669.

ON A PROBLEM OF ARRANGEMENTS.

[From the *Proceedings of the Royal Society of Edinburgh*, t. IX. (1878), pp. 338—342.]

It is a well-known problem to find for n letters the number of the arrangements in which no letter occupies its original place; and the solution of it is given by the following general theorem:—viz., the number of the arrangements which satisfy any r conditions is

$$(1-1)(1-2)\ldots\ldots(1-r),$$
$$=1-\Sigma(1)+\Sigma(12)-\ldots\ldots\pm(12\ldots r),$$

where 1 denotes the whole number of arrangements; (1) the number of them which fail in regard to the first condition; (2) the number which fail in regard to the second condition; (12) the number which fail in regard to the first condition, and also in regard to the second condition; and so on: $\Sigma(1)$ means $(1)+(2)+\ldots+(r)$: $\Sigma(12)$ means $(12)+(13)+(2r)+\ldots+(r-1,\ r)$; and so on, up to $(12\ldots r)$, which denotes the number failing in regard to each of the r conditions.

Thus, in the special problem, the first condition is that the letter in the first place shall not be a; the second condition is that the letter in the second place shall not be b; and so on; taking $r=n$, we have the known result,

$$\text{No.}=\Pi n-\frac{n}{1}\Pi(n-1)+\frac{n\,.\,n-1}{1\,.\,2}\Pi(n-2)\,.+\ldots\pm\frac{n\,.\,n-1\ldots 2\,.\,1}{1\,.\,2\ldots n},$$
$$=1\,.\,2\,.\,3\ldots n\left\{1-\frac{1}{1}+\frac{1}{1\,.\,2}-\frac{1}{1\,.\,2\,.\,3}+\ldots\pm\frac{1}{1\,.\,2\,.\,3\ldots n}\right\},$$

giving for the several cases

$$n=2,\ 3,\ 4,\quad 5,\quad 6,\quad 7,\ldots$$
$$\text{No.}=1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ldots$$

I proceed to consider the following problem, suggested to me by Professor Tait, in connexion with his theory of knots: to find the number of the arrangements of n letters $abc\ldots jk$, when the letter in the first place is not a or b, the letter in the second place not b or $c,\ldots$, the letter in the last place not k or a.

Numbering the conditions 1, 2, 3, ..., n, according to the places to which they relate, a single condition is called [1]; two conditions are called [2] or [1, 1], according as the numbers are consecutive or non-consecutive: three conditions are called [3], [2, 1], or [1, 1, 1], according as the numbers are all three consecutive, two consecutive and one not consecutive, or all non-consecutive; and so on: the numbers which refer to the conditions being always written in their natural order, and it being understood that they follow each other cyclically, so that 1 is consecutive to n. Thus, $n=6$, the set 126 of conditions is [3], as consisting of 3 consecutive conditions; and similarly 1346 is [2, 2].

Consider a single condition [1], say this is 1; the arrangements which fail in regard to this condition are those which contain in the first place a or b; whichever it be, the other $n-1$ letters may be arranged in any form whatever; and there are thus $2\Pi(n-1)$ failing arrangements.

Next for two conditions; these may be [2], say the conditions are 1 and 2: or else [1, 1], say they are 1 and 3. In the former case, the arrangements which fail are those which contain in the first and second places ab, ac, or bc: and for each of these, the other $n-2$ letters may be arranged in any order whatever; there are thus $3\Pi(n-2)$ failing arrangements. In the latter case, the failing arrangements have in the first place a or b, and in the third place c or d,—viz. the letters in these two places are $a.c$, $a.d$, $b.c$, or $b.d$, and in each case the other $n-2$ letters may be arranged in any order whatever: the number of failing arrangements is thus $=2.2.\Pi(n-2)$. And so, in general, when the conditions are $[\alpha, \beta, \gamma, \ldots]$, the number of failing arrangements is

$$=(\alpha+1)(\beta+1)(\gamma+1)\ldots\Pi(n-\alpha-\beta-\gamma\ldots).$$

But for $[n]$, that is, for the entire system of the n conditions, the number of failing arrangements is (not as by the rule it should be $=n+1$, but) $=2$,—viz. the only arrangements which fail in regard to each of the n conditions are (as is at once seen), $abc\ldots jk$, and $bc\ldots jka$.

Changing now the notation so that [1], [2], [1, 1], &c., shall denote the *number* of the conditions [1], [2], [1, 1], &c., respectively, it is easy to see the form of the general result. If, for greater clearness, we write $n=6$, we have

$$\text{No.} = \underset{1}{720} \; \underset{-\Sigma(1)}{-\{([1]=6)\,2\}\,120} \; \underset{+\Sigma(12)}{+\left\{\begin{array}{l}\ \ ([2]\quad\ =6)\,3\\ +([1,1]=9)\,2.2\end{array}\right\}24} \; \underset{-\Sigma(123)}{-\left\{\begin{array}{l}\ \ ([3]\qquad =\ 6)\,4\\ +([2,1]\quad =12)\ \ 3.2\\ +([1,1,1]=\ 2)\,2.2.2\end{array}\right\}6}$$

$$\underset{+\Sigma(1234)}{+\left\{\begin{array}{l}\ \ ([4]\quad\ =6)\,5\\ +([3,1]=6)\,4.2\\ +([2,2]=3)\,3.3\end{array}\right\}2} \qquad \underset{-\Sigma(12345)}{-\{([5]=6)\,6\}\,1} \qquad \underset{+(123456)}{+\{([6]=1)\,2\}};$$

