662.

ON THE DOUBLE Θ-FUNCTIONS IN CONNEXION WITH A 16-NODAL QUARTIC SURFACE.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXIII. (1877), pp. 210—219.]

I HAVE before me Göpel's memoir, "Theoriae transcendentium Abelianarum primi ordinis adumbratio levis," Crelle's Journal, t. XXXV. (1847), pp. 277—312. Writing P_1, P_2, P_3, etc., in place of his P', P'', P''', etc., also α, β, γ, δ, X', Y', Z', W', in place of his t, u, v, w, T, U, V, W, the system of 16 equations (given p. 287) is

$$
\begin{array}{ll}
(1) & P^2 = (\alpha,\ -\beta,\ -\gamma,\ \ \delta)(X',\ Y',\ Z',\ W'),\\
(4) & P_1^2 = (\alpha,\ \ \beta,\ -\gamma,\ -\delta)(X',\ Y',\ Z',\ W'),\\
(9) & P_2^2 = (\alpha,\ -\beta,\ \ \gamma,\ -\delta)(X',\ Y',\ Z',\ W'),\\
(12) & P_3^2 = (\alpha,\ \ \beta,\ \ \gamma,\ \ \delta)(X',\ Y',\ Z',\ W'),\\
(3) & Q^2 = (\beta,\ -\alpha,\ -\delta,\ \ \gamma)(X',\ Y',\ Z',\ W'),\\
(2) & Q_1^2 = (\beta,\ \ \alpha,\ -\delta,\ -\gamma)(X',\ Y',\ Z',\ W'),\\
(11) & Q_2^2 = (\beta,\ -\alpha,\ \ \delta,\ -\gamma)(X',\ Y',\ Z',\ W'),\\
(10) & Q_3^2 = (\beta,\ \ \alpha,\ \ \delta,\ \ \gamma)(X',\ Y',\ Z',\ W'),\\
(13) & R^2 = (\gamma,\ -\delta,\ -\alpha,\ \ \beta)(X',\ Y',\ Z',\ W'),\\
(16) & R_1^2 = (\gamma,\ \ \delta,\ -\alpha,\ -\beta)(X',\ Y',\ Z',\ W'),\\
(5) & R_2^2 = (\gamma,\ -\delta,\ \ \alpha,\ -\beta)(X',\ Y',\ Z',\ W'),\\
(8) & R_3^2 = (\gamma,\ \ \delta,\ \ \alpha,\ \ \beta)(X',\ Y',\ Z',\ W'),\\
(15) & S^2 = (\delta,\ -\gamma,\ -\beta,\ \ \alpha)(X',\ Y',\ Z',\ W'),\\
(14) & S_1^2 = (\delta,\ \ \gamma,\ -\beta,\ -\alpha)(X',\ Y',\ Z',\ W'),\\
(7) & S_2^2 = (\delta,\ -\gamma,\ \ \beta,\ -\alpha)(X',\ Y',\ Z',\ W'),\\
(6) & S_3^2 = (\delta,\ \ \gamma,\ \ \beta,\ \ \alpha)(X',\ Y',\ Z',\ W');
\end{array}
$$

viz. we have $P^2 = \alpha X' - \beta Y' - \gamma Z' + \delta W'$, etc. The reason for the apparently arbitrary manner in which I have numbered these equations, will appear further on. I recall that the sixteen double Θ-functions, that is, Θ-functions of two arguments u, u', are*

$$\begin{array}{cccc} P, & P_1, & P_2, & P_3, \\ iQ, & Q_1, & iQ_2, & Q_3, \\ iR, & iR_1, & R_2, & R_3, \\ S, & iS_1, & iS_2, & S_3, \end{array}$$

the factor $i, = \sqrt{-1}$, being introduced in regard to the six functions which are odd functions of the arguments. But disregarding the sign, I speak of P^2, P_1^2, ..., Q^2, etc., as the squared functions, or simply as the squares; α, β, γ, δ are constants, depending of course on the parameters of the Θ-functions; X', Y', Z', W', which are however to be eliminated, are themselves Θ-functions to a different set of parameters: the 16 equations express that the squared functions P^2, P_1^2, etc., are linear functions of X', Y', Z', W', and they consequently serve to obtain linear relations between the squared functions: viz. by means of them, Göpel expresses the remaining 12 squares, each in terms of the selected four squares P_1^2, P_2^2, S_1^2, S_2^2, which are linearly independent: that is, he obtains linear relations between five squares, and he seems to have assumed that there were not any linear relations between fewer than five squares.

