# 667.

## ON THE BICIRCULAR QUARTIC: ADDITION TO PROFESSOR CASEY'S MEMOIR "ON A NEW FORM OF TANGENTIAL EQUATION."

[From the *Philosophical Transactions of the Royal Society of London,* vol. CLXVII. Part II. (1877), pp. 441—460. Received January 24,—Read February 22, 1877.]

PROFESSOR CASEY communicated to me the MS. of the Memoir referred to, and he has permitted me to make to it the present Addition, containing further developments on the theory of the bicircular quartic.

Starting from his theory of the fourfold generation of the curve, Prof. Casey shows that there exist series of inscribed quadrilaterals $ABCD$ whereof the sides $AB$, $BC$, $CD$, $DA$ pass through the centres of the four circles of inversion respectively; or (as it is convenient to express it) the pairs of points $(A, B)$, $(B, C)$, $(C, D)$, $(D, A)$ belong to the four modes of generation respectively, and may be regarded as depending upon certain parameters (his $\theta$, $\theta'$, $\theta''$, $\theta'''$, or say) $\omega_1$, $\omega_2$, $\omega_3$, $\omega$ respectively, any three of these being in fact functions of the fourth. Considering a given quadrilateral $ABCD$, and giving to it an infinitesimal variation, we have four infinitesimal arcs $AA'$, $BB'$, $CC'$, $DD'$; these are differential expressions, $AA'$ and $BB'$ being of the form $M_1 d\omega_1$, $BB'$ and $CC'$ of the form $M_2 d\omega_2$, $CC'$ and $DD'$ of the form $M_3 d\omega_3$, $DD'$ and $AA'$ of the form $M d\omega$; or, what is the same thing, $AA'$ is expressible in the two forms $M d\omega$ and $M_1 d\omega_1$, $BB'$ in the two forms $M_1 d\omega_1$ and $M_2 d\omega_2$, &c., the identity of the two expressions for the same arc of course depending on the relation between the two parameters. But any such monomial expression $M d\omega$ of an arc $AA'$ would be of a complicated form, not obviously reducible to elliptic functions; Casey does not obtain these monomial expressions at all, but he finds geometrically monomial expressions for the differences and sum $BB' - AA'$, $CC' - BB'$, $DD' + CC'$, $DD' - AA'$ (they cannot be all of them differences), and thence a quadrinomial expression $AA' = N_1 d\omega_1 + N_2 d\omega_2 + N_3 d\omega_3 + N d\omega$ (his $ds' = \rho\, d\theta + \rho'\, d\theta' + \rho''\, d\theta'' + \rho'''\, d\theta'''$); and that without any explicit consideration of the relations which connect the parameters.

I propose to complete the analytical theory by establishing the monomial equations $AA' = Md\omega = M_1 d\omega_1$, &c., and the relations between the parameters $\omega$, $\omega_1$, $\omega_2$, $\omega_3$ which belong to an inscribed quadrilateral $ABCD$, so as to show what the process really is by which we pass from the monomial form to a quadrinomial form

$$AA' \text{ (or } dS) = Nd\omega + N_1 d\omega_1 + N_2 d\omega_2 + N_3 d\omega_3,$$

wherein each term is separately expressible as the differential of an elliptic integral; and further to develop the theory of the transformation to elliptic integrals. We require to establish for these purposes the fundamental formulæ in the theory of the bicircular quartic.

I remark that in the various formulæ $f$, $g$, $\theta$, $\theta_1$, $\theta_2$, $\theta_3$ are constants which enter only in the combinations $f+\theta$, $f-g$, $\theta_1-\theta$, $\theta_2-\theta$, $\theta_3-\theta$: that $X$, $Y$ are taken as current coordinates, and these letters, or the same letters with suffixes, are taken as coordinates of a point or points on the bicircular quartic: and that the letters $(x, y)$, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are used throughout as variable parameters, viz. we have

$$\begin{aligned}(f+\theta)\,x^2 + (g+\theta)\,y^2 &= 1,\\ (f+\theta_1)\,x_1^2 + (g+\theta_1)\,y_1^2 &= 1,\\ (f+\theta_2)\,x_2^2 + (g+\theta_2)\,y_2^2 &= 1,\\ (f+\theta_3)\,x_3^2 + (g+\theta_3)\,y_3^2 &= 1;\end{aligned}$$

so that $x, y = \dfrac{\cos\omega}{\sqrt{f+\theta}}, \dfrac{\sin\omega}{\sqrt{g+\theta}}$, are functions of a single parameter $\omega$, and similarly $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are functions of the parameters $\omega_1$, $\omega_2$, $\omega_3$ respectively. We sometimes use these or similar expressions of $(x, y)$, &c., as trigonometrical functions of a single parameter; but we more frequently retain the pair of quantities, considered as connected by an equation as above and so as equivalent to a single variable parameter.

*Formulæ for the fourfold generation of the Bicircular Quartic.* Art. Nos. 1 to 5.

1. We have four systems of a dirigent conic and circle of inversion, each giving rise to the same bicircular quartic: viz. the bicircular quartic is the envelope of a generating circle, having its centre on a dirigent conic, and cutting at right angles the corresponding circle of inversion; or, what is the same thing, it is the locus of the extremities of a chord of the generating circle, which chord passes through the centre of the circle of inversion, and cuts at right angles the tangent (at the centre of the generating circle) to the dirigent conic; the two extremities of the chord are thus inverse points in regard to the circle of inversion. The four systems are represented by letters without suffixes, or with the suffixes 1, 2, 3 respectively; and we say that the system, or mode of generation, is 0, 1, 2, or 3 accordingly.

2. The dirigent conics are confocal, and their squared semiaxes may therefore be represented by $f+\theta$, $g+\theta$: $f+\theta_1$, $g+\theta_1$: $f+\theta_2$, $g+\theta_2$: $f+\theta_3$, $g+\theta_3$, (which are, in

fact, functions of the five quantities $f+\theta,\ f-g,\ \theta_1-\theta,\ \theta_2-\theta,\ \theta_3-\theta)$; and we can in terms of these data express the equations as well of the dirigent conics as of the circles of inversion; viz. taking $X$, $Y$ as current coordinates, the equations are

$$\frac{X^2}{f+\theta}+\frac{Y^2}{g+\theta}=1,\ (X-\alpha)^2+(Y-\beta)^2-\gamma^2=0,\ \text{or}\ X^2+Y^2-2\alpha X-2\beta Y+k=0,$$

$$\frac{X^2}{f+\theta_1}+\frac{Y^2}{g+\theta_1}=1,\ (X-\alpha_1)^2+(Y-\beta_1)^2-\gamma_1^2=0,\ \text{or}\ X^2+Y^2-2\alpha_1 X-2\beta_1 Y+k_1=0,$$

$$\frac{X^2}{f+\theta_2}+\frac{Y^2}{g+\theta_2}=1,\ (X-\alpha_2)^2+(Y-\beta_2)^2-\gamma_2^2=0,\ \text{or}\ X^2+Y^2-2\alpha_2 X-2\beta_2 Y+k_2=0,$$

$$\frac{X^2}{f+\theta_3}+\frac{Y^2}{g+\theta_3}=1,\ (X-\alpha_3)^2+(Y-\beta_3)^2-\gamma_3^2=0,\ \text{or}\ X^2+Y^2-2\alpha_3 X-2\beta_3 Y+k_3=0,$$

where

$$\sqrt{\frac{f+\theta\,.\,f+\theta_1\,.\,f+\theta_2\,.\,f+\theta_3}{f-g}}=(f+\theta)\,\alpha=(f+\theta_1)\,\alpha_1=(f+\theta_2)\,\alpha_2=(f+\theta_3)\,\alpha_3,$$

$$\sqrt{\frac{g+\theta\,.\,g+\theta_1\,.\,g+\theta_2\,.\,g+\theta_3}{g-f}}=(g+\theta)\,\beta=(g+\theta_1)\,\beta_1=(g+\theta_2)\,\beta_2=(g+\theta_3)\,\beta_3,$$

$$f+\theta\ .\ g+\theta\ .\ \gamma^2=\theta-\theta_1\,.\,\theta-\theta_2\,.\,\theta-\theta_3,$$

$$f+\theta_1\,.\,g+\theta_1\,.\,\gamma_1^2=\theta_1-\theta\ .\ \theta_1-\theta_2\,.\,\theta_1-\theta_3,$$

$$f+\theta_2\,.\,g+\theta_2\,.\,\gamma_2^2=\theta_2-\theta\ .\ \theta_2-\theta_1\,.\,\theta_2-\theta_3,$$

$$f+\theta_3\,.\,g+\theta_3\,.\,\gamma_3^2=\theta_3-\theta\ .\ \theta_3-\theta_1\,.\,\theta_3-\theta_2,$$

$$f+g+\theta+\theta_1+\theta_2+\theta_3=k+2\theta=k_1+2\theta_1=k_2+2\theta_2=k_3+2\theta_3.$$

3. The geometrical relations between the dirigent conics and circles of inversion are all deducible from the foregoing formulæ; in particular, the conics are confocal, and as such intersect each two of them at right angles; the circles intersect each two of them at right angles. Considering a dirigent conic and the corresponding circle of inversion, the centres of the remaining three circles are conjugate points in regard as well to the first-mentioned conic, as to the first-mentioned circle; or, what is the same thing, they are the centres of the quadrangle formed by the intersections of the conic and circle.