or, reducing into numbers, this is

$$\text{No.} = 720 - 1440 + 1296 - 672 + 210 - 36 + 2, \quad = 80.$$

The formula for the next succeeding case, $n = 7$, gives

$$\text{No.} = 5040 - 10080 + 9240 - 5040 + 1764 - 392 + 49 - 2, \quad = 579.$$

Those for the preceding cases, $n = 3$, 4, 5, respectively are

$$\text{No.} = 6 - 12 + 9 - 2 \quad = 1,$$

$$\text{No.} = 24 - 48 + 40 - 16 + 2 \quad = 2,$$

$$\text{No.} = 120 - 240 + 210 - 100 + 25 - 2 \quad = 13.$$

We have in general $[1] = n$, $[2] = n$, $[1, 1] = \frac{1}{2}n(n-3)$; and in the several columns of the formulæ the sums of the numbers thus represented are equal to the coefficients of $(1+1)^2$: thus, when $n = 6$ as above, the sums are 6, 15, 20, 15, 6, 1. As regards the calculation of the numbers in question, any symbol $[\alpha, \beta, \gamma]$ is a sum of symbols $[\alpha - \alpha' + \beta - \beta' + \gamma - \gamma' + \ldots]$, where $\alpha' + \beta' + \gamma' + \ldots$ is any partition of $n - (\alpha + \beta + \gamma + \ldots)$; read, of the series of numbers $1, 2, 3, \ldots, n$, taken in cyclical order beginning with any number, retain α, omit α', retain β, omit β', retain γ, omit $\gamma', \ldots$. Thus in particular, $n = 6$, $[1, 1]$ is a sum of symbols $[1 - 3 + 1 - 1]$ and $[1 - 2 + 1 - 2]$; it is clear that any such symbol $[\alpha - \alpha' + \beta - \beta' + \ldots]$ is $= n$ or a submultiple of n (in particular, if n be prime, the symbol is always $= n$): and we thus in every case obtain the value of $[\alpha, \beta, \gamma, \ldots]$ by taking for the negative numbers the several partitions of

$$n - (\alpha + \beta + \gamma + \ldots),$$

and for each symbol

$$[\alpha - \alpha' + \beta - \beta' + \gamma - \gamma' + \ldots],$$

writing its value, $= n$ or a given submultiple of n, as just mentioned. There would, I think, be no use in pursuing the matter further, by seeking to obtain an analytical expression for the symbols $[\alpha, \beta, \gamma, \ldots]$.

For the actual formation of the required arrangements, it is of course easy, when all the arrangements are written down, to strike out those which do not satisfy the prescribed conditions, and so obtain the system in question. Or introducing the notion of substitutions*, and accordingly considering each arrangement as derived by a substitution from the primitive arrangement *abcd...jk*, we can write down the substitutions which give the system of arrangements in which no letter occupies its original place: viz. we must for this purpose partition the n letters into parts, no part less than 2, and then in each set taking one letter (say the first in alphabetical order) as fixed, permute in every possible way the other letters of the set; we thus obtain

* In explanation of the notation of substitutions, observe that (*abcde*) means that *a* is to be changed into *b*, *b* into *c*, *c* into *d*, *d* into *e*, *e* into *a*; and similarly (*ab*) (*cde*) means that *a* is to be changed into *b*, *b* into *a*, *c* into *d*, *d* into *e*, *e* into *c*.

all the substitutions which move every letter. Thus when $n=5$, we obtain the 44 substitutions for the letters *abcde*, viz. these are

(*abcde*), &c., 24 symbols obtained by permuting in every way the four letters *b, c, d, e*;

(*ab*) (*cde*), &c., 20 symbols corresponding to the 10 partitions *ab, cde*, and for each of them 2 arrangements such as *cde, ced.*

And then if we reject those symbols which contain in any ( ) two consecutive letters, we have the substitutions which give the arrangements wherein the letter in the first place is not *a* or *b*, that in the second place not *b* or *c*, and so on. In particular, when $n=5$, rejecting the substitutions which contain in any ( ), *ab, bc, cd, de*, or *ea*, we have 13 substitutions, which may be thus arranged:—

(*acbed*), (*ac*) (*bed*), (*acebd*), (*adbec*), (*aedbc*),
(*aedbc*), (*bd*) (*aec*),
(*acedb*), (*ce*) (*adb*),
(*aecbd*), (*ad*) (*bec*),
(*adceb*), (*be*) (*adc*).

Here in the first column, performing on the symbol (*acbed*) the substitution (*abcde*), we obtain (*bdcae*), = (*aebdc*), the second symbol; and so again and again operating with (*abcde*), we obtain the remaining symbols of the column; these are for this reason said to be of the same type. In like manner, symbols of the second column are of the same type; but the symbols in the remaining three columns are each of them a type by itself; viz. operating with (*abcde*) upon (*acebd*), we obtain (*bdace*), = (*acebd*); and the like as regards (*adbec*) and (*aedbc*) respectively. The 13 substitutions are thus of 5 different types, or say the arrangements to which they belong, viz.

cebad, ceabd, cdeab, deabc, eabcd,
edacb, edabc,
caebd, daebc,
edbac, debac,
daecb, deacb,

are of 5 different types. The question to determine for any value of n, the number of the different types, is, it would appear, a difficult one, and I do not at present enter upon it.