It appears however by Rosenhain's "Mémoire sur les fonctions de deux variables et à quatre périodes etc.", *Mém. Sav. Étrangers*, t. XI. (1851), pp. 364—468, that there are, in fact, linear relations between four squares, viz. that there exist sixes of squares such that, selecting at pleasure any three out of the six, each of the remaining three squares can be expressed as a linear function of these three squares. Knowing this result, it is easy to verify it by means of the sixteen equations, and moreover to show that there are in all 16 such sixes: these are shown by the following scheme which I copy from Kummer's memoir "Ueber die algebraischen Strahlensysteme u. s. w." *Berlin. Abh.* (1866), p. 66: viz. the scheme is

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
8	7	6	5	4	3	2	1	16	15	14	13	12	11	10	9
7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11.

* The same functions in Rosenhain's notation are

$$\begin{array}{cccc} 00, & 02, & 20, & 22, \\ 01, & 03, & 21, & 23, \\ 10, & 12, & 30, & 32, \\ 11, & 13, & 31, & 33; \end{array}$$

viz. the figures here written down are the suffixes of his ϑ-functions, $00 = \vartheta_{0,0}(v, w)$, etc.

In fact, to show that any four of the squares, for instance 1, 9, 13, 8, that is, P^2, P_2^2, R^2, R_3^2, are linearly connected, it is only necessary to show that the determinant of coefficients

$$\begin{vmatrix} \alpha, & -\beta, & -\gamma, & \delta \\ \alpha, & -\beta, & \gamma, & -\delta \\ \gamma, & -\delta, & -\alpha, & \beta \\ \gamma, & \delta, & \alpha, & \beta \end{vmatrix}$$

is $=0$, or what is the same thing, that there exists a linear function of the new variables (X, Y, Z, W), which will become $=0$ on putting for these variables the values in any line of this determinant: we have such a function, viz. this is

$$\beta X+\alpha Y-\delta Z-\gamma W,$$

or say

$$[1] \quad (\beta, \alpha, -\delta, -\gamma)(X, Y, Z, W).$$

This function also vanishes if for (X, Y, Z, W) we substitute the values

$$\begin{array}{llll} \delta, & -\gamma, & \beta, & -\alpha, \\ \delta, & \gamma, & \beta, & \alpha, \end{array}$$

which belong to 7, 6, that is, S_2^2 and S_3^2 respectively. It thus appears that 1, 9, 13, 8, 7, 6, that is, P^2, P_2^2, R^2, R_3^2, S_2^2, S_3^2, are a set of six squares having the property in question. I remark that the process of forming the linear functions is a very simple one; we write down six lines, and thence directly obtain the result, thus

$$\begin{array}{rllll} 1 & \alpha, & -\beta, & -\gamma, & \delta \\ 9 & \alpha, & -\beta, & \gamma, & -\delta \\ 13 & \gamma, & -\delta, & -\alpha, & \beta \\ 8 & \gamma, & \delta, & \alpha, & \beta \\ 7 & \delta, & -\gamma, & \beta, & -\alpha \\ 6 & \delta, & \gamma, & \beta, & \alpha \\ \hline & \beta, & \alpha, & -\delta, & -\gamma: \end{array}$$

viz. β, α, δ, γ are the letters not previously occurring in the four columns respectively: the first letter β is taken to have the sign $+$, and then the remaining signs are determined by the condition that, combining the last line with any line above it (e.g. with the line next above it $\beta\delta+\alpha\gamma-\delta\beta-\gamma\alpha$), the sum must be zero.

We find in this way, as the conditions for the existence of the 16 sixes respectively,

$$\begin{array}{ll} [1] & (\beta, \quad \alpha, \ -\delta, \ -\gamma)(X, Y, Z, W)=0, \\ [2] & (\alpha, \ -\beta, \ -\gamma, \quad \delta)(X, Y, Z, W)=0, \\ [3] & (\alpha, \quad \beta, \ -\gamma, \ -\delta)(X, Y, Z, W)=0, \\ [4] & (\beta, \ -\alpha, \ -\delta, \quad \gamma)(X, Y, Z, W)=0, \end{array}$$