4. The centre of the conics and the centres of the four circles lie on a rectangular hyperbola, having its asymptotes parallel to the axes of the conics. Given the centres of three of the circles (this determines the centre of the fourth circle) and also the centre of the conic, these four points determine a rectangular hyperbola (which passes also through the centre of the fourth circle); and the axes of the conics are then the lines through the centre, parallel to the asymptotes of the hyperbola.

5. The equation of the bicircular quartic may be expressed in the four forms

$$(X^2+Y^2-k\ )^2-4\,[(f+\theta\ )(X-\alpha\ )^2+(g+\theta\ )(Y-\beta\ )^2]=0,$$
$$(X^2+Y^2-k_1)^2-4\,[(f+\theta_1)(X-\alpha_1)^2+(g+\theta_1)(Y-\beta_1)^2]=0,$$
$$(X^2+Y^2-k_2)^2-4\,[(f+\theta_2)(X-\alpha_2)^2+(g+\theta_2)(Y-\beta_2)^2]=0,$$
$$(X^2+Y^2-k_3)^2-4\,[(f+\theta_3)(X-\alpha_3)^2+(g+\theta_3)(Y-\beta_3)^2]=0,$$

the equivalence of which is easily verified by means of the foregoing relations.

*Determination as to Reality.* Art. Nos. 6 and 7.

6. To fix the ideas, suppose that $f-g$ is positive; then in order that the centres of the four circles of inversion may be real, we must have $f+\theta.f+\theta_1.f+\theta_2.f+\theta_3$ positive, but $g+\theta.g+\theta_1.g+\theta_2.g+\theta_3$ negative; and this will be the case if $f+\theta$, $f+\theta_1$, $f+\theta_2$, $f+\theta_3$ are all positive, but $g+\theta$, $g+\theta_1$, $g+\theta_2$, $g+\theta_3$ one of them negative, and the other three positive. In reference to a figure which I constructed, I found it convenient to take $\theta_3$, $\theta_1$, $\theta_0$, $\theta_2$ to be in order of increasing magnitude: this being so, we have $f+\theta_3$ positive, $g+\theta_3$ negative; and the other like quantities $f+\theta_1$, $f+\theta_0$, $f+\theta_2$, $g+\theta_1$, $g+\theta_0$, $g+\theta_2$ all positive: we then have $\gamma_3^2$ and $\gamma_1^2$ each positive, $\gamma_0^2$ negative, $\gamma_2^2$ positive: viz. the conics and circles are

| | | | |
|---|---|---|---|
| Hyperbola | $H_3$, | corresponding to | real circle $C_3$, |
| Ellipse | $E_1$, | „ | real circle $C_1$, |
| „ | $E_0$, | „ | imaginary circle $C_0$, (viz. the radius is a pure imaginary), |
| „ | $E_2$, | „ | real circle $C_2$, |

and the confocal ellipses $E_1$, $E_0$, $E_2$ are in order of increasing magnitude. The centre $C_0$ is here a point within the triangle formed by the remaining three centres $C_1$, $C_2$, $C_3$. It will be convenient to adopt throughout the foregoing determination as to reality.

7. It may be remarked that a circle of a pure imaginary radius $\gamma$, $=i\lambda$, where $\lambda$ is real, may be indicated by means of the concentric circle radius $\lambda$, which is the concentric orthotomic circle; and that a circle which cuts at right angles the original circle cuts diametrally (that is, at the extremities of a diameter) the substituted circle radius $\lambda$; we have thus a real construction in relation to a circle of inversion of pure imaginary radius.

*Investigation of dS.* Art. Nos. 8 to 17.

8. The coordinates of a point on the dirigent conic $\dfrac{X^2}{f+\theta}+\dfrac{Y^2}{g+\theta}=1$ may be taken to be $(f+\theta)x$ $(g+\theta)y$: and we hence prove as follows the fundamental

theorem for the generation of the bicircular quartic. Consider the generating circle, centre $(f+\theta)x$, $(g+\theta)y$, which cuts at right angles the circle of inversion

$$(X-\alpha)^2+(Y-\beta)^2=\gamma^2.$$

If for a moment the radius is called $\delta$, then the equation of the generating circle is

$$(X-\overline{f+\theta}x)^2+(Y-\overline{g+\theta}y)^2=\delta^2;$$

the condition for the intersection at right angles is

$$(\alpha-\overline{f+\theta}x)^2+(\beta-\overline{g+\theta}y)^2=\gamma^2+\delta^2,$$

and hence eliminating $\delta^2$, the equation of the generating circle is

$$X^2+Y^2-k-2(X-\alpha)(f+\theta)x-2(Y-\beta)(g+\theta)y=0;$$

and considering herein $x$, $y$ as variable parameters connected by the foregoing equation $(f+\theta)x^2+(g+\theta)y^2=1$, we have as the envelope of this circle the required bicircular quartic.

9. It is convenient to write $R=\frac{1}{2}(X^2+Y^2-k)$. The equation then is

$$R-(X-\alpha)(f+\theta)x-(Y-\beta)(g+\theta)y=0;$$

the derived equation is

$$(X-\alpha)(f+\theta)\,dx+(Y-\beta)(g+\theta)\,dy=0;$$

and from these two equations, together with the equation in $(x, y)$ and its derivative, we find $X-\alpha=Rx$, $Y-\beta=Ry$; from these last equations, and the equations $R=\frac{1}{2}(X^2+Y^2-k)$, $(f+\theta)x^2+(g+\theta)y^2=1$, eliminating $x$, $y$, $R$, we have

$$(f+\theta)(X-\alpha)^2+(g+\theta)(Y-\beta)^2=R^2,$$

that is,

$$(X^2+Y^2-k)^2-4[(f+\theta)(X-\alpha)^2+(g+\theta)(Y-\beta)^2]=0,$$

the required equation of the bicircular quartic.

10. We have thus $X-\alpha=Rx$, $Y-\beta=Ry$, as the equations which serve to determine the bicircular quartic: if from these equations, together with $R=\frac{1}{2}(X^2+Y^2-k)$, we eliminate $X$ and $Y$, we have $R$ expressed as a function of $x$, $y$; and thence also $X$, $Y$ expressed in terms of $x$, $y$; that is, in effect the coordinates $X$, $Y$ of a point of the bicircular quartic expressed as functions of a single variable parameter. The process gives $2R+k=(\alpha+Rx)^2+(\beta+Ry)^2$, viz. this is

$$R^2(x^2+y^2)-2(1-\alpha x-\beta y)R+\gamma^2=0,$$

or putting for shortness

$$\Omega=(1-\alpha x-\beta y)^2-\gamma^2(x^2+y^2),$$

this is

$$R=\frac{1-\alpha x-\beta y+\sqrt{\Omega}}{x^2+y^2},$$

or say the two values are

$$R = \frac{1 - \alpha x - \beta y + \sqrt{\Omega}}{x^2 + y^2}, \quad R' = \frac{1 - \alpha x - \beta y - \sqrt{\Omega}}{x^2 + y^2};$$

to preserve the generality it is proper to consider $\sqrt{\Omega}$ as denoting a determinate value (the positive or the negative one, as the case may be) of the radical.

11. Considering the root $R'$, we have $X = \alpha + R'x$, $Y = \beta + R'y$; from these equations we obtain

$$dX = R'dx + x\,dR',$$
$$dY = R'dy + y\,dR'.$$

But from the equation for $R'$ we have

$$[R'(x^2 + y^2) - (1 - \alpha x - \beta y)]\,dR' + R'^2(x\,dx + y\,dy) + R'(\alpha\,dx + \beta\,dy) = 0,$$

that is,

$$-\sqrt{\Omega}\,dR' + R'(Xdx + Ydy) = 0;$$

whence

$$dX = R'dx + \frac{R'x}{\sqrt{\Omega}}(Xdx + Ydy),$$

$$dY = R'dy + \frac{R'y}{\sqrt{\Omega}}(Xdx + Ydy).$$

12. The differentials $dx$, $dy$ can be expressed in terms of a single differential $d\omega$, viz. writing

$$x = \frac{\cos\omega}{\sqrt{f+\theta}}, \quad y = \frac{\sin\omega}{\sqrt{g+\theta}},$$

and

$$\Theta = (f+\theta)(g+\theta),$$

then we have

$$dx = -\frac{g+\theta}{\sqrt{\Theta}}\,y\,d\omega, \quad dy = \frac{f+\theta}{\sqrt{\Theta}}\,x\,d\omega.$$

It is to be observed that, when the dirigent conic is an ellipse, $\omega$ is a real angle, and $\Theta$ is positive (whence also $\sqrt{\Theta}$ is real and positive); but when the dirigent conic is a hyperbola, $\omega$ is imaginary, and $\Theta$ is negative; we have, however, in either case

$$dx^2 + dy^2 = \frac{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}{\Theta}\,d\omega^2,$$

and we may therefore write

$$\frac{d\omega}{\sqrt{\Theta}} = \frac{ds}{\sqrt{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}},$$

where $\sqrt{(f+\theta)^2 x^2 + (g+\theta)^2 y^2}$ is positive; $ds$ is the increment of arc on the conic $(f+\theta)x^2 + (g+\theta)y^2 = 1$, this arc being measured in a determinate sense, and therefore $ds$ being positive or negative as the case may be: $\dfrac{d\omega}{\sqrt{\Theta}}$ has thus a real positive or negative value, even when $\omega$ is imaginary, and it is convenient to retain it in the formulæ.

