_0x_0$, we find

$$p(x-x_0)=q(y-y_0)=r(z-z_0),$$

and consequently $p, q, r=$

$$\sqrt[3]{(1+H)}\,\frac{(y-y_0)^{\frac{1}{3}}(z-z_0)^{\frac{1}{3}}}{(x-x_0)^{\frac{2}{3}}},\quad \sqrt[3]{(1+H)}\,\frac{(z-z_0)^{\frac{1}{3}}(x-x_0)^{\frac{1}{3}}}{(y-y_0)^{\frac{2}{3}}},\quad \sqrt[3]{(1+H)}\,\frac{(x-x_0)^{\frac{1}{3}}(y-y_0)^{\frac{1}{3}}}{(z-z_0)^{\frac{2}{3}}},$$

respectively: whence

$$V=\lambda+\int(p\,dx+q\,dy+r\,dz),$$

$$=\lambda+3\sqrt[3]{(1+H)}\,(x-x_0)^{\frac{1}{3}}(y-y_0)^{\frac{1}{3}}(z-z_0)^{\frac{1}{3}},$$

which is the solution involving the four constants λ, x_0, y_0, z_0.

If in the foregoing value of V we consider x_0, y_0, z_0 as variables, then p, q, r having the values just mentioned, and p_0, q_0, r_0 being equal to these respectively, we obviously have

$$dV=p\,dx+q\,dy+r\,dz-p_0\,dx_0-q_0\,dy_0-r_0\,dz_0.$$

74. Considering now the augmented Hamiltonian system, we join to the foregoing integrals a, b, c, d, e, the new integrals $t-\tau=\dfrac{x}{qr}$ and $V-\lambda=3px$. And then expressing all the quantities in terms of $t-\tau$,

$$x=bc(t-\tau),$$

$$y=ca(t-\tau)+\frac{d}{b},$$

$$z=ab(t-\tau)+\frac{e}{c}.$$

$$p=a,\ q=b,\ r=c,\ H=abc-1,$$

$$V=\lambda+3abc(t-\tau).$$

Forming from these the expression for $dV - p\,dx - q\,dy - r\,dz$, the term in $dt - d\tau$ disappears; there is a term in $t - \tau$, the coefficient of which is

$$3d\,.\,abc - a\,d\,.\,bc - b\,d\,.\,ca - c\,d\,.\,ab,$$

which is $= d\,.\,abc$, or the term is $(t-\tau)\,dH$; and we have, finally,

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH - bd\,\frac{d}{b} - cd\,\frac{e}{c};$$

viz. t enters only in the combination $(t-\tau)\,dH$, which is the fundamental theorem. Considering H as a determinate constant, this term disappears.

We may show how this formula leads to the solution of the partial differential equation $pqr = 1 + H$; treating H as a definite constant, then in order that the formula may give $dV - p\,dx - q\,dy - r\,dz = d\lambda$, or $V = \lambda + \int (p\,dx + q\,dy + r\,dz)$, as before, the last two terms of the formula must disappear; this will be the case if $\frac{d}{b}$ and $\frac{e}{c}$ are constants, or, say, $d = b\beta$, $e = c\gamma$, β and γ being constants. But, this being so, we have $q\beta = qy - px$, $r\gamma = rz - px$, that is, $px = q\,(y-\beta) = r\,(z-\gamma)$, $pqr = 1 + H$, giving the values of p, q, r; and then

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz), \quad = \lambda + 3\,\sqrt[3]{(1+H)}\,x^{\frac{1}{3}}\,(y-\beta)^{\frac{1}{3}}\,(z-\gamma)^{\frac{1}{3}},$$

which is substantially the same solution as is obtained above by a different process. Or, again, observing that we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH - dd - de - \frac{d}{b}\,db - \frac{e}{c}\,dc,$$

then, taking H, b, c constants, we have

$$dV - p\,dx - q\,dy - r\,dz = d\lambda - dd - de,$$

which, changing the value of λ, gives the before-mentioned solution

$$V = \lambda + ax + by + cz, \quad (abc = 1 + H).$$

75. As a second example, suppose

$$H = \tfrac{1}{2}\,(p^2 + q^2 + r^2 - x^2 - y^2 - z^2);$$

the augmented system is

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r} = \frac{dp}{x} = \frac{dq}{y} = \frac{dr}{z} = dt = \frac{dV}{p^2+q^2+r^2},$$

corresponding to the dynamical problem of the motion of a particle acted upon by a repulsive central force equal to the distance.