$$
\begin{array}{ll}
[5] & (\delta, \quad \gamma, \quad \beta, \quad \alpha)(X, Y, Z, W) = 0, \\
[6] & (\gamma, -\delta, \quad \alpha, -\beta)(X, Y, Z, W) = 0, \\
[7] & (\gamma, \quad \delta, \quad \alpha, \quad \beta)(X, Y, Z, W) = 0, \\
[8] & (\delta, -\gamma, \quad \beta, -\alpha)(X, Y, Z, W) = 0, \\
[9] & (\beta, \quad \alpha, \quad \delta, \quad \gamma)(X, Y, Z, W) = 0, \\
[10] & (\alpha, -\beta, \quad \gamma, -\delta)(X, Y, Z, W) = 0, \\
[11] & (\alpha, \quad \beta, \quad \gamma, \quad \delta)(X, Y, Z, W) = 0, \\
[12] & (\beta, -\alpha, \quad \delta, -\gamma)(X, Y, Z, W) = 0, \\
[13] & (\delta, \quad \gamma, -\beta, -\alpha)(X, Y, Z, W) = 0, \\
[14] & (\gamma, -\delta, -\alpha, \quad \beta)(X, Y, Z, W) = 0, \\
[15] & (\gamma, \quad \delta, -\alpha, -\beta)(X, Y, Z, W) = 0, \\
[16] & (\delta, -\gamma, -\beta, \quad \alpha)(X, Y, Z, W) = 0.
\end{array}
$$

I repeat in a new order the sets of coefficients which belong to the several squares, viz. these are

$$
\begin{array}{lll}
(1) & P^2 & (\alpha, -\beta, -\gamma, \quad \delta), \\
(2) & Q_1^2 & (\beta, \quad \alpha, -\delta, -\gamma), \\
(3) & Q^2 & (\beta, -\alpha, -\delta, \quad \gamma), \\
(4) & P_1^2 & (\alpha, \quad \beta, -\gamma, -\delta), \\
(5) & R_2^2 & (\gamma, -\delta, \quad \alpha, -\beta), \\
(6) & S_3^2 & (\delta, \quad \gamma, \quad \beta, \quad \alpha), \\
(7) & S_2^2 & (\delta, -\gamma, \quad \beta, -\alpha), \\
(8) & R_3^2 & (\gamma, \quad \delta, \quad \alpha, \quad \beta), \\
(9) & P_2^2 & (\alpha, -\beta, \quad \gamma, -\delta), \\
(10) & Q_3^2 & (\beta, \quad \alpha, \quad \delta, \quad \gamma), \\
(11) & Q_2^2 & (\beta, -\alpha, \quad \delta, -\gamma), \\
(12) & P_3^2 & (\alpha, \quad \beta, \quad \gamma, \quad \delta), \\
(13) & R^2 & (\gamma, -\delta, -\alpha, \quad \beta), \\
(14) & S_1^2 & (\delta, \quad \gamma, -\beta, -\alpha), \\
(15) & S^2 & (\delta, -\gamma, -\beta, \quad \alpha), \\
(16) & R_1^2 & (\gamma, \quad \delta, -\alpha, -\beta).
\end{array}
$$

And I remark that, if we connect these with the multipliers $(Y, -X, W, -Z)$, we obtain, except that there is sometimes a reversal of all the signs, the *same* linear functions of (X, Y, Z, W) as are written down under the same numbers in square brackets above: thus (1) gives

$$(\alpha, -\beta, -\gamma, \delta)(Y, -X, W, -Z), \text{ which is } (\beta, \alpha, -\delta, -\gamma)(X, Y, Z, W), =[1];$$

and so (2) gives

$$(\beta,\ \alpha,\ -\delta,\ -\gamma)(Y,\ -X,\ W,\ -Z),\ \text{which is}\ (-\alpha,\ \beta,\ \gamma,\ -\delta)(X,\ Y,\ Z,\ W),$$

or, reversing the signs,

$$(\alpha,\ -\beta,\ -\gamma,\ \delta)(X,\ Y,\ Z,\ W),\ =[2].$$

Comparing with the geometrical theory in Kummer's Memoir, it appears that the several systems of values (1), (2), ..., (16) are the coordinates of the nodes of a 16-nodal quartic surface, which nodes lie by sixes in the singular tangent planes, in the manner expressed by the foregoing scheme, wherein each top number may refer to a singular tangent plane, and then the numbers below it show the nodes in this plane: or else the top number may refer to a node, and then the numbers below it show the singular planes through this node.