13. It may further be noticed that, if $\upsilon$ denote the inclination to the axis of $x$ of the tangent to the dirigent conic at the point $\sqrt{f+\theta}\cos\omega$, $\sqrt{g+\theta}\sin\omega$, where $\upsilon$ is Casey's $\theta$, then

$$x=\frac{\cos\upsilon}{\sqrt{U}},\quad y=\frac{\sin\upsilon}{\sqrt{U}},\ \text{ where }\ U=(f+\theta)\cos^2\upsilon+(g+\theta)\sin^2\upsilon,$$

viz. we have

$$\frac{\cos\omega}{\sqrt{f+\theta}}=\frac{\cos\upsilon}{U},\quad \frac{\sin\omega}{\sqrt{g+\theta}}=\frac{\sin\upsilon}{U},$$

giving, as is easily verified, $\dfrac{d\omega}{\sqrt{\Theta}}=\dfrac{d\upsilon}{U}$; we have therefore

$$\frac{d\omega}{(x^2+y^2)\sqrt{\Theta}}=\frac{d\upsilon}{\upsilon\,(x^2+y^2)}=d\upsilon,$$

or

$$\frac{d\omega}{\sqrt{\Theta}}=(x^2+y^2)\,d\upsilon,$$

which is another interpretation of $\dfrac{d\omega}{\sqrt{\Theta}}$.

14. Substituting for $dx$, $dy$ their values, the formulæ become

$$dX=\frac{R'}{\sqrt{\Theta}}\left\{-(g+\theta)\,y+\frac{x}{\sqrt{\Omega}}\big(-(g+\theta)\,yX+(f+\theta)\,xY\big)\right\}d\omega,$$

$$dY=\frac{R'}{\sqrt{\Theta}}\left\{\ \ (f+\theta)\,x+\frac{y}{\sqrt{\Omega}}\big(-(g+\theta)\,yX+(f+\theta)\,xY\big)\right\}d\omega.$$

We have

$$\begin{aligned}xX+yY&=\alpha x+\beta y+(x^2+y^2)\,R'\\&=1-\sqrt{\Omega},\end{aligned}$$

that is,

$$1=\frac{1-xX-yY}{\sqrt{\Omega}};$$

and consequently the foregoing expressions of $dX$, $dY$ become

$$\begin{aligned}dX&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(g+\theta)\,y\,(xX+yY-1)+x\big(-(g+\theta)\,yX+(f+\theta)\,xY\big)\right\}\\&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(\overline{g+\theta}\,y^2+\overline{f+\theta}\,x^2)\,Y-(g+\theta)\,y\right\},\\dY&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(f+\theta)\,x\,(1-xX-yY)+y\,\big(-(g+\theta)\,yX+(f+\theta)\,xY\big)\right\}\\&=\frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\left\{(f+\theta)\,x-\big((f+\theta)\,x^2+(g+\theta)\,y^2\big)\,X\right\},\end{aligned}$$

or finally

$$dX = \frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{Y - (g+\theta)y\} = \frac{R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{R'y + \beta - (g+\theta)y\},$$

$$dY = \frac{-R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{X - (f+\theta)x\} = \frac{-R'd\omega}{\sqrt{\Theta}\sqrt{\Omega}}\{R'x + \alpha - (f+\theta)x\}.$$

15. We have

$$(R'x + \alpha - \overline{f+\theta}\,x)^2 + (R'y + \beta - \overline{g+\theta}\,y)^2$$

$$= R'^2(x^2+y^2) - 2R'(1 - \alpha x - \beta y)$$

$$+ (\alpha - \overline{f+\theta}\,x)^2 + (\beta - \overline{g+\theta}\,y)^2;$$

viz. this is

$$= (\alpha - \overline{f+\theta}\,x)^2 + (\beta - \overline{g+\theta}\,y)^2 - \gamma^2$$

$$= \delta^2, \text{ the radius of the generating circle.}$$

Hence if $dS, = \sqrt{dX^2 + dY^2}$, be the element of arc of the bicircular quartic, this element being taken to be positive, we have

$$dS = \frac{\epsilon' R'\delta\, d\omega}{\sqrt{\Omega}\sqrt{\Theta}},$$

where $\epsilon'$ denotes a determinate sign, + or −, as the case may be.

16. I stop to consider the geometrical interpretation; introducing $dv$, the formula may be written

$$dS = \frac{\epsilon' . R'(x^2+y^2)\,\delta\, dv}{\sqrt{\Omega}},$$

and we have $(x^2+y^2)R' = 1 - \alpha x - \beta y - \sqrt{\Omega}$, or

$$\frac{(x^2+y^2)R'}{\sqrt{\Omega}} = \frac{1 - \alpha x - \beta y}{\sqrt{\Omega}} - 1.$$

Here $\dfrac{1 - \alpha x - \beta y}{\sqrt{x^2+y^2}}$ is the perpendicular from the centre of the circle of inversion upon the tangent to the dirigent conic, and $\dfrac{\sqrt{\Omega}}{\sqrt{x^2+y^2}}$ is the half-chord which this perpendicular forms with the generating circle. Hence $\dfrac{1 - \alpha x - \beta y}{\sqrt{\Omega}} - 1 =$ (perpendicular − half-chord) ÷ half-chord, the numerator being in fact the distance of the element $dS$ (or point $X$, $Y$) from the centre of inversion: the formula thus is

$$dS = \pm\frac{\rho . \delta}{\frac{1}{2}c}\,dv,$$

where $\delta$ is the radius of the generating circle, $\rho$ the distance of the element from the centre of the circle of inversion, and $c$ the chord which this distance forms with the generating circle. If we consider the two points on the generating circle, and write $dS'$ for the element at the other point, then we have

$$(dS \pm dS') = \pm \frac{(\rho - \rho')\,\delta\,dv}{\frac{1}{2}c} = 2\delta\,dv,$$

which is Casey's formula $ds' - ds = 2\rho\,d\phi$ (273).

17. The foregoing forms of $dX$, $dY$ are those which give most directly the required value of $dS$: but I had previously obtained them in a different form. Writing

$$\Delta = \beta x - \alpha y + (f - g)\,xy,$$

then

$$x\Delta = \beta x^2 - \alpha xy + [(f+\theta)\,x^2 - (g+\theta)\,y^2];$$

or since

$$(f+\theta)\,x^2 = 1 - (g+\theta)\,y^2,$$

this is

$$\begin{aligned} x\Delta = \beta x^2 - \alpha xy + [1 - (g+\theta)(x^2+y^2)] &= y(1 - \alpha x - \beta y) + (x^2+y^2)(\beta - (g+\theta)\,y) \\ &= (x^2+y^2)\{yR' + \beta - (g+\theta)\,y\} + y\sqrt{\Omega}, \end{aligned}$$

that is,

$$x\Delta - y\sqrt{\Omega} = (x^2+y^2)\{yR' + \beta - (g+\theta)\,y\};$$

and similarly

$$-y\Delta - x\sqrt{\Omega} = (x^2+y^2)\{xR' + \alpha - (f+\theta)\,x\}.$$

We have therefore

$$dX = \frac{R'd\omega}{(x^2+y^2)\sqrt{\Theta}\sqrt{\Omega}}(x\Delta - y\sqrt{\Omega}),$$

$$dY = \frac{R'd\omega}{(x^2+y^2)\sqrt{\Theta}\sqrt{\Omega}}(y\Delta + x\sqrt{\Omega}),$$

and thence a value of $dS$ which, compared with the former value, gives

$$\Omega + \Delta^2 = (x^2+y^2)\,\delta^2,$$

an equation which may be verified directly.

### *Formulæ for the Inscribed Quadrilateral.* Art. Nos. 18 to 22.

18. We consider on the curve four points, $A$, $B$, $C$, $D$, forming a quadrilateral, $ABCD$. The coordinates are taken to be $(X, Y)$, $(X_1, Y_1)$, $(X_2, Y_2)$, $(X_3, Y_3)$ respectively. It is assumed that $(A, B)$, $(B, C)$, $(C, D)$, $(D, A)$ belong to the generations 1, 2, 3, 0, and depend on the parameters $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x, y)$ respectively.