The integrals of the original system may be expressed in various forms, viz. the quotient of any two of the expressions $x+p$, $y+q$, $z+r$, or of any two of the expressions $x-p$, $y-q$, $z-r$ is an integral, or again the product of any expression of the first set into any expression of the second set is an integral: we may take as integrals

$$\alpha = x^2 - p^2,\quad \beta = y^2 - q^2,\quad \gamma = z^2 - r^2,\quad \delta = \frac{y+q}{x+p},\quad \epsilon = \frac{z+r}{x+p}.$$

We have then

$$dt = \frac{dx}{\sqrt{(x^2-\alpha)}},\text{ that is, } t-\tau = \log\{x+\sqrt{(x^2-\alpha)}\} = \log(x+p),$$

giving $x+p=e^{t-\tau}$, and thence the other quantities $x-p$, $y+q$, &c. For greater symmetry, I introduce a new set of constants a, b, c, a', b', c', and I write also $e^{t-\tau}=T$, $e^{-t+\tau}=T'$ (where $TT'=1$). We then have

$$\begin{aligned} x &= aT + a'T', & p &= aT - a'T', \\ y &= bT + b'T', & q &= bT - b'T', \\ z &= cT + c'T', & r &= cT - c'T'; \end{aligned}$$

also, comparing with the values obtained as above,

$$a = \tfrac{1}{2}\ ,\quad b = \tfrac{1}{2}\delta\ ,\quad c = \tfrac{1}{2}\epsilon\ ,$$

$$a' = \tfrac{1}{2}\alpha,\quad b' = \tfrac{1}{2}\frac{\beta}{\delta},\quad c' = \tfrac{1}{2}\frac{\gamma}{\epsilon}.$$

We have, moreover,

$$H = -2(aa' + bb' + cc') = -\tfrac{1}{2}(\alpha+\beta+\gamma).$$

76. We find

$$p^2+q^2+r^2 = H + (a^2+b^2+c^2)T^2 + (a'^2+b'^2+c'^2)T'^2,$$

and thence

$$\begin{aligned} V &= \lambda + \int (p^2+q^2+r^2)\,dt \\ &= \lambda + H(t-\tau) + \tfrac{1}{2}(a^2+b^2+c^2)T^2 - \tfrac{1}{2}(a'^2+b'^2+c'^2)T'^2. \end{aligned}$$

We may from this obtain the expression for

$$dV - p\,dx - q\,dy - r\,dz,$$

when everything is variable. The terms in $(dt-d\tau)$, as is obvious, disappear; omitting these from the beginning, we have

$$dV = d\lambda + (t-\tau)\,dH + (a\,da + b\,db + c\,dc)T^2 - (a'da' + b'db' + c'dc')T'^2:$$

also

$$\begin{aligned} p\,dx &= (aT - a'T')(Tda + T'da'), \\ &= da\,(aT^2 - a') + da'\,(-a'T'^2 + a): \end{aligned}$$

thence forming the analogous expressions for $q\,dy$ and $r\,dz$, we have

$$\begin{aligned} p\,dx + q\,dy + r\,dz = (a\,da + b\,db + c\,dc)\,T^2 &- (a'da' + b'db' + c'dc')\,T'^2 \\ -(a'da + b'db + c'dc) &+ (a\,da' + b\,db' + c\,dc'), \end{aligned}$$

whence

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + a'da + b'db + c'dc - a\,da' - b\,db' - c\,dc';$$

or, in place of a, b, c, a', b', c', introducing $\alpha, \beta, \gamma, \delta, \epsilon$, and attending to the value of H,

$$dV - p\,dx - q\,dy - r\,dz = d\lambda + (t-\tau)\,dH + \tfrac{1}{2}\,dH + \tfrac{1}{2}\frac{\beta}{\delta}\,d\delta + \tfrac{1}{2}\frac{\gamma}{\epsilon}\,d\epsilon.$$

77. Suppose H, δ, ϵ absolute constants, this becomes

$$d(V-\lambda) = p\,dx + q\,dy + r\,dz,$$

or

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz),$$

and we have thus a solution of the partial differential equation

$$p^2 + q^2 + r^2 = x^2 + y^2 + z^2 + 2H;$$

viz. p, q, r are here to be determined as functions of x, y, z by the equations

$$\begin{aligned} p^2 + q^2 + r^2 &= x^2 + y^2 + z^2 + 2H, \\ y + q &= \delta(x+p), \\ z + r &= \epsilon(x+p). \end{aligned}$$