And, from what precedes, we have the general result: the 16 squared double Θ-functions correspond (one to one) to the nodes of a 16-nodal quartic surface, in such wise that linearly connected squared functions correspond to nodes in the same singular tangent plane.

The question arises, to find the equation of the 16-nodal quartic surface, having the foregoing nodes and singular tangent planes. Starting from one of the irrational forms, say

$$\sqrt{A\,[1]\,[5]}+\sqrt{B\,[2]\,[6]}+\sqrt{C\,[3]\,[7]}=0,$$

the coefficients A, B, C are readily determined; and the result written at full length is

$$\begin{aligned}&\sqrt{2\,(\alpha\beta-\gamma\delta)\,(\alpha\delta+\beta\gamma)\,(\beta X+\alpha Y-\delta Z-\gamma W)\,(\delta X+\gamma Y+\beta Z+\alpha W)}\\ &+\sqrt{(\alpha^2-\beta^2-\gamma^2+\delta^2)\,(\alpha\gamma-\beta\delta)\,(\alpha X-\beta Y-\gamma Z+\delta W)\,(\gamma X-\delta Y+\alpha Z-\beta W)}\\ &+\sqrt{(\alpha^2+\beta^2-\gamma^2-\delta^2)\,(\alpha\gamma+\beta\delta)\,(\alpha X+\beta Y-\gamma Z-\delta W)\,(\gamma X+\delta Y+\alpha Z+\beta W)}=0.\end{aligned}$$

It is a somewhat long, but nevertheless interesting, piece of algebraical work to rationalise the foregoing equation: the result is

$$\begin{aligned}&(\beta^2\gamma^2-\alpha^2\delta^2)\,(\gamma^2\alpha^2-\beta^2\delta^2)\,(\alpha^2\beta^2-\gamma^2\delta^2)\,(X^4+Y^4+Z^4+W^4)\\ &+(\gamma^2\alpha^2-\beta^2\delta^2)\,(\alpha^2\beta^2-\gamma^2\delta^2)\,(\alpha^4+\delta^4-\beta^4-\gamma^4)\,(Y^2Z^2+X^2W^2)\\ &+(\alpha^2\beta^2-\gamma^2\delta^2)\,(\beta^2\gamma^2-\alpha^2\delta^2)\,(\beta^4+\delta^4-\gamma^4-\alpha^4)\,(Z^2X^2+Y^2W^2)\\ &+(\beta^2\gamma^2-\alpha^2\delta^2)\,(\gamma^2\alpha^2-\beta^2\delta^2)\,(\gamma^4+\delta^4-\alpha^4-\beta^4)\,(X^2Y^2+Z^2W^2)\\ &-2\alpha\beta\gamma\delta\,(\alpha^2+\beta^2+\gamma^2+\delta^2)\,(\alpha^2+\delta^2-\beta^2-\gamma^2)\,(\beta^2+\delta^2-\alpha^2-\gamma^2)\,(\gamma^2+\delta^2-\alpha^2-\beta^2)\,XYZW=0\,;\end{aligned}$$

or, if we write for shortness

$$\begin{aligned}L&=\beta^2\gamma^2-\alpha^2\delta^2, & F&=\alpha^2+\delta^2-\beta^2-\gamma^2,\\ M&=\gamma^2\alpha^2-\beta^2\delta^2, & G&=\beta^2+\delta^2-\gamma^2-\alpha^2,\\ N&=\alpha^2\beta^2-\gamma^2\delta^2, & H&=\gamma^2+\delta^2-\alpha^2-\beta^2,\\ & & \Delta&=\alpha^2+\beta^2+\gamma^2+\delta^2,\end{aligned}$$

then the result is

$$\begin{aligned} &LMN(X^4+Y^4+Z^4+W^4)\\ &+MN(F\Delta+2L)(Y^2Z^2+X^2W^2)\\ &+NL(G\Delta+2M)(Z^2X^2+Y^2W^2)\\ &+LM(H\Delta+2N)(X^2Y^2+Z^2W^2)\\ &-2\alpha\beta\gamma\delta FGH\Delta\, XYZW=0. \end{aligned}$$