We write

$$\begin{aligned} \Omega\ &= (1 - \alpha x - \beta y)^2 - \gamma^2(x^2+y^2), \\ \Omega_1 &= (1 - \alpha_1 x_1 - \beta_1 y_1)^2 - \gamma_1^2(x_1^2+y_1^2), \\ \Omega_2 &= (1 - \alpha_2 x_2 - \beta_2 y_2)^2 - \gamma_2^2(x_2^2+y_2^2), \\ \Omega_3 &= (1 - \alpha_3 x_3 - \beta_3 y_3)^2 - \gamma_3^2(x_3^2+y_3^2); \end{aligned}$$

and then, $\sqrt{\Omega}$ denoting as above a determinate value, positive or negative as the case may be, of the radical, and similarly $\sqrt{\Omega_1}$, $\sqrt{\Omega_2}$, $\sqrt{\Omega_3}$ denoting determinate values of these radicals respectively, each radical having its own sign at pleasure, we further write

$$\begin{aligned}
(x^2+y^2)R' &= 1-\alpha x-\beta y-\sqrt{\Omega}, & (x_1^2+y_1^2)R_1 &= 1-\alpha_1x_1-\beta_1y_1+\sqrt{\Omega_1},\\
(x_1^2+y_1^2)R_1' &= 1-\alpha_1x_1-\beta_1y_1-\sqrt{\Omega_1}, & (x_2^2+y_2^2)R_2 &= 1-\alpha_2x_2-\beta_2y_2+\sqrt{\Omega_2},\\
(x_2^2+y_2^2)R_2' &= 1-\alpha_2x_2-\beta_2y_2-\sqrt{\Omega_2}, & (x_3^2+y_3^2)R_3 &= 1-\alpha_3x_3-\beta_3y_3+\sqrt{\Omega_3},\\
(x_3^2+y_3^2)R_3' &= 1-\alpha_3x_3-\beta_3y_3-\sqrt{\Omega_3}, & (x^2+y^2)R &= 1-\alpha x-\beta y+\sqrt{\Omega};
\end{aligned}$$

and this being so, we must have

$$\begin{aligned}
X &= \alpha+R'x=\alpha_1+R_1x_1, & Y &= \beta+R'y=\beta_1+R_1y_1, & R' &= \tfrac{1}{2}(X^2+Y^2-k), & R_1 &= \tfrac{1}{2}(X^2+Y^2-k_1),\\
X_1 &= \alpha_1+R_1'x_1=\alpha_2+R_2x_2, & Y_1 &= \beta_1+R_1'y_1=\beta_2+R_2y_2, & R_1' &= \tfrac{1}{2}(X_1^2+Y_1^2-k_1), & R_2 &= \tfrac{1}{2}(X_1^2+Y_1^2-k_2),\\
X_2 &= \alpha_2+R_2'x_2=\alpha_3+R_3x_3, & Y_2 &= \beta_2+R_2'y_2=\beta_3+R_3y_3, & R_2' &= \tfrac{1}{2}(X_2^2+Y_2^2-k_2), & R_3 &= \tfrac{1}{2}(X_2^2+Y_2^2-k_3),\\
X_3 &= \alpha_3+R_3'x_3=\alpha+Rx, & Y_3 &= \beta_3+R_3'y_3=\beta+Ry, & R_3' &= \tfrac{1}{2}(X_3^2+Y_3^2-k_3), & R &= \tfrac{1}{2}(X_3^2+Y_3^2-k);
\end{aligned}$$

and then from the values of $X$, $Y$, $R'$, $R$, we have

$$\begin{aligned}
\alpha-\alpha_1+R'x-R_1x_1 &= 0,\\
\beta-\beta_1+R'y-R_1y_1 &= 0,\\
(\theta-\theta_1)+R'-R_1 &= 0,
\end{aligned}$$

giving

$$(\beta-\beta_1)(x-x_1)-(\alpha-\alpha_1)(y-y_1)+(\theta-\theta_1)(xy_1-x_1y)=0;$$

and similarly

$$\begin{aligned}
(\beta_1-\beta_2)(x_1-x_2)-(\alpha_1-\alpha_2)(y_1-y_2)+(\theta_1-\theta_2)(x_1y_2-x_2y_1) &= 0,\\
(\beta_2-\beta_3)(x_2-x_3)-(\alpha_2-\alpha_3)(y_2-y_3)+(\theta_2-\theta_3)(x_2y_3-x_3y_2) &= 0,\\
(\beta_3-\beta)(x_3-x)-(\alpha_3-\alpha)(y_3-y)+(\theta_3-\theta)(x_3y-xy_3) &= 0,
\end{aligned}$$

which are the relations connecting the parameters $(x, y)$, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ of the quadrilateral.

19. We have thus apparently four equations for the determination of four quantities, or the number of quadrilaterals would be finite; but if from the first and second equations we eliminate $(x_1, y_1)$, and if from the third and fourth equations we eliminate $(x_3, y_3)$, we find in each case the same relation between $(x, y)$, $(x_2, y_2)$, viz. this is found to be

$$\Omega\Omega_2=(1-\alpha x_2-\beta y_2)^2(1-\alpha_2x-\beta_2y)^2;$$

and we have thus the singly infinite series of quadrilaterals. We have, of course, between $(x_1, y_1)$, $(x_3, y_3)$ the like relation,

$$\Omega_1\Omega_3=(1-\alpha_1x_3-\beta_1y_3)^2(1-\alpha_3x_1-\beta_3y_1)^2.$$

20. The relation between $(x, y)$, $(x_1, y_1)$ may be expressed also in the two forms:

$$1-\alpha\,(x+x_1)-\beta\,(y+y_1)+(f+\theta_1)\,xx_1+(g+\theta_1)\,yy_1+\frac{x^2+y^2}{xy_1-x_1y}(\overline{\alpha-\alpha_1}y_1-\overline{\beta-\beta_1}x_1)=0,$$

$$1-\alpha_1\,(x+x_1)-\beta_1\,(y+y_1)+(f+\theta)\,xx_1+(g+\theta)\,yy_1+\frac{x_1^2+y_1^2}{x_1y-xy_1}(\overline{\alpha_1-\alpha}y-\overline{\beta_1-\beta}x)=0.$$

In fact, the first of these equations is

$$\{1+(f+\theta_1)\,xx_1+(g+\theta_1)\,yy_1\}\,(xy_1-x_1y)-\{\alpha\,(x+x_1)+\beta\,(y+y_1)\}\,(xy_1-x_1y)$$
$$+\{(\alpha-\alpha_1)\,y_1-(\beta-\beta_1)\,x_1\}\,(x^2+y^2)=0,$$

which, by virtue of the original form of relation, is

$$-\{1+(f+\theta_1)\,xx_1+(g+\theta_1)\,yy_1\}\,\frac{(\beta-\beta_1)\,(x-x_1)-(\alpha-\alpha_1)\,(y-y_1)}{\theta-\theta_1}$$
$$-\{\alpha\,(x+x_1)+\beta\,(y+y_1)\}\,(xy_1-x_1y)+\{(\alpha-\alpha_1)\,y_1-(\beta-\beta_1)\,x_1\}\,(x^2+y^2)=0;$$

or, in the first term, writing

$$-\frac{\beta-\beta_1}{\theta-\theta_1}=\frac{\beta}{g+\theta_1},\quad \frac{\alpha-\alpha_1}{\theta-\theta_1}=\frac{-\alpha}{f+\theta_1},$$

and in the third term

$$\alpha-\alpha_1=-\frac{(\theta-\theta_1)\,\alpha}{f+\theta_1},\quad -(\beta-\beta_1)=\frac{(\theta-\theta_1)\,\beta}{g+\theta_1},$$

this is

$$(1+(f+\theta_1)\,xx_1+(g+\theta_1)\,yy_1)\left(\frac{\beta\,(x-x_1)}{g+\theta_1}-\frac{\alpha\,(y-y_1)}{f+\theta_1}\right)$$
$$-\{\alpha\,(x+x_1)+\beta\,(y+y_1)\}\,(xy_1-x_1y)-\left\{\frac{\alpha\,(\theta-\theta_1)}{f+\theta_1}y_1-\frac{\beta\,(\theta-\theta_1)}{g+\theta_1}x_1\right\}(x^2+y^2)=0.$$

In this equation the coefficients of $\alpha$ and of $\beta$ are separately $=0$: in fact, the coefficient of $\beta$ is

$$\frac{x-x_1}{g+\theta_1}+\frac{f+\theta_1}{g+\theta_1}xx_1(x-x_1)+(x-x_1)\,yy_1-(y+y_1)\,(xy_1-x_1y)+\frac{\theta-\theta_1}{g+\theta_1}x_1\,(x^2+y^2)$$
$$=\frac{x}{g+\theta_1}\{1-(f+\theta_1)\,x_1^2-(g+\theta_1)\,y_1^2\}-\frac{x_1}{g+\theta_1}\{1-(f+\theta)\,x^2-(g+\theta)\,y^2\}=0;$$

and similarly the coefficient of $\alpha$ is $=0$.

And in like manner the second equation may be verified.

