# 670.

## [NOTE ON MR MUIR'S SOLUTION OF A "PROBLEM OF ARRANGEMENT."]

[From the *Proceedings of the Royal Society of Edinburgh*, t. IX. (1878), pp. 388—391.]

THE investigation may be carried further: writing for shortness $u_3$, $u_4$, &c., in place of $\Psi(3)$, $\Psi(4)$, &c., the equations are

$$\begin{aligned} u_3 &= 1, \\ u_4 &= 2u_3, \\ u_5 &= 3u_4 + \ \ 6u_3 + \ \ 1, \\ u_6 &= 4u_5 + \ \ 8u_4 + 12u_3, \\ u_7 &= 5u_6 + 10u_5 + 15u_4 + 18u_3 + 1. \end{aligned}$$

Hence assuming

$$u = u_3 + u_4 x + u_5 x^2 + u_6 x^3 + u_7 x^4 + \dots,$$

we have

$$\begin{aligned} u = \frac{1}{1-x^2} &+ u_3(2x + \ \ 6x^2 + 12x^3 + 18x^4 + \dots) \\ &+ u_4(3x^2 + \ \ 8x^3 + 15x^4 + 22x^5 + \dots) \\ &+ u_5(4x^3 + 10x^4 + 18x^5 + 26x^6 + \dots) \\ &+ u_6(5x^4 + 12x^5 + 21x^6 + 30x^7 + \dots); \end{aligned}$$

so that, forming the equation

$$\begin{aligned} u' \frac{x^2}{(1-x)^2} = \ \ & u_4(\ x^2 + \ \ 2x^3 + \ \ 3x^4 + \ \ 4x^5 + \dots) \\ &+ u_5(2x^3 + \ \ 4x^4 + \ \ 6x^5 + \ \ 8x^6 + \dots) \\ &+ u_6(3x^4 + \ \ 6x^5 + \ \ 9x^6 + 12x^7 + \dots), \end{aligned}$$

where $u'$ denotes $\frac{du}{dx}$, we have

$$u - u' \frac{x^3}{(1-x)^2} = \frac{1}{1-x^2} + (u_3 + u_4 x + u_5 x^2 + \dots)(2x + 6x^2 + 12x^3 + 18x^4 + \dots)$$

$$= \frac{1}{1-x^2} + u(2x + 6x^2 + 12x^3 + 18x^4 + \dots);$$

or, what is the same thing,

$$u - u' \frac{x^2}{(1-x)^2} = \frac{1}{1-x^2} + u\left\{\frac{2x}{(1-x)^3} - \frac{2x^4}{(1-x)^3(1+x)}\right\};$$

that is,

$$\left\{1 - \frac{2x}{(1-x)^3} + \frac{2x^4}{(1-x)^3(1+x)}\right\} u - \frac{x^2}{(1-x)^2} u' = \frac{1}{1-x^2}.$$

This equation may be simplified: write

$$u = -\frac{1-x^2}{x^4} Q, \quad = \left(-\frac{1}{x^4} + \frac{1}{x^2}\right) Q,$$

then

$$u' = \left(\frac{4}{x^5} - \frac{2}{x^3}\right) Q + \frac{1-x^2}{x^4} Q',$$

and the equation is

$$\left\{-\frac{1-x^2}{x^4} + \frac{2}{x^3}\frac{1+x}{(1+x)^2} - \frac{2}{(1-x)^2} - \frac{4}{x^3}\frac{1}{(1-x)^2} + \frac{2}{x(1-x)^2}\right\} Q + \frac{1+x}{(1+x)x^2} Q' = \frac{1}{1-x^2};$$

that is,

$$\left\{-\frac{1}{x^4} + \frac{1}{x^2} - \frac{2}{x^3(1-x)^2} + \frac{2}{x^2(1-x)^2} + \frac{2}{x(1-x)^2} - \frac{2}{(1-x)^2}\right\} Q + \frac{1+x}{(1-x)x^2} Q' = \frac{1}{1-x^2},$$