It may be easily verified that any one of the sixteen points, for instance $(\alpha, \beta, \gamma, \delta)$, is a node of the surface. Thus to show that the derived function in respect to X, vanishes for $X, Y, Z, W = \alpha, \beta, \gamma, \delta$; the derived function here divides by 2α, and omitting this factor, the equation to be verified is

$$LMN \,.\, 2\alpha^2 + MN(F\Delta+2L)\delta^2 + NL(G\Delta+2M)\gamma^2 + LM(H\Delta+2N)\beta^2 - \beta^2\gamma^2\delta^2 FGH\Delta = 0,$$

viz. the whole coefficient of LMN is $2(\alpha^2+\beta^2+\gamma^2+\delta^2), = 2\Delta$; hence throwing out the factor Δ, the equation becomes

$$2LMN + MNF\delta^2 + NLG\gamma^2 + LMH\beta^2 - \beta^2\gamma^2\delta^2 FGH = 0.$$

Writing this in the form

$$L(2MN + NG\gamma^2 + MH\beta^2) = F\delta^2(GH\beta^2\gamma^2 - MN),$$

we find without difficulty $GH\beta^2\gamma^2 - MN = -(\beta^2-\gamma^2)^2 L$; hence throwing out the factor L, the equation becomes

$$N(2M + G\gamma^2) + MH\beta^2 + F\delta^2(\beta^2-\gamma^2)^2 = 0;$$

we find

$$\begin{aligned} MH\beta^2 + F\delta^2(\beta^2-\gamma^2)^2 &= (\alpha^2\beta^2 - \gamma^2\delta^2)(2\beta^2\delta^2 - \gamma^2(\alpha^2+\beta^2+\delta^2) + \gamma^4)\\ &= N(2\beta^2\delta^2 - \gamma^2(\alpha^2+\beta^2+\delta^2) + \gamma^4), \end{aligned}$$

or throwing out the factor N, the equation becomes

$$2M + G\gamma^2 + 2\beta^2\delta^2 - \gamma^2(\alpha^2+\beta^2+\delta^2) + \gamma^4 = 0,$$

which is at once verified: and similarly it can be shown that the other derived functions vanish, and the point $(\alpha, \beta, \gamma, \delta)$ is thus a node.

The surface seems to be the general 16-nodal surface, viz. replacing X, Y, Z, W by any linear functions of four coordinates, we have thus $4 \,.\, 4 - 1, = 15$ constants, and the equation contains besides the three ratios $\alpha : \beta : \gamma : \delta$, that is, in all 18 constants: the general quartic surface has 34 constants, and therefore the general 16-nodal surface $34 - 16, = 18$ constants: but the conclusion requires further examination.

Göpel and Rosenhain each connect the theory with that of the ultra-elliptic functions involving the radical $\sqrt{X}, = \sqrt{x \,.\, 1-x \,.\, 1-lx \,.\, 1-mx \,.\, 1-nx}$; viz. it appears by their formulæ (more completely by those of Rosenhain) that the ratios of the 16 squares can be expressed rationally in terms of the two variables x, x', and the radicals

$\sqrt{X}$, $\sqrt{X'}$, X' being the same function of x' that X is of x. We may instead of the preceding form take X to be the general quintic function, or what is better take it to be the sextic function $a-x \,.\, b-x \,.\, c-x \,.\, d-x \,.\, e-x \,.\, f-x$; and we thus obtain a remarkable algebraical theorem: viz. I say that the 16 squares, each divided by a proper constant factor, are proportional to six functions of the form

$$a-x \,.\, a-x',$$

and ten functions of the form

$$\frac{1}{(x-x')^2}\{\sqrt{a-x \,.\, b-x \,.\, c-x \,.\, d-x' \,.\, e-x' \,.\, f-x'} - \sqrt{a-x' \,.\, b-x' \,.\, c-x' \,.\, d-x \,.\, e-x \,.\, f-x}\}^2,$$

and consequently that these 16 algebraical functions of x, x' are linearly connected in the manner of the 16 squares; viz. there exist 16 sixes such that, in each six, the remaining three functions can be linearly expressed in terms of any three of them.

To further develop the theory, I remark that the six functions may be represented by A, B, C, D, E, F respectively: any one of the ten functions would be properly represented by $ABC \,.\, DEF$, but isolating one letter F, and writing DE to denote DEF, this function $ABC \,.\, DEF$ may be represented simply as DE; and the ten functions thus are AB, AC, AD, AE, BC, BD, BE, CD, CE, DE.