21. The two equations are:

$$1-\alpha x-\beta y-(x^2+y^2)\,R'=\alpha x_1+\beta y_1-(f+\theta_1)\,xx_1-(g+\theta_1)\,yy_1,$$

$$1-\alpha_1x_1-\beta_1y_1-(x_1^2+y_1^2)\,R_1=\alpha_1x+\beta_1y-(f+\theta)\,xx_1-(g+\theta)\,yy_1;$$

or, substituting for $R'$ and $R_1$ their values, these are

$$\sqrt{\Omega}=\alpha x_1+\beta y_1-(f+\theta_1)\,xx_1-(g+\theta_1)\,yy_1,\quad \sqrt{\Omega_1}=-\alpha_1x-\beta_1y+(f+\theta)\,xx_1+(g+\theta)\,yy_1;$$

and similarly

$$\sqrt{\Omega_1} = \alpha_1 x_2 + \beta_1 y_2 - (f+\theta_2)\, x_1 x_2 - (g+\theta_2)\, y_1 y_2, \quad \sqrt{\Omega_2} = -\alpha_2 x_1 - \beta_2 y_1 + (f+\theta_1)\, x_1 x_2 + (g+\theta_1)\, y_1 y_2$$

$$\sqrt{\Omega_2} = \alpha_2 x_3 + \beta_2 y_3 - (f+\theta_3)\, x_2 x_3 - (g+\theta_3)\, y_2 y_3, \quad \sqrt{\Omega_3} = -\alpha_3 x_2 - \beta_3 y_2 + (f+\theta_2)\, x_2 x_3 + (g+\theta_2)\, y_2 y_3,$$

$$\sqrt{\Omega_3} = \alpha_3 x + \beta_3 y - (f+\theta)\, x_3 x - (g+\theta)\, y_3 y, \quad \sqrt{\Omega} = -\alpha x_3 - \beta y_3 + (f+\theta_3)\, x_3 x + (g+\theta_3)\, y_3 y.$$

Differentiating the equation

$$(\beta-\beta_1)(x-x_1) - (\alpha-\alpha_1)(y-y_1) + (\theta-\theta_1)(xy_1 - x_1 y) = 0,$$

we have

$$[(\beta-\beta_1) + (\theta-\theta_1)\, y_1]\, dx - [(\alpha-\alpha_1) + (\theta-\theta_1)\, x_1]\, dy$$
$$- [(\beta-\beta_1) + (\theta-\theta_1) y]\, dx_1 + [(\alpha-\alpha_1) + (\theta-\theta_1)\, x]\, dy_1 = 0;$$

and writing herein

$$dx = -\frac{(g+\theta)}{\sqrt{\Theta}}\, y\, d\omega, \quad dx_1 = \frac{-(g+\theta_1)}{\sqrt{\Theta_1}}\, y_1\, d\omega_1,$$

$$dy = \frac{(f+\theta)}{\sqrt{\Theta}}\, x\, d\omega, \quad dy_1 = \frac{(f+\theta_1)}{\sqrt{\Theta_1}}\, x_1\, d\omega_1,$$

we find

$$-\frac{d\omega}{\sqrt{\Theta}}\{(g+\theta)(\beta-\beta_1)\, y + (f+\theta)(\alpha-\alpha_1)\, x + (\theta-\theta_1)((f+\theta)\, xx_1 + (g+\theta)\, yy_1)\}$$

$$+\frac{d\omega_1}{\sqrt{\Theta_1}}\{(g+\theta_1)(\beta-\beta_1)\, y_1 + (f+\theta_1)(\alpha-\alpha_1)\, x_1 + (\theta-\theta_1)((f+\theta_1)\, xx_1 + (g+\theta_1)\, yy_1)\} = 0;$$

viz., dividing by $\theta-\theta_1$, this becomes

$$-\sqrt{\Omega_1}\frac{d\omega}{\sqrt{\Theta}} - \sqrt{\Omega}\frac{d\omega_1}{\sqrt{\Theta_1}} = 0, \text{ that is, } \frac{d\omega}{\sqrt{\Theta}\sqrt{\Omega}} + \frac{d\omega_1}{\sqrt{\Theta_1}\sqrt{\Omega_1}} = 0;$$

or, completing the system, we have

$$\frac{d\omega}{\sqrt{\Theta}\sqrt{\Omega}} = \frac{-d\omega_1}{\sqrt{\Theta_1}\sqrt{\Omega_1}} = \frac{d\omega_2}{\sqrt{\Theta_2}\sqrt{\Omega_2}} = \frac{-d\omega_3}{\sqrt{\Theta_3}\sqrt{\Omega_3}},$$

which are the differential relations between the parameters $\omega$, $\omega_1$, $\omega_2$, $\omega_3$, or $(x, y)$, $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$.

22. From the equations $X = \alpha + R'x$, $Y = \beta + R'y$, we found

$$dX = \frac{R'd\omega}{\sqrt{\Omega}\sqrt{\Theta}}\{Y - (g+\theta)\, y\},$$

$$dY = \frac{R'd\omega}{\sqrt{\Omega}\sqrt{\Theta}}\{X - (f+\theta) x\};$$

the new values, $X = \alpha_1 + R_1 x_1$ and $Y = \beta_1 + R_1 y_1$, give in like manner

$$dX = -\frac{R_1 d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}\{Y - (g+\theta_1)\, y_1\},$$

$$dY = -\frac{R_1 d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}\{X - (f+\theta_1)\, x_1\};$$

in virtue of the relation just found between $d\omega$ and $d\omega_1$, these two sets of values will agree together if only

$$R'\{Y-(g+\theta)y\}=R_1\{Y-(g+\theta_1)y_1\},$$
$$R'\{X-(f+\theta)x\}=R_1\{X-(f+\theta_1)x_1\}.$$

These are easily verified: the first is

$$R'Y-(g+\theta)(Y-\beta)=(R'-\theta+\theta_1)Y-(g+\theta_1)(Y-\beta_1),$$

viz. this is $(g+\theta)\beta-(g+\theta_1)\beta_1=0$, which is right; and similarly the second equation gives $(f+\theta)\alpha-(f+\theta_1)\alpha_1=0$, which is right.

From the first values of $dX$, $dY$, we have, as above,

$$dS=\frac{\epsilon' R'\delta\, d\omega}{\sqrt{\Omega}\sqrt{\Theta}};$$

and the second values give in like manner

$$dS=\frac{\epsilon_1 R_1\delta_1\, d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},$$

where $\epsilon_1$ is $=\pm 1$. It will be observed that we have in effect, by means of the relation $(\beta-\beta_1)(x-x_1)-(\alpha-\alpha_1)(y-y_1)+(\theta-\theta_1)(xy_1-x_1y)=0$, proved the identity of the two values of $dS$.

Considering the quadrilateral $ABCD$, and giving it an infinitesimal variation, so as to change it into $A'B'C'D'$, then $dS$ is the element of arc $AA'$; and writing in like manner $dS_1$, $dS_2$, $dS_3$ for the elements of arc $BB'$, $CC'$, $DD'$, we have, of course, a like pair of values for each of the elements $dS_1$, $dS_2$, $dS_3$.

*Formulæ for the elements of Arc $dS$, $dS_1$, $dS_2$, $dS_3$.* Art. Nos. 23 to 27.

23. The formulæ are

$$dS=\epsilon' R'\delta\,\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}}=\epsilon_1 R_1\delta_1\frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},$$

$$dS_1=\epsilon_1' R_1'\delta_1\frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}=\epsilon_2 R_2\delta_2\frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}},$$

$$dS_2=\epsilon_2' R_2'\delta_2\frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}}=\epsilon_3 R_3\delta_3\frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}},$$

$$dS_3=\epsilon_3' R_3'\delta_3\frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}}=\epsilon R\,\delta\,\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}},$$

where the $\epsilon$'s each denote $\pm 1$. Supposing as above that $\gamma^2$ is negative, but that $\gamma_1^2$, $\gamma_2^2$, $\gamma_3^2$ are positive; then $R'$, $R$ have opposite signs: but $R_1'$, $R_1$ have the same sign, as have also $R_2'$ and $R_2$, and $R_3'$ and $R_3$. We may take $\delta$, $\delta_1$, $\delta_2$, and $\delta_3$ as each of them positive: the signs of

$$\frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}}, \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}}, \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}}, \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}} \text{ are } +, -, +, -, \text{ or } -, +, -, +:$$

hence to make $dS$, $dS_1$, $dS_2$, $dS_3$ all positive,

$$\epsilon', \quad \epsilon_1', \quad \epsilon_2', \quad \epsilon_3', \quad \epsilon_1, \quad \epsilon_2, \quad \epsilon_3, \quad \epsilon,$$

must have either the signs of

$$R', \ -R_1', \ R_2', \ -R_3', \ -R_1, \ R_2, \ -R_3, \ R,$$

or else the reverse signs: hence in either case $\epsilon' = -\epsilon$, $\epsilon_1' = \epsilon_1$, $\epsilon_2' = \epsilon_2$, $\epsilon_3' = \epsilon_3$; or the equations are

$$dS = -\epsilon R'\delta \frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}} = \epsilon_1 R_1 \delta_1 \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}},$$

$$dS_1 = \epsilon_1 R_1' \delta_1 \frac{d\omega_1}{\sqrt{\Omega_1}\sqrt{\Theta_1}} = \epsilon_2 R_2 \delta_2 \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}},$$

$$dS_2 = \epsilon_2 R_2' \delta_2 \frac{d\omega_2}{\sqrt{\Omega_2}\sqrt{\Theta_2}} = \epsilon_3 R_3 \delta_3 \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}},$$

$$dS_3 = \epsilon_3 R_3' \delta_3 \frac{d\omega_3}{\sqrt{\Omega_3}\sqrt{\Theta_3}} = \epsilon R \delta \frac{d\omega}{\sqrt{\Omega}\sqrt{\Theta}}.$$