We have

$$2H + x^2 + y^2 + z^2 = p^2 + \{y - \delta(x+p)\}^2 + \{z - \epsilon(x+p)\}^2;$$

or, on the right-hand side, writing $p^2 = (x+p)^2 - 2x(x+p) + x^2$,
„ left „ „ $x^2 = (x-p)^2 - 2x(x+p) + p^2$,

the equation is

$$(1 + \delta^2 + \epsilon^2)(x+p)^2 - 2(x + \delta y + \epsilon z)(x+p) - 2H = 0,$$

which gives p as a function of x, y, z. But the result is a complicated one, except in the case $H = 0$; we then have

$$x + p = \frac{2(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

$$y + q = \frac{2\delta(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

$$z + r = \frac{2\epsilon(x + \delta y + \epsilon z)}{1 + \delta^2 + \epsilon^2},$$

and thence

$$V = \lambda - \tfrac{1}{2}(x^2 + y^2 + z^2) + \frac{(x + \delta y + \epsilon z)^2}{1 + \delta^2 + \epsilon^2},$$

a complete solution of the partial differential equation

$$p^2 + q^2 + r^2 = x^2 + y^2 + z^2.$$

More symmetrically, we have the solution

$$V = \lambda - \tfrac{1}{2}(x^2+y^2+z^2) + \frac{(ax+by+cz)^2}{a^2+b^2+c^2},$$

as can be at once verified.

78. In the same particular case $H=0$, introducing the corresponding values $p_0, q_0, r_0, x_0, y_0, z_0$, we find a very simple expression for $V-V_0$, as a function of x, y, z, x_0, y_0, z_0. We have, writing $T_0 = e^{t_0-\tau}$, $T_0' = e^{-t_0+\tau}$, and therefore $T_0T_0'=1$,

$$x_0 = aT_0 + a'T_0', \quad p_0 = aT_0 - a'T_0',$$

$$y_0 = bT_0 + b'T_0', \quad q_0 = bT_0 - b'T_0',$$

$$z_0 = cT_0 + c'T_0', \quad r_0 = cT_0 - c'T_0',$$

and thence

$$x - x_0 = a(T-T_0) + a'\left(\frac{1}{T}-\frac{1}{T_0}\right), = (T-T_0)\left(a - \frac{a'}{TT_0}\right),$$

$$x + x_0 = a(T+T_0) + a'\left(\frac{1}{T}+\frac{1}{T_0}\right), = (T-T_0)\left(a + \frac{a'}{TT_0}\right).$$

Forming the analogous quantities $y-y_0$, &c., we deduce

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = (T-T_0)^2\left\{a^2+b^2+c^2+(a'^2+b'^2+c'^2)\frac{1}{T^2T_0^2}\right\},$$

$$(x+x_0)^2+(y+y_0)^2+(z+z_0)^2 = (T+T_0)^2\left\{a^2+b^2+c^2+(a'^2+b'^2+c'^2)\frac{1}{T^2T_0^2}\right\}.$$

But we have

$$V - V_0 = \tfrac{1}{2}\left\{(a^2+b^2+c^2)(T^2-T_0^2) - (a'^2+b'^2+c'^2)\left(\frac{1}{T^2}-\frac{1}{T_0^2}\right)\right\}$$

$$= \tfrac{1}{2}(T^2-T_0^2)\left\{a^2+b^2+c^2+(a'^2+b'^2+c'^2)\frac{1}{T^2T_0^2}\right\},$$

and hence the required formula

$$V - V_0 = \tfrac{1}{2}\sqrt{\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}}\sqrt{\{(x+x_0)^2+(y+y_0)^2+(z+z_0)^2\}},$$

or, say, for shortness,

$$= \tfrac{1}{2}\sqrt{(R)}\sqrt{(S)}.$$

79. We ought, therefore, to have

$$\tfrac{1}{2}d\sqrt{(R)}\sqrt{(S)} = p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0,$$

where p, q, r, p_0, q_0, r_0 denote as above, and consequently

$$p^2+q^2+r^2 = x^2+y^2+z^2, \quad p_0^2+q_0^2+r_0^2 = x_0^2+y_0^2+z_0^2.$$

We have in fact

$$p = \tfrac{1}{2}\left\{ \frac{\sqrt{(S)}}{\sqrt{(R)}}(x - x_0) + \frac{\sqrt{(R)}}{\sqrt{(S)}}(x + x_0)\right\},\ \text{\&c.},$$