Writing for shortness a, b, c, d, e, f, to denote $a-x$, $b-x$, etc., and similarly a', b', c', d', e', f', to denote $a-x'$, $b-x'$, etc., we thus have

(13) $A = aa'$,

(9) $B = bb'$,

(7) $C = cc'$,

(8) $D = dd'$,

(6) $E = ee'$, $(= E)$,

(1) $F = ff'$, $(= F)$,

(3) $DE = \frac{1}{(x-x')^2}\{\sqrt{abcd'e'f'} - \sqrt{a'b'c'def}\}^2$, $(= \bar{D})$,

(4) $CE = \frac{1}{(x-x')^2}\{\sqrt{abdc'e'f'} - \sqrt{a'b'd'cef}\}^2$, $(= \bar{E})$,

(2) $CD = \frac{1}{(x-x')^2}\{\sqrt{abec'd'f'} - \sqrt{a'b'e'cdf}\}^2$,

(14) $BE = \frac{1}{(x-x')^2}\{\sqrt{acdb'e'f'} - \sqrt{a'c'd'bef}\}^2$, $(= \bar{B})$,

(16) $BD = \frac{1}{(x-x')^2}\{\sqrt{aceb'd'f'} - \sqrt{a'c'e'bdf}\}^2$,

(15) $BC = \frac{1}{(x-x')^2}\{\sqrt{adeb'c'f'} - \sqrt{a'd'e'bcf}\}^2$,

(10) $AE = \frac{1}{(x-x')^2}\{\sqrt{bcda'e'f'} - \sqrt{b'c'd'aef}\}^2$, $(= \bar{A})$,

$$(12)\quad AD = \frac{1}{(x-x')^2}\{\sqrt{bcea'd'f'} - \sqrt{b'c'e'adf}\}^2,$$

$$(11)\quad AC = \frac{1}{(x-x')^2}\{\sqrt{bdea'c'f'} - \sqrt{b'd'e'acf}\}^2,$$

$$(5)\quad AB = \frac{1}{(x-x')^2}\{\sqrt{cdea'b'f'} - \sqrt{c'd'e'abf}\}^2,$$

where the numbers are in accordance with the foregoing scheme; viz. the scheme becomes

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
F	*CD*	*DE*	*CE*	*AB*	*E*	*C*	*D*	*B*	*AE*	*AC*	*AD*	*A*	*BE*	*BC*	*BD*
B	*AE*	*AC*	*AD*	*A*	*BE*	*BC*	*BD*	*F*	*CD*	*DE*	*CE*	*AB*	*E*	*C*	*D*
A	*BE*	*BC*	*BD*	*B*	*AE*	*AC*	*AD*	*AB*	*E*	*C*	*D*	*F*	*CD*	*DE*	*CE*
D	*C*	*E*	*AB*	*CE*	*DE*	*CD*	*F*	*BD*	*BC*	*BE*	*A*	*AD*	*AC*	*AE*	*B*
C	*D*	*AB*	*E*	*DE*	*CE*	*F*	*CD*	*BC*	*BD*	*A*	*BE*	*AC*	*AD*	*B*	*AE*
E	*AB*	*D*	*C*	*CD*	*F*	*CE*	*DE*	*DE*	*A*	*BD*	*BC*	*AE*	*B*	*AD*	*AC*.

There is of course the six A, B, C, D, E, F; for each of these is a linear function of $1, x+x', xx'$, and there is thus a linear relation between any four of them. It would at first sight appear that the remaining sixes were of two different forms, A, B, AB, CE, CD, DE, and F, A, AB, AC, AD, AE; but these are really identical, for taking *any* two letters E, F, the six is E, F, AE, BE, CE, DE, or, as this might be written, E, F, AEF, BEF, CEF, DEF, where AEF means $BCD\,.\,AEF$, etc.; and we thus obtain each of the remaining fifteen sixes. The six just referred to, viz. E, F, AE, BE, CE, DE, or changing the notation say $E, F, \bar{A}, \bar{B}, \bar{C}, \bar{D}$ as indicated in the table, thus represents any one of the sixes other than the rational six A, B, C, D, E, F; and there is no difficulty in actually finding each of the fifteen relations between four functions of the six in question, $E, F, \bar{A}, \bar{B}, \bar{C}, \bar{D}$. It is to be observed that every such function as $\bar{A}$ contains the same irrational part

$$\frac{2}{(x-x')^2}\sqrt{abcdefa'b'c'd'e'f'},$$

and that the linear relations involve therefore only the differences $\bar{A}-\bar{B}$, $\bar{A}-\bar{C}$, etc., which are rational. Proceeding to calculate these differences, we have for instance

