661.

ON THE DOUBLE ϑ-FUNCTIONS.

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PROF. CAYLEY gave an account of researches* on which he is engaged upon the double ϑ-functions. In regard to these, he establishes in a strictly analogous manner the theory of the single ϑ-functions, the process for the single functions being in fact as follows:—Considering u, x as connected by the differential relation

$$\delta u = \frac{\delta x}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}},$$

then, if A, B, C, D, Ω denote functions of u, viz. for shortness, the single letters are used, instead of writing them as functional symbols, $A(u)$, $B(u)$, &c., then, by way of definition of these functions (called, the first four of them ϑ-functions, and the last an ω-function), we assume

$$A,\ B,\ C,\ D = \Omega\sqrt{a-x},\ \Omega\sqrt{b-x},\ \Omega\sqrt{c-x},\ \Omega\sqrt{d-x}$$

respectively, together with one other equation, as presently mentioned. Without in any wise defining the meaning of Ω, we then obtain a set of equations of the form

$$A\delta B - B\delta A = \Omega^2\sqrt{c-x\,.\,d-x}\,\delta u,$$

(mere constant coefficients are omitted), or, what is the same thing,

$$A\delta B - B\delta A = CD\,\delta u,$$

which are differential equations defining the nature of the ratio-functions $A:B:C:D$. If, proceeding to second differential coefficients, we attempt to form the expressions for $A\delta^2 A - (\delta A)^2$, &c., these involve multiples of $\Omega\delta^2\Omega - (\delta\Omega)^2$; in order to obtain a con-

[* See paper, number 665.]

venient form, we assume $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2$, where M is a function of x. We thus obtain an equation $A\delta^2 A - (\delta A)^2 = \Omega^2\mathfrak{A}(\delta u)^2$, where the value of $\mathfrak{A}$ depends upon that of M. The value of M has to be taken so as to simplify as much as may be the expression of $\mathfrak{A}$, but so that M shall be a symmetrical function of the constants a, b, c, d: this last condition is assigned in order that the like simplification may present itself in the analogous relations $B\delta^2 B - (\delta B)^2 = \Omega^2\mathfrak{B}(\delta u)^2$, &c. The proper expression of M is found to be

$$M = -2x^2 + x(a+b+c+d) + a^2+b^2+c^2+d^2-2bc-2ca-2ab-2ad-2bd-2cd,$$

viz. M having this value, the one other equation above referred to is

$$\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2;$$

and we then have a system of four equations such as

$$A\delta^2 A - (\delta A)^2 = \Omega^2\mathfrak{A}(\delta u)^2,$$

where $\mathfrak{A}$ is a linear function of x, and where consequently $\Omega^2\mathfrak{A}$ can be expressed as a linear function of any two of the four squares A^2, B^2, C^2, D^2.

To establish the connexion with the Jacobian H and Θ functions, the differential relation between u, x may be taken to be

$$\delta u = \frac{\delta x}{\sqrt{x \, . \, 1-x \, . \, 1-k^2x}};$$

viz. we have here $d=\infty$, and to adapt the formulæ to this value it is necessary to write $\frac{u}{\sqrt{d}}$ instead of u, and make other easy changes. The result is that Ω differs from D by a constant factor only, so that, instead of the five functions A, B, C, D, Ω, we have only the four functions A, B, C, D. The equation $\Omega\delta^2\Omega - (\delta\Omega)^2 = \Omega^2 M(\delta u)^2$ is replaced by an equation $D\delta^2 D - (\delta D)^2 = D^2\mathfrak{D}(\delta u)^2$, or say $\delta^2(\log D) = \mathfrak{D}(\delta u)^2$, which gives a result of the form

$$D = e^{u+\lambda^2\int\delta u\int\delta u\frac{A^2}{D^2}},$$

showing that D differs from Jacobi's $\Theta(u)$ only by an exponential factor of the form $Ce^{\lambda u^2}$. And it then further appears that A, B, C differ only by factors of the like form from the three numerator functions $\mathrm{H}(u)$, $\mathrm{H}(u+K)$, $\Theta(u+K)$, so that, neglecting constant factors, the functions

$$\frac{A}{D}, \; \frac{B}{D}, \; \frac{C}{D} \text{ are equal to } \frac{\mathrm{H}(u)}{\Theta(u)}, \; \frac{\mathrm{H}(u+K)}{\Theta(u)}, \; \frac{\Theta(u+K)}{\Theta(u)};$$

that is, to the elliptic functions sn u, cn u, dn u.