$$p_0 = \tfrac{1}{2}\left\{ -\frac{\sqrt{(S)}}{\sqrt{(R)}}(x - x_0) + \frac{\sqrt{(R)}}{\sqrt{(S)}}(x + x_0)\right\},\ \text{\&c.};$$

and thence

$$p^2 + q^2 + r^2 = \tfrac{1}{4}\{R + S + 2(x^2 + y^2 + z^2 - x_0^2 - y_0^2 - z_0^2)\},\ = x^2 + y^2 - z^2,$$

$$p_0^2 + q_0^2 + r_0^2 = \tfrac{1}{4}\{R + S - 2(x^2 + y^2 + z^2 - x_0^2 - y_0^2 - z_0^2)\},\ = x_0^2 + y_0^2 + z_0^2,$$

or the last-mentioned results are thus verified.

Partial Differential Equation containing the Dependent Variable: Reduction to Standard Form. Art. Nos. 80, 81.

80. The equation $H = \text{const.}$ is the most general form of a partial differential equation not containing the dependent variable V; but if a partial differential equation does contain the independent variable, we can, by regarding this as one of the dependent variables, and in place of it introducing a new independent variable, exhibit the equation in the standard form $H = \text{const.}$, H being here a homogeneous function of the order zero in the differential coefficients. Thus, if the independent variables are x, y, the dependent variable z, and its differential coefficients p, q, then the given partial differential equation may be $H, = H(\mathrm{p}, \mathrm{q}, x, y, z), = \text{const.}$ But we may determine z as a function of x, y by an equation $V = \text{const.}$, V being a desired function of x, y, z; and then writing p, q, r for the differential coefficients $\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$, we have $\mathrm{p} = -\frac{p}{r}$, $\mathrm{q} = -\frac{q}{r}$, and the proposed partial differential equation becomes

$$H\left(-\frac{p}{r}, -\frac{q}{r}, x, y, z\right) = \text{const.}$$

viz. this is an equation containing only the differential coefficients p, q, r of the dependent variable V, a function of x, y, z. And, moreover, H is homogeneous of the order zero in p, q, r; consequently

$$p\frac{dH}{dp} + q\frac{dH}{dq} + r\frac{dH}{dr} = 0,$$

or, in the augmented Hamiltonian system, the last equality is $= \frac{dV}{0}$, so that an integral is $V = \text{const.}$; as already stated, this is the equation by which z is determined as a function of x, y.

81. Thus, if the given partial differential equation be $\mathrm{pq} - z = H$, we here consider the equation $\frac{pq}{r^2} - z = H$. The Hamiltonian system is

$$\frac{r^2 dx}{q} = \frac{r^2 dy}{p} = \frac{-r^3 dz}{2pq} = \frac{dp}{0} = \frac{dq}{0} = \frac{dr}{1}\left(= \frac{dV}{0}\right),$$

having the integrals

$$a = p,$$

$$b = q,$$

$$c = px - qy,$$

$$d = \frac{1}{r} + \frac{x}{q},$$

$$e = \frac{z}{pq} - \frac{1}{r^2},$$

(whence $H = -abe$). We have H, a, b, a system of conjugate integrals and, in terms of these,

$$p = a,\ q = b,\ r = \sqrt{\left(\frac{ab}{z+H}\right)};$$

hence, writing λ for the constant value of V, we have

$$\lambda = \int\left\{a\,dx + b\,dy + \sqrt{\left(\frac{ab}{z+H}\right)}\,dz\right\},$$

that is,

$$\lambda = ax + by + 2\sqrt{\{ab\,(z+H)\}},$$

or say,

$$4ab\,(z+H) = (ax + by - \lambda)^2,$$

a solution containing really the two constants λ and $\frac{a}{b}$, and thus a complete solution of the given equation $\mathrm{pq} - z = H$. We have, in fact,

$$2ab\,\mathrm{p} = a\,(ax + by - \lambda),$$

$$2ab\,\mathrm{q} = b\,(ax + by - \lambda);$$

that is,

$$4a^2b^2\mathrm{pq} = ab\,(ax + by - \lambda)^2,\ = 4a^2b^2\,(z+H),$$

or

$$\mathrm{pq} = z + H,$$

as it should be.