$$\bar{C}-\bar{D} = \frac{1}{(x-x')^2}(cefa'b'd' + c'e'f'abd - defa'b'c' - d'e'f'abc) = \frac{1}{(x-x')^2}(cd'-c'd)(efa'b'-e'f'ab);$$

or, substituting for a, a', etc. their values $a-x$, $a-x'$, etc., we have

$$cd'-c'd = (x-x')(c-d),$$

$$efa'b'-e'f'ab = (x-x')\begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+b, & ab \\ 1, & e+f, & ef \end{vmatrix},$$

or say for shortness

$$= (x-x')\,[xx'abef].$$

We have therefore

$$\bar{C}-\bar{D}=(c-d)\,[xx'abef]\,;$$

and in like manner we obtain the equations

$$\bar{B}-\bar{C}=(b-c)\,[xx'adef],\quad \bar{A}-\bar{D}=(a-d)\,[xx'bcef],$$
$$\bar{C}-\bar{A}=(c-a)\,[xx'bdef\,],\quad \bar{B}-\bar{D}=(b-d)\,[xx'caef\,],$$
$$A-\bar{B}=(a-b)\,[xx'cdef\,],\quad \bar{C}-\bar{D}=(c-d)\,[xx'abef].$$

It is now easy to form the system of formulæ

E	F	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{D}$	
		$ae\,.\,af\,.\,bcd$	$-be\,.\,bf\,.\,cda$	$+\,ce\,.\,cf\,.\,dab$	$-\,de\,.\,df\,.\,abc$	$=0$
$ad\,.\,bf\,.\,cf$	$-\,ad\,.\,be\,.\,ce$	$+\,ef$			$-\,ef$	$=0$
$bd\,.\,cf\,.\,af$	$-\,bd\,.\,ce\,.\,ae$		$+\,ef$	$-\,ef$		$=0$
$cd\,.\,af\,.\,bf$	$-\,cd\,.\,ae\,.\,be$			$+\,ef$	$-\,ef$	$=0$
$bc\,.\,af\,.\,df$	$-\,bc\,.\,ae\,.\,de$		$+\,ef$	$-\,ef$		$=0$
$ca\,.\,bf\,.\,df$	$-\,ca\,.\,be\,.\,de$	$-\,ef$		$+\,ef$		$=0$
$ab\,.\,cf\,.\,df$	$-\,ab\,.\,ce\,.\,de$	$+\,ef$	$-\,ef$			$=0$
$-\,af\,.\,bcd$			$+\,be\,.\,cd$	$+\,ce\,.\,db$	$+\,de\,.\,bc$	$=0$
$-\,bf\,.\,cda$		$+\,ae\,.\,cd$		$+\,ce\,.\,da$	$+\,de\,.\,ac$	$=0$
$-\,cf\,.\,dab$		$+\,ae\,.\,bd$	$+\,be\,.\,da$		$+\,de\,.\,ab$	$=0$
$-\,df\,.\,abc$		$+\,ae\,.\,bc$	$+\,be\,.\,ca$	$+\,ce\,.\,ab$		$=0$
	$-\,ae\,.\,bcd$		$+\,bf\,.\,cd$	$+\,cf\,.\,db$	$+\,df\,.\,bc$	$=0$
	$-\,be\,.\,cda$	$+\,af\,.\,cd$		$+\,cf\,.\,da$	$+\,df\,.\,ac$	$=0$
	$-\,ce\,.\,dab$	$+\,af\,.\,bd$	$+\,bf\,.\,da$		$+\,df\,.\,ab$	$=0$
	$-\,df\,.\,abc$	$+\,af\,.\,bc$	$+\,bf\,.\,ca$	$+\,cf\,.\,ab$		$=0,$

where for shortness ab, ac, etc., are written to denote $a-b$, $a-c$, etc.; also abc, etc., to denote $(b-c)(c-a)(a-b)$, etc.: the equations contain all of them only the differences of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$; thus the first equation is equivalent to

$$ae\,.\,af\,.\,bcd\,(\bar{A}-\bar{D})-be\,.\,bf\,.\,cde\,(\bar{B}-\bar{D})+ce\,.\,cf\,.\,dab\,(\bar{C}-\bar{D})=0,$$

and so in other cases.

Cambridge, 14 *March*, 1877.