24. But we have $R' - R = \dfrac{-2\sqrt{\Omega}}{x^2 + y^2}$, &c.; and hence, putting for shortness

$$\frac{\delta}{(x^2+y^2)\sqrt{\Theta}}, \ \frac{\delta_1}{(x_1^2+y_1^2)\sqrt{\Theta_1}}, \ \frac{\delta_2}{(x_2^2+y_2^2)\sqrt{\Theta_2}}, \ \frac{\delta_3}{(x_3^2+y_3^2)\sqrt{\Theta_3}} = P, \ P_1, \ P_2, \ P_3,$$

$$\begin{aligned} dS + dS_3 &= +2\epsilon P \, d\omega, \\ dS_1 - dS &= -2\epsilon_1 P_1 d\omega_1, \\ dS_2 - dS_1 &= -2\epsilon_2 P_2 d\omega_2, \\ dS_3 - dS &= -2\epsilon_3 P_3 d\omega_3, \end{aligned}$$

and consequently

$$\begin{aligned} dS &= \epsilon P d\omega + \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3, \\ dS_1 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3, \\ dS_2 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 - \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3, \\ dS_3 &= \epsilon P d\omega - \epsilon_1 P_1 d\omega_1 - \epsilon_2 P_2 d\omega_2 - \epsilon_3 P_3 d\omega_3, \end{aligned}$$

which are the required formulæ for the elements of arc.

25. The determination of the signs has been made by means of the particular figure; but it is easy to see that the pairs of terms could not for instance be $dS - dS_3$, $dS_1 - dS$, $dS_2 - dS_1$, $dS_3 - dS$, or any other pairs such that it would be possible to eliminate $dS$, $dS_1$, $dS_2$, $dS_3$, and thus obtain an equation such as

$$\epsilon P d\omega + \epsilon_1 P_1 d\omega_1 + \epsilon_2 P_2 d\omega_2 + \epsilon_3 P_3 d\omega_3 = 0;$$

this would, by virtue of the relations between $d\omega$, $d\omega_1$, $d\omega_2$, $d\omega_3$, become

$$\epsilon \frac{\delta\sqrt{\Omega}}{x^2+y^2} - \epsilon_1 \frac{\delta_1\sqrt{\Omega_1}}{x_1{}^2+y_1{}^2} + \epsilon_2 \frac{\delta_2\sqrt{\Omega_2}}{x_2{}^2+y_2{}^2} - \epsilon_3 \frac{\delta_3\sqrt{\Omega_3}}{x_3{}^2+y_3{}^2} = 0,$$

an equation not deducible from the relations which connect $\omega$, $\omega_1$, $\omega_2$, $\omega_3$, and which therefore cannot be satisfied by the variable quadrilateral.

26. The differentials of the formulæ are, it will be observed, of the form $Pd\omega$

$$= \frac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}},$$

where $\sqrt{\Theta}, = \sqrt{f+\theta\,.\,g+\theta}$, is a mere constant,

$$x,\ y = \frac{\cos\omega}{\sqrt{f+\theta}},\ \frac{\sin\omega}{\sqrt{g+\theta}},$$

and

$$\delta^2 = \{(f+\theta)\,x - \alpha\}^2 + \{(g+\theta)\,y - \beta\}^2 - \gamma^2;$$

viz. the form is

$$\frac{\sqrt{(\cos\omega\sqrt{f+\theta}-\alpha)^2 + (\sin\omega\sqrt{g+\theta}-\beta)^2 - \gamma^2}}{\sqrt{\Theta}\,.\left(\dfrac{\cos^2\omega}{f+\theta} + \dfrac{\sin^2\omega}{g+\theta}\right)}\,d\omega,$$

which is, in fact, the same as Casey's form in $\phi$, equation (300), his $\phi$ being $= 90° - \omega$.

Writing as before $v$ in place of his $\theta$, the differential expression becomes simply $= \delta\, dv$: but $\delta^2$ expressed as a function of $v$ is an irrational function $M + N\sqrt{U}$, and $\delta$ would be the root of such a function; so that, if the form originally obtained had been this form $\delta\, dv$, it would have been necessary to transform it into the first-mentioned form $\dfrac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}}$, in which $\delta$ is expressed as a function of $(x, y)$, that is, of $\omega$.

27. The system of course is

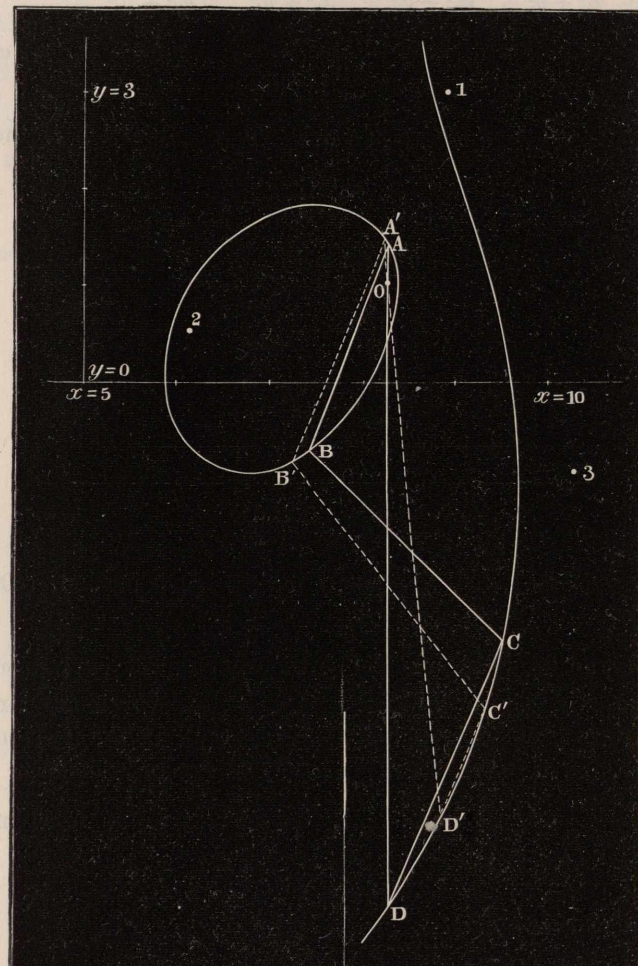
$$\begin{aligned}
dS\ &= \epsilon\delta dv + \epsilon_1\delta_1 dv_1 + \epsilon_2\delta_2 dv_2 + \epsilon_3\delta_3 dv_3,\\
dS_1 &= \epsilon\delta dv - \epsilon_1\delta_1 dv_1 + \epsilon_2\delta_2 dv_2 + \epsilon_3\delta_3 dv_3,\\
dS_2 &= \epsilon\delta dv - \epsilon_1\delta_1 dv_1 - \epsilon_2\delta_2 dv_2 + \epsilon_3\delta_3 dv_3,\\
dS_3 &= \epsilon\delta dv - \epsilon_1\delta_1 dv_1 - \epsilon_2\delta_2 dv_2 - \epsilon_3\delta_3 dv_3,
\end{aligned}$$

where $dv = \dfrac{d\omega}{(x^2+y^2)\sqrt{\Theta}}$, &c.; and this is the most convenient way of writing it.

*Reference to Figure.* Art. No. 28.

28. I constructed a bicircular quartic consisting of an exterior and interior oval with the following numerical data: $(f+\theta_3 = 48,\ f+\theta_1 = 56,\ f+\theta_0 = 60,\ f+\theta_2 = 80;$ $g+\theta_3 = -6,\ g+\theta_1 = 2,\ g+\theta_0 = 6,\ g+\theta_2 = 26)$,—not very convenient ones, inasmuch as

the exterior oval came out too large. The annexed figure shows 0, 1, 2, 3, the centres of the circles of inversion, the interior oval, and a portion of the exterior




oval, also the origin and axes; it will be seen that the centres 0, 2 lie inside the interior oval, the centres 1, 3 outside the exterior oval: I add further the values

$$\begin{aligned}
&\sqrt{f+\theta_3}=6{\cdot}93, && \sqrt{-(g+\theta_3)}=2{\cdot}45, && \alpha_3=10{\cdot}18, && \beta_3=-\ {\cdot}98,\\
&\sqrt{f+\theta_1}=7{\cdot}48, && \sqrt{g+\theta_1}\quad\ =1{\cdot}41, && \alpha_1=\ 8{\cdot}73, && \beta_1=+\,2{\cdot}94,\\
&\sqrt{f+\theta_0}=7{\cdot}75, && \sqrt{g+\theta_0}\quad\ =2{\cdot}45, && \alpha_0=\ 8{\cdot}15, && \beta_0=+\ {\cdot}98,\\
&\sqrt{f+\theta_2}=8{\cdot}94, && \sqrt{g+\theta_2}\quad\ =5{\cdot}09, && \alpha_2=\ 6{\cdot}10, && \beta_2=+\ {\cdot}23.
\end{aligned}$$