viz. this is

$$\left\{-\frac{(1-x)^2}{x^4} + \frac{(1-x)^2}{x^2} - \frac{2}{x^3} + \frac{2}{x^2} + \frac{2}{x} - 2\right\} Q + \frac{1-x^2}{x^2} Q' = \frac{1-x}{1+x},$$

that is,

$$\left\{-\frac{1}{x^4} + \frac{2}{x^2} - 1\right\} Q + \frac{1-x^2}{x^2} Q' = \frac{1-x}{1+x},$$

or

$$-\frac{(1-x^2)^2}{x^4} Q + \frac{1-x^2}{x^2} Q' = \frac{1-x}{1+x};$$

or finally,

$$Q\left(1 - \frac{1}{x^2}\right) + Q' = \frac{x^2}{(1+x)^2},$$

giving

$$Q = e^{-\left(x+\frac{1}{x}\right)} \int \frac{x^2}{(x+1)^2} e^{x+\frac{1}{x}} \, dx,$$

and thence

$$u = \frac{x^2-1}{x^4} e^{-\left(x+\frac{1}{x}\right)} \int \frac{x^2}{(x+1)^2} e^{\left(x+\frac{1}{x}\right)} \, dx,$$

which is the value of the generating function

$$u = u_3 + u_4 x + u_5 x^2 + \&\text{c}.$$

But for the purpose of calculation it is best to integrate by a series the differential equation for $Q$: assuming

$$Q = -q_3x^4 - q_4x^5 - q_5x^6 - \ldots,$$

we find

$$\begin{aligned} q_4 &= 4q_3 \qquad\quad - 2, \\ q_5 &= 5q_4 \quad + q_3 \quad + 3, \\ q_6 &= 6q_5 \quad + q_4 \quad - 4, \\ q_7 &= 7q_6 \quad + q_5 \quad + 5, \\ &\vdots \\ q_n &= nq_{n-1} + q_{n-2} + (-)^{n-1}(n-2). \end{aligned}$$

We have thus for $q_3$, $q_4$, $q_5$, ... the values 1, 2, 14, 82, 593, 4820, ..., and thence

$$u = (1 - x^2)(1 + 2x + 14x^2 + 82x^3 + 593x^4 + 4820x^5 + \ldots),$$

viz. writing

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 14 | 82 | 593 | 4820... |
| | | | −1 | −2 | −14 | −82 |
| the values of $u_3$, $u_4$, ... are | 1, | 2, | 13, | 80, | 579, | 4738, ..., |

agreeing with the results found above.

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In the more simple problem, where the arrangements of the $n$ things are such that no one of them occupies its original place, if $u_n$ be the number of arrangements, we have

$$\begin{aligned} u_2 &= 1 &&= 1, \\ u_3 &= 2\,u_2 &&= 2, \\ u_4 &= 3\,(u_3 + u_2) &&= 9, \\ u_5 &= 4\,(u_4 + u_3) &&= 44, \\ &\vdots \\ u_{n+1} &= n\,(u_n + u_{n-1}), \end{aligned}$$

and writing

$$u = u_2 + u_3x + u_4x^2 + \ldots,$$

we find

$$u = 1 + (2x + 3x^2)\,u + (x^2 + x^3)\,u';$$

that is,

$$(-1 + 2x + 3x^2)\,u + (x^2 + x^3)\,u' = -1,$$

or, what is the same thing,

$$u' + \left(\frac{3}{x} - \frac{1}{x^2}\right) u = -\frac{1}{x^2(1+x)},$$

whence

$$u = x^{-3} e^{-\frac{1}{x}} \int \frac{-x}{1+x} e^{\frac{1}{x}}\, dx.$$

But the calculation is most easily performed by means of the foregoing equation of differences, itself obtained from the differential equation written in the foregoing form,

$$(-1 + 2x + 3x^2)\,u + (x^2 + x^3)\,u' = -1.$$