We thus see how there exists a series of quadrilaterals $ABCD$, where $A$, $B$ are situate on the interior oval, $C$, $D$ on the exterior oval. Considering the sides as drawn in the senses $A$ to $B$, $B$ to $C$, $C$ to $D$, $D$ to $A$: and representing the inclinations, measured from the positive infinity on the axis of $x$ in the sense $x$ to $y$, by $v_1$, $v_2$, $v_3$, $v$ respectively: then, in passing to the consecutive quadrilateral $A'B'C'D'$, we have $v_1$ and $v_2$ decreasing, $v_3$ and $v$ increasing, that is, $dv_1$ and $dv_2$ negative, $dv_3$ and $dv$ positive; so that, reckoning the elements $AA'$, $BB'$, $CC'$, $DD'$, that is, $dS_1$, $dS_2$, $dS_3$, $dS$, as each of them positive, we have

$$\begin{aligned} dS_2 - dS_1 &= -2\delta_1 dv_1, \\ dS_3 - dS_2 &= -2\delta_2 dv_2, \\ dS \ - dS_3 &= +2\delta_3 dv_3, \\ dS_1 + dS \ &= +2\delta\, dv, \end{aligned}$$

and thence

$$\begin{aligned} dS \ &= \delta dv - \delta_1 dv_1 - \delta_2 dv_2 + \delta_3 dv_3, \\ dS_1 &= \delta dv + \delta_1 dv_1 + \delta_2 dv_2 - \delta_3 dv_3, \\ dS_2 &= \delta dv - \delta_1 dv_1 + \delta_2 dv_2 - \delta_3 dv_3, \\ dS_3 &= \delta dv - \delta_1 dv_1 - \delta_2 dv_2 - \delta_3 dv_3, \end{aligned}$$

which are the correct signs in regard to the particular figure.

### *Reduction of* $\int \frac{\delta\, d\omega}{(x^2+y^2)\sqrt{\Theta}}$ *to Elliptic Integrals.* Art. No. 29.

29. The expression in question is

$$\int d\omega \,.\, \frac{\sqrt{(\cos\omega\sqrt{f+\theta}-\alpha)^2+(\sin\omega\sqrt{g+\theta}-\beta)^2-\gamma^2}}{\left\{\frac{\cos^2\omega}{f+\theta}+\frac{\sin^2\omega}{g+\theta}\right\}\sqrt{\Theta}},$$

where $\sqrt{\Theta}$ is a mere constant; and we may apply it to the Gaussian transformation,

$$\cos\omega = \frac{a+a'\cos T+a''\sin T}{c+c'\cos T+c''\sin T},$$

$$\sin\omega = \frac{b+b'\cos T+b''\sin T}{c+c'\cos T+c''\sin T},$$

where the coefficients $a$, $b$, $c$, $a'$, $b'$, $c'$, $a''$, $b''$, $c''$ are such that identically

$$\cos^2\omega + \sin^2\omega - 1 = \frac{1}{(c+c'\cos T+c''\sin T)^2}\{\cos^2 T+\sin^2 T-1\}:$$

and also

$$(\cos\omega\sqrt{f+\theta}-\alpha)^2+(\sin\omega\sqrt{g+\theta}-\beta)^2-\gamma^2,$$

that is,

$$\cos^2\omega\,(f+\theta)+\sin^2\omega\,(g+\theta)-2\alpha\sqrt{f+\theta}\cos\omega-2\beta\sqrt{g+\theta}\sin\omega+k,$$

$$=\frac{1}{(c+c'\cos T+c''\sin T)^2}(G_1-G_2\cos^2 T-G_3\sin^2 T).$$

30. It is found that $G_1$, $G_2$, $G_3$ are the roots of a cubic equation

$$(G+\theta-\theta_1)(G+\theta-\theta_2)(G+\theta-\theta_3),$$

which being so, we may assume $G_1=\theta_1-\theta$, $G_2=\theta_2-\theta$, $G_3=\theta_3-\theta$; the second condition, in fact, then is

$$(f+\theta)\cos^2\omega+(g+\theta)\sin^2\omega-2\alpha\sqrt{f+\theta}\cos\omega-2\beta\sqrt{g+\theta}\sin\omega+k$$

$$=\frac{1}{(c+c'\cos T+c''\sin T)^2}\{\theta_1-\theta-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T\};$$

and this being so, we find without difficulty the values

$$a^2=\frac{g+\theta_1\,.\,f+\theta_2\,.\,f+\theta_3}{f-g\,.\,\theta_1-\theta_2\,.\,\theta_1-\theta_3},\quad b^2=\frac{f+\theta_1\,.\,g+\theta_2\,.\,g+\theta_3}{g-f\,.\,\theta_1-\theta_2\,.\,\theta_1-\theta_3},\quad c^2=\frac{f+\theta_1\,.\,g+\theta_1}{\theta_1-\theta_2\,.\,\theta_1-\theta_3},$$

$$a'^2=-\frac{g+\theta_2\,.\,f+\theta_1\,.\,f+\theta_3}{f-g\,.\,\theta_2-\theta_1\,.\,\theta_2-\theta_3},\quad b'^2=-\frac{f+\theta_2\,.\,g+\theta_1\,.\,g+\theta_3}{g-f\,.\,\theta_2-\theta_1\,.\,\theta_2-\theta_3},\quad c'^2=-\frac{f+\theta_2\,.\,g+\theta_2}{\theta_2-\theta_1\,.\,\theta_2-\theta_3},$$

$$a''^2=-\frac{g+\theta_3\,.\,f+\theta_1\,.\,f+\theta_2}{f-g\,.\,\theta_3-\theta_1\,.\,\theta_3-\theta_2},\quad b''^2=-\frac{f+\theta_3\,.\,g+\theta_1\,.\,g+\theta_2}{g-f\,.\,\theta_3-\theta_1\,.\,\theta_3-\theta_2},\quad c''^2=-\frac{f+\theta_3\,.\,g+\theta_3}{\theta_3-\theta_1\,.\,\theta_3-\theta_2}.$$

To make these positive, the order of ascending magnitude must, however, be not as heretofore $\theta_3$, $\theta_1$, $\theta_2$, but $\theta_3$, $\theta_2$, $\theta_1$, viz. we must have $f+\theta_1$, $f+\theta_2$, $f+\theta_3$, $g+\theta_1$, $g+\theta_2$, $-(g+\theta_3)$, $\theta_1-\theta_3$, $\theta_1-\theta_2$, $\theta_2-\theta_3$ all positive.

31. The above are the values of the squares of the coefficients; we must have definite relations between the signs of the products $aa'$, $bb'$, $ab$, &c., viz. we may have

$$a'a''=\frac{f+\theta_1}{f-g\,.\,\theta_2-\theta_3}\sqrt{\frac{\Theta_2\Theta_3}{\theta_3-\theta_1\,.\,\theta_1-\theta_2}},\quad a''a=\frac{f+\theta_2}{f-g\,.\,\theta_3-\theta_1}\sqrt{\frac{-\Theta_3\Theta_1}{\theta_1-\theta_2\,.\,\theta_2-\theta_3}},$$

$$b'b''=\frac{g+\theta_1}{g-f\,.\,\theta_2-\theta_3}\sqrt{\quad\text{,,}\quad},\quad b''b=\frac{g+\theta_2}{g-f\,.\,\theta_3-\theta_1}\sqrt{\quad\text{,,}\quad},$$

$$c'c''=\frac{1}{\theta_2-\theta_3}\sqrt{\quad\text{,,}\quad},\quad c''c=\frac{1}{\theta_3-\theta_1}\sqrt{\quad\text{,,}\quad},$$

$$aa'=\frac{f+\theta_2}{f-g\,.\,\theta_1-\theta_2}\sqrt{\frac{-\Theta_1\Theta_2}{\theta_2-\theta_3\,.\,\theta_3-\theta_1}},$$

$$bb'=\frac{g+\theta_2}{g-f\,.\,\theta_1-\theta_2}\sqrt{\quad\text{,,}\quad},$$

$$cc'=\frac{1}{\theta_1-\theta_2}\sqrt{\quad\text{,,}\quad},$$

and further

$$ab = \frac{1}{f-g\,.\,\theta_3-\theta_1\,.\,\theta_1-\theta_2}\sqrt{-\Theta_1\Theta_2\Theta_3}, \quad bc = -\frac{f+\theta_1}{\theta_3-\theta_1\,.\,\theta_1-\theta_2}\sqrt{\frac{g+\theta_1\,.\,g+\theta_2\,.\,g+\theta_3}{g-f}},$$

$$a'b' = \frac{-1}{f-g\,.\,\theta_1-\theta_2\,.\,\theta_2-\theta_3}\sqrt{\quad ,, \quad}, \quad b'c' = \frac{f+\theta_2}{\theta_1-\theta_2\,.\,\theta_2-\theta_3}\sqrt{\quad ,, \quad},$$

$$a''b'' = \frac{-1}{f-g\,.\,\theta_2-\theta_3\,.\,\theta_3-\theta_1}\sqrt{\quad ,, \quad}, \quad b''c'' = \frac{f+\theta_3}{\theta_2-\theta_3\,.\,\theta_3-\theta_1}\sqrt{\quad ,, \quad},$$

$$ca = -\frac{g+\theta_1}{\theta_3-\theta_1\,.\,\theta_1-\theta_2}\sqrt{\frac{f+\theta_1\,.\,f+\theta_2\,.\,f+\theta_3}{f-g}},$$

$$c'a' = \frac{g+\theta_2}{\theta_1-\theta_2\,.\,\theta_2-\theta_3}\sqrt{\quad ,, \quad},$$

$$c''a'' = \frac{g+\theta_3}{\theta_2-\theta_3\,.\,\theta_3-\theta_1}\sqrt{\quad ,, \quad};$$

and also

$$b'c''+b''c' = \frac{2g+\theta_2+\theta_3}{\theta_2-\theta_3}\sqrt{\frac{g+\theta_1.f+\theta_2.f+\theta_3}{g-f.\theta_3-\theta_1.\theta_1-\theta_2}}, \quad c'a''+c''a' = \frac{2f+\theta_2+\theta_3}{\theta_2-\theta_3}\sqrt{\frac{f+\theta_1.g+\theta_2.g+\theta_3}{f-g.\theta_3-\theta_1.\theta_1-\theta_2}},$$

$$b''c+bc'' = \frac{2g+\theta_3+\theta_1}{\theta_3-\theta_1}\sqrt{-\frac{g+\theta_2.f+\theta_3.f+\theta_1}{g-f.\theta_1-\theta_2.\theta_2-\theta_3}}, \quad c''a+ca'' = \frac{2f+\theta_3+\theta_1}{\theta_3-\theta_1}\sqrt{-\frac{f+\theta_2.g+\theta_3.g+\theta_1}{f-g.\theta_1-\theta_2.\theta_2-\theta_3}},$$

$$bc'+b'c = \frac{2g+\theta_1+\theta_2}{\theta_1-\theta_2}\sqrt{-\frac{g+\theta_3.f+\theta_1.f+\theta_2}{g-f.\theta_2-\theta_3.\theta_3-\theta_1}}, \quad ca'+c'a = \frac{2f+\theta_1+\theta_2}{\theta_1-\theta_2}\sqrt{-\frac{f+\theta_3.g+\theta_1.g+\theta_2}{f-g.\theta_2-\theta_3.\theta_3-\theta_1}}.$$

32. These values, in fact, satisfy the several relations which exist between the nine coefficients; viz. the original expressions of $\cos\omega$, $\sin\omega$, in terms of $\cos T$, $\sin T$ give conversely expressions of $\cos T$, $\sin T$ in terms of $\cos\omega$, $\sin\omega$, the two sets being

$$\cos\omega = \frac{a+a'\cos T+a''\sin T}{c+c'\cos T+c''\sin T}, \quad \cos T = -\frac{a'\cos\omega+b'\sin\omega-c'}{a\cos\omega+b\sin\omega-c},$$

$$\sin\omega = \frac{b+b'\cos T+b''\sin T}{c+c'\cos T+c''\sin T}, \quad \sin T = -\frac{a''\cos\omega+b''\sin\omega-c''}{a\cos\omega+b\sin\omega-c}:$$

and we have then the relations

$$\cos^2\omega+\sin^2\omega-1 = \frac{1}{(c+c'\cos T+c''\sin T)^2}(\cos^2 T+\sin^2 T-1),$$

$$\cos^2 T+\sin^2 T-1 = \frac{1}{(a\cos\omega+b\sin\omega-c)^2}(\cos^2\omega+\sin^2\omega-1),$$

$$(\theta+f)\cos^2\omega+(\theta+g)\sin^2\omega-2\alpha\sqrt{\theta+f}\cos\omega-2\beta\sqrt{\theta+g}\sin\omega+k$$

$$= \frac{1}{(c+c'\cos T+c''\sin T)^2}\{(\theta_1-\theta)-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T\},$$

$$(\theta_1-\theta)-(\theta_2-\theta)\cos^2 T-(\theta_3-\theta)\sin^2 T$$

$$= \frac{1}{(a\cos\omega+b\sin\omega-c)^2}\{(\theta+f)\cos^2\omega+(\theta+g)\sin^2\omega-2\alpha\sqrt{\theta+f}\cos\omega-2\beta\sqrt{\theta+g}\sin\omega+k\},$$

giving the four sets each of six equations

$$\begin{aligned}
a^2+b^2-c^2&=-1, & a'a''+b'b''-c'c''&=0,\\
a'^2+b'^2-c'^2&=+1, & a''a+b''b-c''c&=0,\\
a''^2+b''^2-c''^2&=+1, & aa'+bb'-cc'&=0,\\
-a^2+a'^2+a''^2&=+1, & -bc+b'c'+b''c''&=0,\\
-b^2+b'^2+b''^2&=+1, & -ca+c'a'+c''a''&=0,\\
-c^2+c'^2+c''^2&=-1, & -ab+a'b'+a''b''&=0,
\end{aligned}$$

$$\begin{aligned}
(\theta+f)a^2+(\theta+g)b^2-2\alpha\sqrt{\theta+f}\,ac-2\beta\sqrt{\theta+g}\,bc+kc^2&=\theta_1+\theta,\\
(\theta+f)a'^2+(\theta+g)b'^2-2\alpha\sqrt{\theta+f}\,a'c'-2\beta\sqrt{\theta+g}\,b'c'+kc'^2&=-\theta_2+\theta,\\
(\theta+f)a''^2+(\theta+g)b''^2-2\alpha\sqrt{\theta+f}\,a''c''-2\beta\sqrt{\theta+g}\,b''c''+kc''^2&=-\theta_3+\theta,
\end{aligned}$$

$$\begin{aligned}
(\theta+f)a'a''+(\theta+g)b'b''-\alpha\sqrt{\theta+f}(a'c''+a''c')-\beta\sqrt{\theta+g}(b'c''+b''c')+kc'c''&=0,\\
(\theta+f)a''a+(\theta+g)b''b-\alpha\sqrt{\theta+f}(a''c+ac'')-\beta\sqrt{\theta+g}(b''c+bc'')+kc''c&=0,\\
(\theta+f)aa'+(\theta+g)bb'-\alpha\sqrt{\theta+f}(ac'+a'c)-\beta\sqrt{\theta+g}(bc'+b'c)+kcc'&=0,
\end{aligned}$$

$$\begin{aligned}
(\theta_1-\theta)a^2-(\theta_2-\theta)a'^2-(\theta_3-\theta)a''^2&=\theta+f, \text{ or say } (\theta_1+f)a^2-(\theta_2+f)a'^2-(\theta_3+f)a''^2=0,\\
(\theta_1-\theta)b^2-(\theta_2-\theta)b'^2-(\theta_3-\theta)b''^2&=\theta+g, \quad ,, \quad (\theta_1+g)b^2-(\theta_2+g)b'^2-(\theta_3+g)b''^2=0,\\
(\theta_1-\theta)c^2-(\theta_2-\theta)c'^2-(\theta_3-\theta)c''^2&=k, \quad ,, \quad \theta_1c^2-\theta_2c'^2-\theta_3c''^2=k+\theta,
\end{aligned}$$

$$\begin{aligned}
-(\theta_1-\theta)bc+(\theta_2-\theta)b'c'+(\theta_3-\theta)b''c''&=-\beta\sqrt{\theta+g},\\
-(\theta_1-\theta)ca+(\theta_2-\theta)c'a'+(\theta_3-\theta)c''a''&=-\alpha\sqrt{\theta+f},\\
-(\theta_1-\theta)ab+(\theta_2-\theta)a'b'+(\theta_3-\theta)a''b''&=0;
\end{aligned}$$

all which formulæ are in fact satisfied by the foregoing values of the expressions $a^2$, $b^2$, $a'^2$, &c.

33. We then have

$$d\omega=\frac{dT}{c+c'\cos T+c''\sin T};$$

the radical which multiplies $d\omega$ being

$$=\frac{1}{c+c'\cos T+c''\sin T}\sqrt{\theta_1-\theta_2\cos^2 T-\theta_3\sin^2 T},$$

the differential becomes

$$=\frac{dT\sqrt{\theta_1-\theta_2\cos^2 T-\theta_3\sin^2 T}}{\left(\dfrac{\cos^2\omega}{f+\theta}+\dfrac{\sin^2\omega}{g+\theta}\right)(c+c'\cos T+c''\sin T)^2\sqrt{\Theta}},$$

that is,

$$=\frac{dT\sqrt{\theta_1-\theta_2\cos^2 T-\theta_3\sin^2 T}}{\left\{\dfrac{1}{f+\theta}(a+a'\cos T+a''\sin T)^2+\dfrac{1}{g+\theta}(b+b'\cos T+b''\sin T)^2\right\}\sqrt{\Theta}}.$$

The denominator could, of course, be reduced to the form $(*\between 1, \cos T, \sin T)^2$; but the actual form seems preferable, inasmuch as it puts in evidence the linear factors

$$\frac{1}{\sqrt{f+\theta}}(a+a'\cos T+a''\sin T)\pm\frac{i}{\sqrt{g+\theta}}(b+b'\cos T+b''\sin T),$$

and there seems to be no advantage in further reducing the integral.